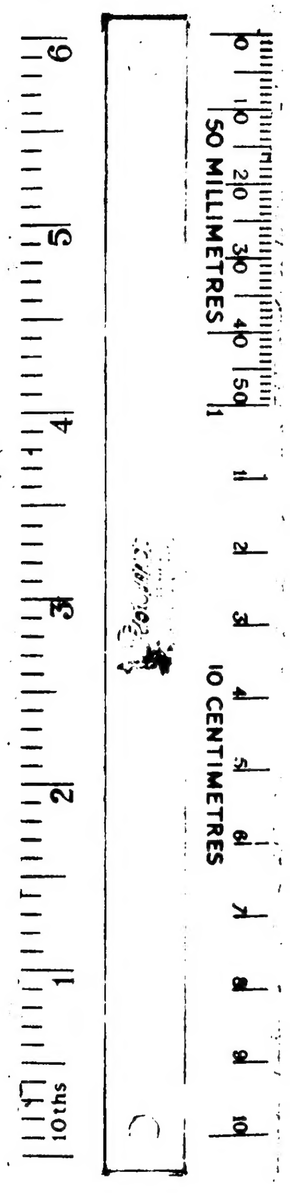
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THE THIRD & FOURTH BOOKS OF THE ELEMENTS OF ALGEBRA.

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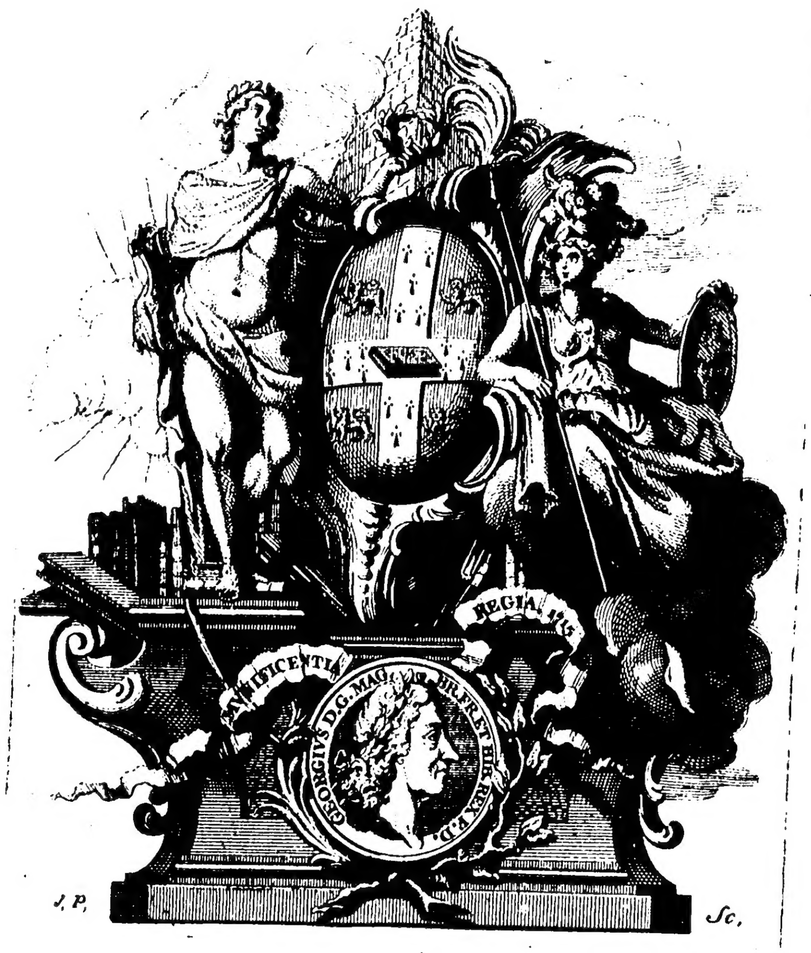
JOHN KERSEY.

——— *Si quid novisti rectius istis,*
Candidus imperti; si non, his utere mecum.

BIBL. CANT. CAD.

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The CONTENTS of the Third Book.

THE Third Book is an Analyſis *in* Species *of the choiceſt of* Diophantus's *much admired Queſtions concerning Squares, Cubes, and right-angled Triangles in rational numbers; with other Queſtions of like nature. To which is alſo added a brief Expoſition upon Monſieur* Fermat's *Analytical Invention, prefix'd to Monſieur* Bachet's *Comment upon* Diophantus, *Printed at* Toloze, *Ann.* 1670. *The number of Queſtions in the Third Book is* 130.

The CONTENTS of the Fourth Book.

THE Fourth Book is an Introduction to Mathematical Reſolution and Compoſition; where, the excellent uſe of the Algebraical or Analytical Art is evidently ſhewn, both in finding out the Solutions of Geometrical Plane Problems, (viz. *of ſuch whoſe Delineations require only the drawing of Right and Circular lines;*) *as alſo in diſcovering Synthetical Demonſtrations from the Analytical ſteps, to prove the truth of thoſe Solutions by a* Series *of Conſequences deduced altogether from things really given or granted. All which are clearly expounded, and copiouſly exemplified both Geometrically and Arithmetically, by a choice Collection of uſeful and delightfull Problems. The Fourth Book is divided into Ten Chapters, the Contents whereof are theſe,* viz.

CHAP.

1. The Explication of Characters, *&c.*
2. The Explication of Axioms.
3. The Explication of Definitions concerning the uſual ways of arguing to deduce one Analogy from another.
4. Various Fundamental Theorems demonſtrated.
5. A Collection of Canonical Geometrical Effections.
6. Algebraical Fractions Geometrically expounded.
7. 8. 9. 10. Four *Claſſes* of Examples of the Reſolution and Compoſition of Plane Problems.

Faults to be Corrected in the Third and Fourth Books, before they be read.

Pag.	*Lin.*	*Faults,*	*thus to be Corrected.*
7	27	part,	pair.
23	24	To the given difference 60,	The given difference is 60.
55	38	adding 2,	adding 3.
63	10	*Chapt.*	*Book.*
67	37	} upon the fourth *Book*,	upon *Queſt.* 2. *Book* IV.
71	48	}	
104	54	$\dashv \frac{228}{72}$,	$\dashv \frac{228}{729}$.
109	49	160989184000,	160989184000.
214	4	right-angle,	right line.
221	2	greater,	leſſer.
228	16	36,	26.
228	19	30,	40.
229	51	and the extremes of,	of the extremes and.

THE

THE ELEMENTS OF THE ALGEBRAICAL ART.

Book III.

AMong all the Writers upon the *Algebraical Art*, there hath none been hitherto known more antient, nor any more famous for ſhewing the admirable force of that Art in ſolving Arithmetical Problems, than *Diophantus* of *Alexandria*, who lived, (as Authors compute,) above thirteen hundred years ago. He wrote thirteen Books of *Arithmetick*, and one concerning *Multangular numbers*; but of thoſe thirteen, ſix only are extant, which contain two hundred and eight Queſtions, many of which are ſo knotty and abſtruſe, that they can hardly be ſolved without the help of *Diophantus* his peculiar Method; whoſe Speculations are ſo ſublime, that where there ſeems to be an impoſſibility of finding out a ſingle Anſwer to a Queſtion, he ſhews how to find out innumerable Anſwers in rational (or ordinary) numbers. Now to give the ingenious Reader of theſe *Elements* a delightful Proſpect of the rare Inventions of that Prince of Arithmeticians, I have with no ſmall labour framed this Third Book, and therein explain'd the Reſolutions of the hardeſt and choiceſt of his Queſtions, in the ſecond, third, fourth, fifth and ſixth Books of his Arithmetick before-mentioned; as alſo of divers other ſubtil Queſtions invented upon his grounds by *Vieta*, *Bachet*, (the beſt Commentator hitherto upon *Diophantus*,) *Fermat* and others; among which alſo divers Queſtions of my own are inſerted, to wit, thoſe which have no citation referring to any Author.

Note, That △ ſtands for the word *Triangle*, and □ for a *Square number*; but as to the reſt of the Characters uſed in this third Book, they have already been explain'd in *Chap.* 1. *Book* 1. of theſe Elements.

QUEST. 1. (This is the 9*th* of the ſecond Book of *Diophantus.*)

To divide a given ſquare number into two Squares.

RESOLUTION 1.

1. Let the ſquare number given to be divided be 16
2. The Root or ſide thereof is 4
3. For the Root or ſide of the firſt of the two Squares ſought put . . . . . *a*
4. Therefore the firſt Square is . . . . . *aa*
5. And conſequently, (by the firſt and fourth ſteps,) the ſecond Square muſt be equal to . . . . . 16 — *aa*
6. Now let the ſide of the ſecond Square be feigned to be . . . . . 2*a* — 4, or 4 — 2*a*
7. Therefore the Square of the ſaid feigned ſide is . . . . . 4*aa* — 16*a* + 16
8. Which Square muſt be equal to 16 — *aa* in the fifth ſtep, hence this following Equation ariſeth, *viz.*

$$4aa - 16a + 16 = 16 - aa;$$

9. From which Equation, after due Reduction, the ſide of the firſt Square will be made known, *viz.* . . . . . } $a = \frac{16}{5}$

10. And by the ninth and ſixth ſteps, the ſide of the ſecond Square will likewiſe be diſcovered, *viz.* . . . . . } $2a - 4 = \frac{12}{5}$

So the ſides of the two Squares ſought are found $\frac{16}{5}$ and $\frac{12}{5}$; which will ſolve the Queſtion; for the Square of $\frac{16}{5}$ is $\frac{256}{25}$, and the Square of $\frac{12}{5}$ is $\frac{144}{25}$, both which Squares added together make $\frac{400}{25}$, that is, 16; as was required.

Note. That which is moſt remarkable in the foregoing Reſolution of *Diophantus*, is, his ingenious and peculiar way of feigning the ſide of a Square to be equated to $16 - aa$ in ſuch manner, that after the Equation is duely reduced, the number repreſented by a will neceſſarily be Rational. Now becauſe he makes great uſe of the like manner of feigning the ſides of Squares to be equal to Algebraical quantities, in reſolving divers hard Queſtions, (as will copiouſly appear in this third Book,) the Learner muſt endeavour to be very well acquainted with the ſaid Method; for his eaſe therefore I ſhall explain it in the following Obſervations.

Obſervations upon Queſt. 1.

1. Concerning the feigned ſide $2a - 4$ in the ſixth ſtep of the foregoing Reſolution, it may be asked, why -4, and not -5, -6, or ſome other number? to this I anſwer, There is a neceſſity that this number be alwayes the ſide of the Square given to be divided into two Squares; ſo *Diophantus* feigns the ſecond ſide to be $2a - 4$, (4 being the ſide or ſquare Root of 16 the given Square,) to the end that in the Square of $2a - 4$ there may be found the Abſolute number 16, to wit, the Square given to be divided; for then the Square of $2a - 4$ being equated to $16 - aa$, (as in the eighth ſtep of the Reſolution,) there will be found $+16$ on each part of the Equation, whence by ſubtracting 16 from each part, there ariſeth $5aa = 16a$; and conſequently each part of this Equation being divided by a, the Quotients give $5a = 16$, wherefore $a = \frac{16}{5}$.

2. One of the parts of the ſaid feigned ſide of the ſecond Square muſt (in this *Queſt.* 1.) neceſſarily have the ſign $-$ prefixt to it; ſo *Diophantus* feigns the ſaid ſecond ſide to be $2a - 4$, for if it were $2a + 4$, all the parts or terms of its Square would be Affirmative, and conſequently no poſſible Equation would ariſe; as will eaſily appear by comparing the Square of $2a + 4$, that is, $4aa + 16a + 16$ to $16 - aa$, whence $5aa + 16a = 0$.

3. The Learner may alſo demand, why in the ſixth ſtep of the foregoing Reſolution, the ſide of the ſecond Square ſought is feigned to be $2a - 4$, or $4 - 2a$, and not $a - 4$, or $4 - a$, *viz.* why is 2, and not 1 or ſome other number prefixt to a? to this I anſwer, If the ſide of the firſt Square ſought be aſſumed or ſuppoſed to be a or $1a$, (as it is in the third ſtep of the Reſolution,) then the ſide of the ſecond Square cannot be $a - 4$, or $4 - a$, as will be evident by a due proceſs upon that ſuppoſition; for the Square of $a - 4$, or $4 - a$, that is, $aa - 8a + 16$ being equated to $aa - 16$, there will ariſe, after due Reduction, $a = 4$, and conſequently, $a - 4$, or $4 - a$, which was put for the ſecond Square, will be equal to nothing: The like abſurdity will follow as often as the numbers prefixt to a in the feigned ſides of the two Squares ſought are equal to one another, *viz.* if the firſt ſide be feigned to be $5a$, and the ſecond $5a - 4$, or $4 - 5a$; or if the firſt ſide be $8a$, and the ſecond $8a - 4$, or $4 - 8a$, &c. from ſuch ſuppoſitions a fruitleſs Equation will enſue, for the ſide of the ſecond Square will be found equal to nothing. Now for the avoiding of ſuch abſurdity, the Learner may take this for a Rule, (the reaſon whereof will hereafter appear by *Obſervat.* 1. of the following *Reſolution* 2. of this *Queſt.* 1.) Let the numbers prefixt to a in the feigned ſides of the two Squares ſought be any two unequal numbers, *viz.* if a or $1a$ be put for the firſt ſide, the ſecond may be $2a - 4$, or $3a - 4$, &c. Again, if we put $3a$ for the firſt ſide, the Square thereof will be $9aa$; and conſequently becauſe the Square given to be divided into two Squares is 16, the ſecond Square ſhall be equal to $16 - 9aa$, whoſe ſide we may feign to be $2a - 4$, or $4 - 2a$, the Square whereof is $4aa - 16a + 16$; this æquated to the ſaid $16 - 9aa$, gives after due Reduction $a = \frac{16}{13}$, therefore $3a$ which was put for the ſide of the firſt Square ſhall be $\frac{48}{13}$, and $4 - 2a$ which was put for the ſide of the ſecond Square will be found $\frac{20}{13}$, and conſequently, the two Squares ſought ſhall be $\frac{2304}{169}$ and $\frac{400}{169}$, whoſe ſumm makes 16, as the Queſtion requires.

In which laſt Example (which is worthy of the Learner's obſervation) it happens, that in reſolving the Poſitions, the ſecond ſide is expounded by $4 - 2a$, not by $2a - 4$, although the

the Reſolution be juſtly formed from either of them; for the Square $4aa - 16a + 16$, whiles a is unknown, may have for its ſide either $2a - 4$, or $4 - 2a$, and which ſo ever of theſe ſides be feigned, the ſame Equation will ariſe to find out the number a, which, after it is diſcovered, is to be compared to ſuch of thoſe two feigned ſides as will produce a number greater than nothing; ſo the number a being before found out to be $\frac{16}{13}$, it is manifeſt that $2a - 4$ is leſs than nothing, but $4 - 2a$ gives $\frac{20}{13}$, which is the true ſide of the ſecond Square. So likewiſe in the Example of *Diophantus*, the ſide of the ſecond Square cannot be expounded by $4 - 2a$, (although the value of a may be rightly found out from that ſuppoſition, as well as from $2a - 4$;) for a being found equal to $\frac{16}{5}$, the ſaid $4 - 2a$ is leſs than nothing.

4. Here I ſhall recommend to the Learner one general Obſervation, *viz.* The principal ſcope in feigning the ſide of a Square to be equated to ſome Algebraick Quantity wherein the higheſt unknown Power is aa, muſt be to feign the ſaid ſide in ſuch manner, that after due Reduction, either the Abſolute numbers may vaniſh, and conſequently an Equation remain between ſome number of aa, and ſome number of a, whence by Diviſion the number a will neceſſarily be Rational; or elſe, (as hereafter will fully appear in this Book,) that aa may vaniſh out of each part, and conſequently an Equation remain between ſome number of a, and ſome Abſolute number; hence alſo the number a will be Rational.

Having explain'd *Diophantus* his Reſolution of *Queſt.* 1. by Numeral *Algebra*, I ſhall in the next place reſolve the ſame by Literal *Algebra*, whence divers uſeful Canons will be brought to light.

RESOLUTION 2. of *Queſt.* 1. which is here repeated, *viz.*

To divide a given ſquare number into two Squares.

1. For the ſide of the given Square put	d
2. Therefore the ſaid Square is	dd
3. Take any two unequal numbers, ſuppoſe s the greater, and r the leſſer, (which s and r are to be uſed inſtead of the numbers prefixt to a in the foregoing Reſolution,) then for the ſide of the firſt Square ſought put	ra
4. And for the ſide of the ſecond Square ſought, put	$sa - d$, or, $d - sa$
5. Therefore, from the third ſtep, the firſt Square is	$rraa$
6. And from the fourth ſtep, the ſecond Square is . .	$ssaa - 2sda + dd$
7. Therefore the ſumm of thoſe Squares is . . .	$ssaa + rraa - 2sda + dd$
8. Which ſumm muſt be equal to the given Square dd, hence this Equation ariſeth, *viz.*	$ssaa + rraa - 2sda + dd = dd$
9. Which Equation, after due Reduction, gives . .	$a = \frac{2sd}{ss + rr}$
10. Therefore out of the ninth and third ſteps, the ſide of the firſt Square ſought is now made known, for it is equal to	$\frac{2rsd}{ss + rr}$
11. And from the ninth and fourth ſteps the ſide of the ſecond Square is alſo known; for it's equal to . .	$\frac{ssd - rrd}{ss + rr}$

Obſervations upon the preceding Reſolution 2. *of* Queſt. 1.

1. The eleventh ſtep of the ſaid Reſolution diſcovers that the known numbers s and r muſt be unequal, to the end the difference of their Squares may be greater than nothing.

2. After the number a is made known, (as in the ninth ſtep,) it will be manifeſt that the ſide of the ſecond Square is to be expounded by $sa - d$, not by $d - sa$; for ſince a is found equal to $\frac{2sd}{ss + rr}$, (as appears by the ninth ſtep of *Reſolut.* 2.) it follows that $sa = \frac{2ssd}{ss + rr}$, and $d - sa = \frac{rrd - ssd}{ss + rr}$, which is leſs than nothing, for s is greater than r by ſuppoſition: But whiles a is unknown, the ſide of the ſecond Square may be feigned $d - sa$ as well as $sa - d$, for each of theſe produceth the ſame Square $ssaa - 2sda + dd$.

3. The ſide of the firſt Square may be feigned sa, and the ſide of the ſecond $ra - d$, or $d - ra$, for from theſe Poſitions the true ſides of the two Squares ſought will be found

the ſame as before are expreſt in the tenth and eleventh ſteps of *Reſolut.* 2. but in this latter way of feigning the ſides, the ſide of the ſecond Square will be expounded by $d - ra$, not by $ra - d$, for this will be found leſs than nothing.

4. The tenth and eleventh ſteps of the foregoing *Reſolution* 2. give this

CANON.

Take any two unequal numbers; multiply ſeverally the double of the Product of their multiplication, and the difference of their Squares by the ſide of the given Square; laſtly, divide thoſe Products ſeverally by the ſumm of the Squares of the two numbers firſt taken, and the Quotients ſhall be the ſides of the two Squares ſought.

An Example in Numbers.

Let the ſide of the given Square be 4, then take two unequal numbers at pleaſure, as 1 and 2; the double Product of their multiplication is 4, the difference of their Squares is 3, then by multiplying the ſaid 4 and 3 ſeverally by 4, (the ſide of the given Square,) the Products are 16 and 12; theſe divided ſeverally by 5, (that is, by the ſumm of the Squares of 1 and 2 the numbers firſt taken,) give the Quotients $\frac{16}{5}$ and $\frac{12}{5}$, which are the ſides of the two Squares ſought; for the Squares of $\frac{16}{5}$ and $\frac{12}{5}$ added together make 16, which was given to be divided into two Squares.

5. For as much as (by *Prop.* 47. *Elem.* 1. Euclid.) when a Square is equal to two Squares, the ſides of thoſe three Squares will make a right-angled Triangle, the preceding *Queſt.* 1. may be thus ſtated, *viz.*

A Rational number being given for the *Hypothenuſal* of a right-angled Triangle, to find Rational numbers to expreſs the Baſe and Perpendicular, *viz.* the ſides about the right-angle.

This may be ſolved by the preceding Canon; for if d be put to repreſent the given Hypothenuſal, and s and r any two unequal numbers, r being the leſſer, theſe three following numbers will conſtitute a right-angled Triangle having d for its Hypothenuſal, *viz.*

Hypothenuſal,	Baſe,	Perpendicular.
d ,	$\frac{ssd - rrd}{ss + rr}$,	$\frac{2rsd}{ss + rr}$.

6. Moreover, if thoſe three ſides of a right-angled Triangle be ſeverally multiplied by the Denominator $ss + rr$, the Products ſhall alſo be the ſides of a right-angled Triangle, to wit, theſe following;

Hypothen.	Baſe,	Perp.
$ssd + rrd$,	$ssd - rrd$,	$2rsd$.

7. And by dividing every one of the three ſides laſt expreſt, by their common Factor d, the Quotients will give theſe three following ſides of a right-angled Triangle, *viz.*

Hypothen.	Baſe,	Perp.
$ss + rr$,	$ss - rr$,	$2rs$.

8. Which three ſides laſt above expreſt are in words the following uſeful Canon, to form a right-angled Triangle in numbers by the help of any two unequal numbers given.

CANON.

Take any two unequal numbers, (ſuppoſe s the greater, and r the leſſer,) then the ſumm of their Squares ſhall be the Hypothenuſal, the difference of the ſame Squares ſhall be one of the ſides about the right-angle, and the double Product of the multiplication of the ſaid two numbers, the other ſide.

The Proof of this Canon.

The Square of $ss + rr$ is . . . . . $s^4 + 2ssrr + r^4$,
The Square of $ss - rr$ is . . . . . $s^4 - 2ssrr + r^4$,
The Square of $2rs$ is . . . . . $+ 4ssrr$.

The firſt of thoſe three Squares is manifeſtly equal to the ſumm of the other two, and therefore the ſides of thoſe three Squares, if they be expreſt by numbers, ſhall be the meaſures of the ſides of a right-angled Triangle.

An Example of the ſaid Canon *in Numbers.*

Take two unequal numbers at pleaſure, as 1 and 2; then the ſumm of their Squares is 5 for the Hypothenuſal, the difference of the Squares of the ſame two numbers is 3 for the Baſe, (that is, either of the ſides about the right-angle,) and the double Product of the two numbers

numbers is 4 for the Perpendicular; but that the numbers 5, 3, 4 may be taken for the meaſures of the ſides of a right-angled Triangle is evident, for the Square of the firſt is equal to the Squares of the two latter.

9. Three Corollaries *deduced from the laſt preceding* Canon.

Firſt, in every right-angled Triangle in ſuch whole numbers which are Prime between themſelves, the ſumm of the Hypothenuſal $(ss + rr)$ and $(2rs)$ one of the ſides about the right-angle is a ſquare number, to wit, $(ss + rr + 2rs)$ the Square of the ſumm of (s and r) the two numbers by which the ſaid Triangle may be formed according to the laſt preceding Canon.

Secondly, the ſumm of the Hypothenuſal $(ss + rr)$ and $(ss - rr)$ the other of the ſides about the right-angle is the double of a ſquare number, to wit, the double of (ss) the Square of (s) the greater of the two numbers by which the Triangle may be formed; And the exceſs of the Hypothenuſal $(ss + rr)$ above the ſaid ſide $(ss - rr)$ is the double of the Square of (r) the leſſer of the ſame two numbers; therefore,

Thirdly, the three ſides of any right-angled Triangle in ſuch Rational whole numbers as are Prime between themſelves being ſeverally given, we may find two whole numbers by which the ſaid Triangle may be formed according to the Canon in *Obſervat.* 8. As, for example, to find two numbers to form theſe three ſides of a right-angled Triangle, to wit, 65, 33, 56, (which are Prime between themſelves, for they have no common Diviſor but Unity,) I add the Hypothenuſal 65 to 33 and 56 ſeverally, and it makes 98 and 121, which latter ſumm is a Square, and therefore (*per Coroll.* 1.) its Root 11 is the ſumm of the two numbers ſought, and the firſt ſumm 98 is the double of the Square 49, whoſe Root 7 ſhall be the greater of the two numbers ſought, (*per Coroll.* 2.) laſtly, by ſubtracting 7 from 11, the Remainder 4 is the leſſer number ſought; whence I conclude, that the right-angled Triangle propoſed may be formed out of 7 and 4; for the ſumm of their Squares makes the Hypothenuſal 65; the difference of the ſame Squares is 33 one of the ſides about the right-angle, and the double Product of 7 and 4, to wit, 56 is the other ſide.

10. From the two preceding Canons (in *Obſervat.* 4, and 8.) another may be deduced to ſolve *Queſt.* 1. *viz.* to divide a given ſquare number into two Squares; or a Rational number being given for the Hypothenuſal of a right-angled Triangle, to find the Baſe and Perpendicular in Rational numbers.

CANON.

By the foregoing Canon in *Obſervat.* 8. let a right-angled Triangle be formed out of any two unequal numbers, and call this Triangle the firſt; then it ſhall be, as the Hypothenuſal of the ſaid firſt Triangle is to its Baſe, ſo is the given Hypothenuſal of a ſecond Triangle deſired to its Baſe; and as the Hypothenuſal of the firſt Triangle is to its Perpendicular, ſo is the Hypothenuſal of the ſecond to its Perpendicular.

An Example in Numbers.

Let it be required to find the Baſe and Perpendicular of a right-angled Triangle in numbers whoſe Hypothenuſal ſhall be 7.

First, by the Canon in *Obſervat.* 8. I form a right-angled Triangle in numbers, as, ... 5 , 3 , 4
Then, by the Rule of Three, I find $4\frac{1}{5}$ for the deſired Baſe, thus, ... 5 . 3 :: 7 . $4\frac{1}{5}$
Likewiſe $5\frac{2}{5}$ for the deſired Perpendicular, thus, . . . 5 . 4 :: 7 . $5\frac{2}{5}$

Therefore 7, $4\frac{1}{5}$, $5\frac{2}{5}$ will conſtitute a right-angled Triangle whoſe Hypothenuſal is 7, as was deſired.

11. After the ſame manner, as many right-angled Triangles in numbers as ſhall be deſired may be found out, which ſhall have one common Hypothenuſal given: As, for example, if three right-angled Triangles in Rational numbers be deſired, that 2 may be a Hypothenuſal to every one of them, they may be found out thus;

First, by the Canon in the foregoing *Obſervat.* 8. let three right-angled Triangles be formed, ſuppoſe theſe, ... 5 , 3 , 4
13 , 5 , 12
17 , 15 , 8

Then,

Then by the Rule of Three, the Baſes and Perpendiculars of the three right-angled Triangles ſought may be found out thus;

I. $\begin{cases} 5 \cdot 3 :: 2 \cdot 1\frac{1}{5} & (\textit{Baſe.}) \\ 5 \cdot 4 :: 2 \cdot 1\frac{1}{5} & (\textit{Perp.}) \end{cases}$

II. $\begin{cases} 13 \cdot 5 :: 2 \cdot \frac{10}{13} & (\textit{Baſe.}) \\ 13 \cdot 12 :: 2 \cdot 1\frac{11}{13} & (\textit{Perp.}) \end{cases}$

III. $\begin{cases} 17 \cdot 15 :: 2 \cdot 1\frac{11}{17} & (\textit{Baſe.}) \\ 17 \cdot 8 :: 2 \cdot \frac{16}{17} & (\textit{Perp.}) \end{cases}$

Whence the deſired ſides of the three right-angled Triangles having 2 for a common Hypothenuſal are found to be theſe, *viz.*

Hypoth.	*Baſes.*	*Perp.*
2	$1\frac{1}{5}$	$1\frac{1}{5}$
2	$\frac{10}{13}$	$1\frac{11}{13}$
2	$1\frac{11}{17}$	$\frac{16}{17}$

12. But note well, that in the ſearch of the Triangles laſt mentioned, the preparatory right-angled Triangles firſt found out by the Canon in the preceding *Obſervat.* 8. muſt not be like, (that is, ſuch as have proportional Sides,) for it will not be difficult to apprehend, that if from them, other Triangles be deduced by the Rule of Three in ſuch manner as before hath been ſhewn, there will be but one right-angled Triangle found out, when many are deſired to have a common Hypothenuſal: That your labour therefore may not be in vain, the preparatory right-angled Triangles muſt be unlike; to which end they muſt be formed from pairs of numbers expreſſing different Reaſons, and ſuch, that the two numbers by which any one of the preparatory Triangles is formed, muſt not be in ſuch proportion to one another as the ſumm is to the difference of two numbers by which any other of the preparatory Triangles is formed. As, for example, if a right-angled Triangle be formed from 1 and 2, then another right-angled Triangle muſt not be formed from 2 and 4, 3 and 6, *&c.* becauſe each of theſe pairs of numbers expreſſing the ſame Reaſon as 1 and 2 will produce a right-angled Triangle like to the firſt; nor from 3 and 1, 6 and 2, *&c.* becauſe 3 having ſuch proportion to 1, likewiſe 6 to 2, as the ſumm of 1 and 2 to their difference, thoſe pairs alſo will produce right-angled Triangles like to the firſt. But that two right-angled Triangles formed from pairs of numbers expreſſing the ſame Reaſon, or from two ſuch pairs, that one number of the one pair hath ſuch proportion to its yoak-fellow, as the ſumm of the two numbers of the other pair hath to their difference, are like, I prove thus;

	Hyp.	Baſe,	Perp.
Firſt, let a right-angled Triangle be formed from two numbers s and r, ſo the three ſides will be theſe, *viz.*	$ss+rr$	$ss-rr$	$2sr$
Then let a ſecond right-angled Triangle be formed from ds and dr, which have the ſame proportion to one another as s and r; ſo the three ſides will be theſe, *viz.*	$ddss+ddrr$	$ddss-ddrr$	$2ddsr$
Again, let a third right-angled Triangle be formed from $s+r$ and $s-r$, *viz.* the ſumm and difference of the two numbers by which the firſt Triangle was formed, ſo the three ſides will be theſe, *viz.* . . .	$2ss+2rr$	$4sr$	$2ss-2rr$

Now I ſay, that the ſecond Triangle is like to the firſt, for the ſides of the ſecond are the Products of the ſides of the firſt multiplied by the common Factor dd. The third Triangle is alſo like to the firſt, for the ſides of the third are the doubles of the ſides of the firſt, and conſequently Proportionals to them, but in this order, *viz.* As the Hypothenuſal of the firſt is to its Baſe, ſo is the Hypothenuſal of the third to its Perpendicular; and, As the Hypothenuſal of the firſt is to its Perpendicular, ſo is the Hypothenuſal of the third to its Baſe.

13. By the help of the preceding Canon in *Obſervat.* 8. as many right-angled Triangles in whole numbers as ſhall be deſired, and which ſhall have a common Hypothenuſal, may be found out in manner following, *viz.*

Let it be required to find out three right-angled Triangles in whole numbers, which ſhall have one common Hypothenuſal.

First,

Firſt, by the Canon in the foregoing *Obſervat.* 8. with reſpect alſo to the *Note* in the laſt preceding Obſervation, let three unlike right-angled Triangles be formed, ſuppoſe theſe, ...	5, 3, 4 13, 5, 12 17, 15, 8
Secondly, multiply ſeverally the three ſides of the firſt Triangle 5, 3, 4, by 221, that is, the Product of the ſecond and third Hypothenuſals 13 and 17; ſo the three Products ſhall be the ſides of a right-angled Triangle, to wit, ...	1105, 663, 884
Thirdly, multiply ſeverally the three ſides of the ſecond right-angled Triangle 13, 5, 12, by 85, that is, the Product of the firſt and third Hypothenuſals 5 and 17; ſo the three Products ſhall be the ſides of this right-angled Triangle, *viz.*	1105, 425, 1020
Laſtly, multiply ſeverally the three ſides of the third right-angled Triangle 17, 15, 8, by 65, that is, the Product of the firſt and ſecond Hypothenuſals 5 and 13; ſo the three Products ſhall be alſo the ſides of a right-angled Triangle, to wit, ...	1105, 975, 520

From the premiſſes it is manifeſt that three right-angled Triangles are found out in whole numbers, having 1105 for a common Hypothenuſal, and by the ſame Method you may find out as many as you pleaſe.

14. But the ſmaller the numbers are that expreſs the ſides of thoſe preparatory right-angled Triangles the better, and therefore I think it not amiſs in this place to ſhew, how to find out all the unlike right-angled Triangles in whole numbers orderly enumerated, according as their Hypothenuſals increaſe in greatneſs, ſo, as that the greateſt Hypothenuſal may not exceed a given number, ſuppoſe 180: To which end,

Firſt, I extract the ſquare Root of 180, and find it falls between 13 and 14, and conſequently a right-angled Triangle formed from 14 and 1, will have its Hypothenuſal greater than 180; therefore all the pairs of whole numbers, which have the greater number of each pair, either 14 or greater than 14, will be unfit for our preſent ſearch.

Secondly, I ſubtract 169 the Square of the ſaid 13 from 180 the given limit, and the Remainder is 11, whoſe ſquare Root falls between 3 and 4; whence 'tis evident that a right-angled Triangle formed from 13 and 4 will have a Hypothenuſal greater than 180, but 13 and 3 will give an Hypothenuſal leſs than 180; and therefore I proceed to make an orderly choice of pairs of whole numbers, from the firſt pair 2 and 1, until I come to 13 and 3 incluſive, and no farther, in this manner, *viz.*

Thirdly, I write in the firſt Column of the following Table a Series of whole numbers proceeding from 1, according to the natural order of numbers, as, 1, 2, 3, 4, 5, *&c.* then at the top of the ſecond Column I ſet 2 and 1 for the firſt pair, that done, I combine every number following or ſtanding underneath 2 in the firſt Column, with every one of the numbers that ſtands above ſuch following number, except in theſe two Caſes, *viz.* Firſt, when two numbers ſo combined are ſuch, that their ſumm and difference have the ſame proportion to one another as the two numbers of any pair already ſet in Column 2. then the two numbers ſo combined are to be caſt out of Column 2. As, for example, becauſe the ſumm of 3 and 1, to wit, 4, is to their difference 2, as 2 to 1, which 2 and 1 make the firſt pair already ſet in Column 2; I omit the writing of the pair 3 and 1 in the ſecond Column: And for the ſame Reaſon the pair 5 and 1 is not inſerted in the ſecond Column; for the ſumm of 5 and 1, to wit, 6, is to their difference 4, as 3 to 2, which 3 and 2 made the ſecond pair before written in Column 2. and in like manner all other pairs cauſing that effect are to be excluded out of the ſecond Column. Again, when two numbers combined as aforeſaid happen to be in the ſame proportion as the two numbers of any pair already ſet in the ſecond Column, then alſo the two numbers ſo combined are to be excluded out of the ſaid Column 2; ſo 4 and 2 having the ſame Reaſon as the firſt pair 2 and 1, are not inſerted in the ſecond Column: the reaſon of excluding all pairs in thoſe two Caſes is, for that they would produce right-angled Triangles like to others before produced, which is contrary to the import of the Propoſition. So at length I find only thirty-two pairs of numbers that are fit to be inſerted in the ſaid ſecond Column.

Fourthly, from every one of thoſe thirty-two pairs of numbers in the ſecond Column, (the laſt of which pairs is 13 and 2,) I form a right-angled Triangle (by the Canon in the foregoing *Obſervat.* 8.) and inſert thoſe Triangles into Column 3, among which I find five, to wit, thoſe formed from the pairs 10 and 9; 11 and 8; 11 and 10; 12 and 7;

12 and 7; and 12 and 11, whoſe Hypothenuſals exceed 180 the preſcribed limit, and therefore I caſt away thoſe five Triangles, and transferr the reſt, which are 27 in multitude, into the fourth Column, in ſuch order as the Hypothenuſals do increaſe in greatneſs. So 27 unlike right-angled Triangles are found out, which are all that can be given in whole numbers, ſo as that the greateſt Hypothenuſal may not exceed 180, as was required. But for further illuſtration of the premiſſes view the following Table.

A Table whoſe fourth Column contains 27 unlike right-angled Triangles in numbers, orderly enumerated according as their Hypothenuſals *increaſe in greatneſs.*

		H. B. P	H. B. P	H. B. P	
1	2, 1	5. 3. 4	5. 3. 4	5. 3. 4	8
2	3, 2	13. 5. 12	13. 5. 12	13. 5. 12	12
3	4, 1	17. 15. 8	17. 15. 8		
4	4, 3	25. 7. 24	25. 7. 24	25. 7. 24	16
5	5, 2	29. 21. 20	29. 21. 20		
6	5, 4	41. 9. 40	37. 35. 12	41. 9. 40	20
7	6, 1	37. 35. 12	41. 9. 40		
8	6, 5	61. 11. 60	53. 45. 28	61. 11. 60	24
9	7, 2	53. 45. 28	61. 11. 60		
10	7, 4	65. 33. 56	65. 33. 56		
11	7, 6	85. 13. 84	65. 63. 16	85. 13. 84	28
12	8, 1	65. 63. 16	73. 55. 48		
13	8, 3	73. 55. 48	85. 13. 84		
14	8, 5	89. 39. 80	85. 77. 36		
15	8, 7	113. 15. 112	89. 39. 80	113. 15. 112	32
16	9, 2	85. 77. 36	97. 65. 72		
17	9, 4	97. 65. 72	101. 99. 20		
18	9, 8	145. 17. 144	109. 91. 60	145. 17. 144	36
19	10, 1	101. 99. 20	113. 15. 112		
20	10, 3	109. 91. 60	125. 117. 44		
21	10, 7	149. 51. 140	137. 105. 88		
22	10, 9	181. 19. 180	145. 17. 144	181. 19. 180	40
23	11, 2	125. 117. 44	145. 143. 24		
24	11, 4	137. 105. 88	149. 51. 140		
25	11, 6	157. 85. 132	157. 85. 132		
26	11, 8	185. 57. 176	169. 119. 120		
27	11, 10	221. 21. 220	173. 165. 52	221. 21. 220	44
28	12, 1	145. 143. 24			
29	12, 5	169. 119. 120			
30	12, 7	193. 95. 168			
31	12, 11	265. 23. 264		265. 23. 264	
32	13, 2	173. 165. 52			

15. By inſpecting the preceding Table we may perceive that the unlike right-angled Triangles in Column 5. which are formed from 2 and 1; 3 and 2; 4 and 3; 5 and 4; *&c. viz.* from pairs of ſuch whole numbers as differ by Unity, have theſe properties, namely;

Firſt, their Baſes 3, 5, 7, 9, 11, 13, *&c.* which you ſee ſtanding under *B* in the fifth Column are in an Arithmetical Progreſſion proceeding from the Baſe 3 of the Primitive right-angled Triangle 5, 3, 4 by the common difference 2.

Secondly, if an Arithmetical Progreſſion be formed from 8 as the firſt and leaſt Term, and the common difference of the Terms be 4; as this Progreſſion 8, 12, 16, 20, *&c.* (which is placed in the laſt Columel of the Table,) then 8 the firſt Term added to the firſt Hypothenuſal 5, makes the ſecond Hypothenuſal 13 ſtanding under *H* in the fifth Column; alſo 12 the ſecond Term of the ſame Progreſſion added to 13 the ſecond Hypothenuſal, gives 25 the third Hypothenuſal in the ſame Column; and 16 the third Term added to 25 the third Hypothenuſal, gives 41 the fourth Hypothenuſal, and ſo forwards continually.

continually. In like manner, 8 the firſt Term of the ſame Progreſſion added to 4 the firſt Perpendicular, gives 12 the ſecond Perpendicular, ſtanding under *P* in the ſaid fifth Column; alſo 12 the ſecond Term added to 12 the ſecond Perpendicular, gives 24 the third Perpendicular; and 16 the third Term added to 24 the third Perpendicular, makes 40 the fourth Perpendicular; and ſo forwards perpetually. So that by the help of the Primitive right-angled Triangle 5, 3, 4, and the ſaid Progreſſion 8, 12, 16, 20, 24, *&c.* innumerable unlike right-angled Triangles may be found out by Addition only.

Thirdly, the difference between the Hypothenuſal and Perpendicular of every one of the ſaid Triangles in Column 5. is Unity.

Fourthly, the Baſe is equal to the ſumm of the two numbers forming the Triangle.

Fifthly, the ſumm of the Hypothenuſal and Perpendicular is a Square, whoſe ſide is equal to the Baſe, or ſumm of the two numbers forming the Triangle; therefore,

Sixthly, if the ſumm of the Hypothenuſal and Perpendicular be multiplied into the Baſe, the Product ſhall be a Cube, whoſe ſide is equal to the Baſe.

Seventhly, the difference of the Hypothenuſal and Baſe is equal to the double of the Square of the leſſer of the two numbers forming the Triangle.

The certainty of all the ſaid Properties will be apparent, if you form right-angled Triangles from theſe following pairs of numbers, and compare thoſe Triangles to one another, according to the import of the ſaid Properties.

a, $a+1$, the firſt pair; || $a+2$, $a+3$, the third pair;
$a+1$, $a+2$, the ſecond pair; || $a+3$, $a+4$, the fourth pair, *&c.*

QUEST. 2. (Quæſt. 10. Lib. 2. *Diophant.*)

To divide (13) a number given, which is compos'd of two Squares, (9 and 4,) into two other Squares.

RESOLUTION 1.

1. The ſide or ſquare Root of 9 the greater Square given is $\succ$ 3
2. The ſide of the leſſer Square 4 is . . . . . $\succ$ 2
3. Let the ſide of the firſt of the two Squares ſought be aſſumed to be . . . . . $\succ$ $a+2$
4. And let the ſide of the ſecond Square ſought be feigned $\succ$ $2a-3$; or, $3-2a$
5. Therefore from the third ſtep the firſt Square deſired is $\succ$ $aa+4a+4$
6. And from the fourth ſtep the ſecond Square ſought is . . $\succ$ $4aa-12a+9$
7. Therefore the ſumm of the two Squares ſought is . . $\succ$ $5aa-8a+13$
8. Which ſumm laſt expreſt muſt be equal to the given number 13, hence this Equation ariſeth, *viz.* . . . . . $\succ$ $5aa-8a+13=13$
9. And that Equation, after due Reduction, gives . . . $\succ$ $a=\frac{8}{5}$
10. Therefore from the ninth and third ſteps, the ſide of the firſt Square ſought is made known, *viz.* . . . . . $\succ$ . . $1\frac{3}{5}$
11. And from the ninth and fourth ſteps, the ſide of the ſecond Square ſought is likewiſe diſcovered, *viz.* . . . $\succ$ . . $\frac{1}{5}$

So the ſides of the two Squares ſought are found $1\frac{3}{5}$ and $\frac{1}{5}$; for $3\frac{2}{25}$ the Square of $1\frac{3}{5}$, added to $\frac{1}{25}$ the Square of $\frac{1}{5}$, makes $3\frac{2}{25}$, that is, 13; as was required.

This Queſtion is of the ſame nature with the foregoing, and deſerves to be ranked among the moſt excellent Problems; for it affords divers admirable Canons concerning the conſtruction of Right-angled Triangles, and is of great uſe for the underſtanding of many of *Diophantus*'s Queſtions, eſpecially in his fifth Book; I ſhall therefore firſt explain the preceding Numeral Reſolution of the Queſtion, and afterwards reſolve the ſame by Literal *Algebra.*

Obſervations upon Queſt. 2.

1. It is evident by the foregoing Reſolution of *Diophantus*, That after $a+2$ and $2a-3$, or $3-2a$ are feigned to be the ſides of the two Squares ſought, the ſumm of thoſe Squares, that is, $5aa-8a+13$, is equated to the given number 13. *viz.* $5aa-8a+13=13$; which Equation, if there were not the ſame Abſolute number 13 in each part, could not be reduced to an Equation between ſome number of aa and ſome number of a, and conſequently the number a would not be Rational, unleſs by meer chance:

Whence then comes it to paſs, that the ſame Abſolute number 13 is found in each part of the ſaid Equation? If the Operation be well examin'd, it will appear that the numbers 2 and 3 in the feigned ſides of the two Squares ſought are the ſides of the two given Squares 4 and 9; which 2 and 3 are the only numbers that can be uſed in the ſaid feigned ſides, to cauſe the number 13 to be found in the ſumm of their Squares.

2. As to the Signs to be prefixt before the given ſides 2 and 3 in the feigned ſides of the two Squares ſought, they muſt neceſſarily be either both —, or one of them +, and the other —, to the end that in the ſumm of the feigned Squares there may be ſome number of *a* with the ſign — prefixt; whence it will follow that the ſaid number of *a* may be transferr'd to the other part of the Equation with the ſign +, and then the Abſolute numbers vaniſhing by Subtraction, becauſe they are one and the ſame number as hath been ſhewn in the preceding *Obſervat.* 1. there will remain an Equation between ſome number of *aa* and ſome number of *a*; whence by due Diviſion the number *a* will be Rational.

3. The numbers to be prefixt before *a* in the feigned ſides of the two Squares ſought, may be variouſly choſen according to divers particular Rules that might be given, among which I ſhall recommend but two to the Learner's practice: The firſt Rule is this;

Let two unequal numbers be taken to be prefixt before *a* in the feigned ſides, but with this Caution, *viz.* That the greater of the two numbers taken may not have the ſame proportion to the leſſer as the ſumm of the ſides of the two Squares given in the Queſtion hath to their difference: As, if the two Squares given be 4 and 9, whoſe ſides are 2 and 3, the greater of the two numbers taken muſt not be to the leſſer as 5 to 1, becauſe 5 is the ſumm, and 1 the difference of the ſaid 2 and 3. Suppoſe therefore that 5 and 3 be taken; then let the firſt feigned ſide be $3a + 2$, (3 being the leſſer of the ſaid two numbers taken, and 2 the leſſer of the ſides of the two Squares given,) and let the ſecond feigned ſide be $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers taken, and 3 the ſide of the greater Square given:) Now if from thoſe feigned ſides the Operation be proſecuted like as in the preceding Reſolution of *Queſt.* 2. an Equation will rightly enſue to find out two Squares different from thoſe given, but ſuch as being added together ſhall make the ſame ſumm as thoſe given.

The ſecond Rule is this; Let two unequal numbers be taken with this Caution, *viz.* That they be not in the ſame Reaſon (or Proportion) as the ſides of the two Squares given: As, if the two Squares given in the Queſtion be 9 and 4, whoſe ſides are 3 and 2, then the two numbers taken muſt not be 3 and 2, 6 and 4, 9 and 6, nor any numbers in the ſame Reaſon: Suppoſe therefore that 5 and 4 be choſen; then for the ſide of the firſt Square ſought put $4a - 2$, or $2 - 4a$, (4 being the leſſer of the ſaid two numbers choſen, and 2 the leſſer of the ſides of the two Squares given,) and for the ſide of the ſecond Square ſought put $5a - 3$, or $3 - 5a$, (5 being the greater of the two numbers before choſen, and 3 the greater of the ſides of the two Squares given;) then if from the ſaid feigned ſides the Operation be proſecuted like as in the foregoing Reſolution of *Queſt.* 2. an Equation will enſue, to find out two Squares different from thoſe given, but ſuch as being added together ſhall make the ſame ſumm as thoſe given. The reaſon of theſe two Cautions will hereafter appear.

The preceding Obſervations may ſuffice for explication of the Reſolution of *Queſt.* 2. by Numeral *Algebra*; I ſhall in the next place ſhew how to reſolve the ſame by Literal *Algebra*, and among various ways that might be uſed, I ſhall chuſe but two, which correſpond with the Rules before given in *Obſervat.* 3. and do produce divers excellent Canons.

RESOLUTION 2. of *Queſt.* 2. which is here repeated, *viz.*

To divide a number given which is compos'd of two known Squares, into two other Squares.

1. For the ſide of the greater Square given, put . . . d
2. And for the ſide of the leſſer Square given, put . . . b
3. Therefore the greater Square is . . . . . dd
4. And the leſſer Square is . . . . . bb
5. Take two unequal numbers, *s* the greater, and *r* the leſſer, with this Caution, *viz.* that *s* be not in ſuch proportion to *r*, as $d + b$ to $d - b$; which numbers *s* and *r* are to be uſed inſtead of the numbers that were prefixt before the unknown number *a* in the

the foregoing Numeral Reſolution of this Queſtion, and the reaſon of the Caution will be ſhewn in the ſixteenth ſtep of this Reſolution.

6. For the ſide of the firſt of the two Squares ſought, put . . . . . $ra + b$
7. And for the ſide of the ſecond Square ſought put $sa - d$, or, $d - sa$
8. Therefore from the ſixth ſtep the firſt Square ſought is . . . . . $rraa + 2rba + bb$
9. And from the ſeventh ſtep the ſecond Square ſought is . . . . . $ssaa - 2sda + dd$
10. Therefore the ſumm of thoſe two Squares is $ssaa + rraa + 2rba - 2sda + bb + dd$
11. But the ſaid ſumm muſt be equal to $dd + bb$ the ſumm of the two Squares given in the Queſtion, and before expreſt in the third and fourth ſteps; hence the following Equation ariſeth, *viz.*

$$ssaa + rraa + 2rba - 2sda + bb + dd = bb + dd.$$

12. Which Equation, after due Reduction, gives $a = \frac{2sd - 2rb}{ss + rr}$
13. Therefore from the twelfth and ſixth ſteps, the ſide of the firſt Square ſought is now made known, and found equal to this following Quantity,

$$\frac{2rsd + ssb - rrb}{ss + rr}.$$

14. And from the twelfth and ſeventh ſteps, the ſide of the ſecond Square ſought is likewiſe known, and found equal to

$$\frac{ssd - rrd - 2rsb}{ss + rr}, \text{ or, } \frac{2rsb + rrd - ssd}{ss + rr};$$

That is to ſay, The former of thoſe two Quantities expreſt Fraction-wiſe ſhall be the ſide of the ſecond Square when $ssd - rrd$ is greater than $2rsb$, but the latter of thoſe Quantities ſhall be the ſaid ſide when $ssd - rrd$ is leſs than $2rsb$. For if $ssd - rrd$ be greater than $2rsb$, then by ſubtracting $2rsb$ from $ssd - rrd$, the Remainder is the ſame with the Numerator of the firſt of the two Fractions above expreſt; but if $2rsb$ be greater than $ssd - rrd$, then by ſubtracting $ssd - rrd$ from $2rsb$, the Remainder is the ſame with the Numerator of the latter of the ſaid Fractions; therefore the ſide of the ſecond Square may be expreſt thus,

$$\frac{\overline{ssd - rrd} \text{ or } 2rsb}{ss + rr},$$

That is to ſay, If the difference between $ssd - rrd$ and $2rsb$ be divided by $ss + rr$, the Quotient ſhall be the ſide of the ſecond Square ſought.

From the premiſſes ariſeth this following

CANON 1.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the ſame proportion to the leſſer, as the ſumm of the ſides of the two Squares given hath to the difference of the ſame ſides: Multiply the double Product of the multiplication of thoſe two numbers firſt taken by each of the ſaid two ſides given, and reſerve the Products; multiply alſo the difference of the Squares of the ſaid two numbers firſt taken by each of the ſaid two ſides given, and reſerve theſe Products; then add the greater of the two firſt reſerved Products to the leſſer of the two latter, and reſerve the ſumm for a Dividend; take alſo the difference between the leſſer of the two firſt Products and the greater of the two latter for a ſecond Dividend; laſtly, divide ſeverally the ſaid Dividends by the ſumm of the Squares of the two numbers firſt taken, ſo ſhall the Quotients be the ſides of the two Squares ſought.

Example 1. *Where the number given is compos'd of two unequal Squares.*

Let it be required to divide 13 which is compos'd of two Squares, 9 and 4, into two other Squares.

The ſide of the greater Square given is 3
The ſide of the leſſer Square given is 2
Take two unequal numbers, with reſpect to the Caution in the Canon, as, theſe, ... 2 and 1
Then by uſing thoſe four numbers as the Canon doth direct, the ſides of the two Squares ſought will be found theſe, ... $\frac{18}{5}$ and $\frac{1}{5}$

The Squares of which ſides being added together make 13, as was required.

Example 2.

Let it again be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares different from thoſe found out in *Example* 1.

The ſides of the two Squares given are . . . . . } 3 and 2

Take two unequal numbers with reſpect to the Caution in the foregoing *Canon* 1. as theſe, ... } 4 and 3

Then by uſing the four numbers laſt before expreſt, as the ſaid Canon doth direct, the ſides of the two Squares ſought will be found theſe, . . } $\frac{86}{25}$ and $\frac{27}{25}$

The Proof.

The Square of $\frac{86}{25}$ is . . . . . $\frac{7396}{625}$,

The Square of $\frac{27}{25}$ is . . . . . $\frac{729}{625}$,

The ſumm of thoſe Squares is . . . . . $\frac{8125}{625}$, or 13.

Example 3. *Where the given number is compos'd of two equal Squares.*

Let it be required to divide 2, which is compos'd of two equal Squares, 1 and 1, into two unequal Squares.

The ſide of either of the Squares given is . . . . . } 1

Take in this Caſe any two unequal numbers, as . . . . . } 1 and 2

Then by working with thoſe three numbers according to the direction of *Canon* 1. the ſides of the two Squares ſought will be found theſe, . . } $\frac{7}{5}$ and $\frac{1}{5}$

The Squares of which ſides being added together make 2, as may eaſily be proved.

16. Now that the neceſſity of the Caution preſcribed in the foregoing *Canon* 1. about chuſing the unequal numbers s and r may appear, I ſhall prove, That if s the greater of them hath the ſame proportion to r the leſſer, as $d+b$ the ſumm of the ſides of the two unequal Squares given in *Queſt.* 2. hath to $d-b$ the difference of the ſame ſides, then the ſaid Canon will produce the ſame ſides d and b for the ſides of the two Squares ſought, and conſequently the Operation in ſuch Caſe will be in vain. Firſt, it is manifeſt by the thirteenth ſtep, that one of the ſides found out by the Canon is $\frac{2rsd+ssb-rrb}{ss+rr}$; ſo that if we prove this ſide to be equal to d the ſide of the greater of the two Squares given, then conſequently the other ſide found out by the Canon, that is, the ſide expreſt by the fourteenth ſtep, ſhall be equal to the ſide of the leſſer of the two Squares given; for the ſumm of the Squares found out is equal to the ſumm of thoſe given.

17. Let it therefore be ſuppoſed that . . . . . } $d+b \,.\, d-b :: s \,.\, r$

18. And then, we are to demonſtrate that . . . . } $\frac{2rsd+ssb-rrb}{ss+rr} = d$

Demonſtration.

19. By ſuppoſition in the ſeventeenth ſtep, ... } $d+b \,.\, d-b :: s \,.\, r$

20. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, ... } $rd+rb = sd-sb$

21. And by adding sb to each part of the laſt Equation, this ariſeth, ... } $sb+rd+rb = sd$

22. And by ſubtracting rd from each part, it makes . . } $sb+rb = sd-rd$

23. And by reſolving the laſt Equation into Proportionals, this Analogy ariſeth, *viz.* . . . . . } $s+r \,.\, s-r :: d \,.\, b$

24. And by drawing $s-r$ as a common Factor into the two firſt Terms of that Analogy, this ariſeth,

$$ss-rr \,.\, ss+rr-2rs :: d \,.\, b.$$

25. Therefore, by comparing the Product of the extremes in the laſt Analogy to the Product of the means, this Equation ariſeth, *viz.*

$$ssb-rrb = ssd+rrd-2rsd.$$

26. Whence by equal Addition of $2rsd$, this Equation ariſeth, *viz.*

$$2rsd+ssb-rrb = ssd+rrd.$$

27. Wherefore by dividing each part of the laſt Equation by $ss+rr$, this ariſeth, *viz.*

$$\frac{2rsd+ssb-rrb}{ss+rr} = d.$$ Which was to be demonſtrated.

Reſolution 3.

RESOLUTION 3. of *Queſt.* 2. which is here repeated, *viz.*

To divide a given number which is compos'd of two known Squares, into two other Squares.

1. For the ſide of the greater Square given, put . . . . . } d
2. And for the ſide of the leſſer Square given, put . . . . } b
3. Therefore the greater of thoſe Squares ſhall be . . . . } dd
4. And the leſſer . . . . . } bb
5. Take two unequal numbers, s the greater, and r the leſſer, with this Caution, *viz.* That s may not be in ſuch proportion to r, as d to b; which s and r do repreſent the numbers to be prefixt to the unknown number a, according to the ſecond Rule before mentioned in *Obſervat.* 3. *Reſolut.* 1. *Queſt.* 2. and the reaſon of the Caution will be ſhewn in the ſixteenth ſtep of this Reſolution.
6. Then for the ſide of the firſt Square ſought, put . . . . } $ra-b$, or, $b-ra$
7. And for the ſide of the ſecond Square ſought, put . . . } $sa-d$, or, $d-sa$
8. Therefore from the ſixth ſtep the firſt Square ſought is . . } $rraa-2rba+bb$
9. And from the ſeventh ſtep the ſecond Square ſought is . . } $ssaa-2sda+dd$
10. Therefore the ſumm of the Squares in the eighth and ninth ſteps is

$$ssaa+rraa-2rba-2sda+bb+dd.$$

11. But the ſaid ſumm muſt be equal to the two Squares given, to wit, dd and bb, hence therefore ariſeth the following Equation, *viz.*

$$ssaa+rraa-2rba-2sda+bb+dd = bb+dd.$$

12. Which Equation, after due Reduction, gives . . . . . } $a = \frac{2rb+2sd}{ss+rr}$

13. Therefore from the twelfth and ſixth ſteps the ſide of the firſt Square ſought is now made known, and equal to one of theſe two Quantities, to wit,

$$\frac{2rsd+rrb-ssb}{ss+rr}, \text{ or, } \frac{ssb-rrb-2rsd}{ss+rr};$$

That is to ſay, the former of thoſe two Quantities expreſt Fraction-wiſe ſhall be the ſide of the firſt Square ſought, when $ssb-rrb$ is leſs than $2rsd$; but the latter ſhall be the ſaid ſide when $ssb-rrb$ is greater than $2rsd$. For if $ssb-rrb$ be leſs than $2rsd$, then by ſubtracting $ssb-rrb$ from $2rsd$, the Remainder will be the ſame with the Numerator of the firſt of the two Quantities above expreſt Fraction-wiſe; but if $ssb-rrb$ be greater than $2rsd$, then by ſubtracting $2rsd$ from $ssb-rrb$, the Remainder will be the ſame with the Numerator of the latter of the ſaid Quantities: Therefore the ſide of the firſt Square ſought may be expreſt thus,

$$\frac{ssb-rrb \text{ or } 2rsd}{ss+rr};$$

That is to ſay, If the difference between $ssb-rrb$ and $2rsd$, be divided by $ss+rr$ the Quotient ſhall be the ſide of the firſt Square ſought.

14. But from the twelfth and ſeventh ſteps the ſide of the ſecond Square will be found equal to this known Quantity, *viz.*

$$\frac{2rsb+ssd-rrd}{ss+rr}.$$

From the premiſſes ariſeth this following

CANON 2.

15. Take two unequal numbers, with this Caution, *viz.* That the greater may not have the ſame proportion to the leſſer, as the ſide of the greater of the two Squares given hath to the leſſer ſide: Multiply the double Product of the multiplication of the two unequal numbers firſt taken by each of the ſaid two ſides given, and reſerve the Products; multiply alſo the difference of the Squares of the two numbers firſt taken, by each of the ſaid two ſides given, and reſerve theſe Products; then take the difference between the greater of the two firſt reſerved Products and the leſſer of the two latter for a Dividend; take alſo the ſumm of the leſſer of the two firſt Products and the greater of the two latter for a ſecond Dividend; laſtly, divide each of thoſe Dividends by the ſumm of the Squares of the two numbers firſt taken, ſo ſhall the Quotients be the ſides of the two Squares ſought.

Example 1.

Example 1. *Where the number given to be divided is compos'd of two unequal Squares.*

Let it be required to divide 13, which is compos'd of two Squares, 9 and 4, into two other Squares.

The ſide of the greater Square given is 3
The ſide of the leſſer Square given is 2
Take two unequal numbers, with reſpect to the Caution in *Canon* 2. as theſe, ... 2 and 1
Then by uſing thoſe four numbers according to the direction of *Canon* 2. the ſides of the two Squares ſought will be found theſe, *viz.* . . . . $\frac{6}{5}$ and $\frac{17}{5}$

The Squares of which ſides $\frac{6}{5}$ and $\frac{17}{5}$ being added together make $\frac{325}{25}$ or 13, as was required.

Example 2.

Let it be again required to divide 13, which is compos'd of 9 and 4 into two other Squares different from thoſe found out in *Example* 1.

The ſides of the two given Squares, 9 and 4, are . . . . . 3 and 2
Take two unequal numbers with reſpect to the Caution in *Canon* 2. as theſe, ... 4 and 1
Then by working with thoſe four numbers as the ſaid *Canon* 2. doth direct, the ſides of the two Squares ſought will be found theſe, . . . $\frac{6}{17}$ and $\frac{61}{17}$

The Squares of which ſides are $\frac{36}{289}$ and $\frac{3721}{289}$, whoſe ſumm makes $\frac{3757}{289}$, that is, 13, as was required.

Example 3. *Where the number given to be divided is compos'd of two equal Squares.*

Let it be required to divide 18, which is compos'd of two equal Squares, 9 and 9, into two unequal Squares.

The ſide of either of the Squares given is 3
Take in this Caſe any two unequal numbers, as . . . . . 1 and 2
Then by uſing thoſe three numbers according to the direction of the foregoing *Canon* 2. the ſides of the two Squares ſought will be found theſe, *viz.* . . . . . $\frac{3}{5}$ and $\frac{21}{5}$

The Squares of which ſides are $\frac{9}{25}$ and $\frac{441}{25}$, whoſe ſumm makes $\frac{450}{25}$, that is 18, as was required.

16. Now that the neceſſity of the Caution preſcribed in the foregoing *Canon* 2. about chuſing the unequal numbers s and r may appear, I ſhall prove, That if s the greater of them hath the ſame proportion to r the leſſer, as d the ſide of the greater of the two Squares given in *Queſt.* 2. hath to b the ſide of the leſſer of the ſame Squares, then the ſaid Canon will produce the ſame ſides d and b for the ſides of the two Squares ſought, and conſequently the Operation in ſuch Caſe will be in vain: Firſt, it is manifeſt by the fourteenth ſtep, that one of the ſides found out by the Canon is $\frac{ssd - rrd + 2rsb}{ss + rr}$; ſo that if we prove this ſide to be equal to d the ſide of the greater of the two Squares given, then conſequently the other ſide found out by the Canon, that is, the ſide expreſt in the thirteenth ſtep ſhall be equal to the ſide of the leſſer of the two Squares given, for the ſumm of the Squares found out is equal to the ſumm of thoſe given.

17. Let it therefore be ſuppoſed that . . . . . $d . b :: s . r$
18. And then we are to demonſtrate that . . . . $\frac{ssd - rrd + 2rsb}{ss + rr} = d$

Demonſtration.

19. By ſuppoſition in the ſixteenth ſtep, ... $d . b :: s . r$
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, ... $sb = rd$
21. And by drawing $2r$ into each part of the laſt Equation, this ariſeth, *viz.* . . . . . $2rsb = 2rrd$
22. And by adding ssd to each part of the laſt Equation, it gives . . . . . $ssd + 2rsb = ssd + 2rrd$
23. And by ſubtracting rrd from each part of the laſt Equation, there remains . . . . . $ssd - rrd + 2rsb = ssd + rrd$

24. Where-

24. Wherefore by dividing each part of the laſt Equation by $ss + rr$, there ariſeth . . . . . $\frac{ssd - rrd + 2rsb}{ss + rr} = d$

Which was to be proved.

Obſervations upon the preceding Reſolutions 2, *and* 3. *of* Queſt. 2. *by Literal* Algebra.

1. If z be put equal to $\sqrt{:bb + dd}$: that is, the ſquare Root of the number compos'd of two Squares given in *Queſt.* 2. that Queſtion may be ſtated thus, *viz.*

Two Rational numbers, b and d, being given for the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal is z, Rational or Irrational; to find out other Rational numbers to expreſs the Baſe and Perpendicular of a ſecond right-angled Triangle whoſe Hypothenuſal ſhall be z likewiſe.

The Baſe and Perpendicular of the Triangle ſought ſhall be given either by *Canon* 1. in the fifteenth ſtep of *Reſolution* 2. of *Queſt.* 2. or by *Canon* 2. in the fifteenth ſtep of *Reſolution* 3. and may be expreſt by Letters, as before in the thirteenth and fourteenth ſteps of *Reſolution* 2. or by the thirteenth and fourteenth ſteps of *Reſolution* 3. *viz.*

	By *Canon* 1.		By *Canon* 2.	
ΔK.	Hyp.	z, (or, $\sqrt{:bb + dd}$:)	Hyp.	z, (or, $\sqrt{:bb + dd}$:)
	Perp.	$\frac{2rsd + ssb - rrb}{ss + rr}$	Perp.	$\frac{ssb - rrb \text{ or } 2rsd}{ss + rr}$
	Baſe,	$\frac{ssd - rrd \text{ or } 2rsb}{ss + rr}$	Baſe,	$\frac{2rsb + ssd - rrd}{ss + rr}$
				ΔL.

2. If the Baſes and Perpendiculars of thoſe two right-angled Triangles above-expreſt, which I call ΔK and ΔL, be well examined, another way will be diſcovered to find out the ſame Baſes and Perpendiculars by the help of the Baſes and Perpendiculars of two like right-angled Triangles whoſe Hypothenuſals are b and d. For,

Firſt, it is manifeſt by *Obſervat.* 5. *Reſolut.* 2. *Queſt.* 1. of this Book, that theſe three following numbers will conſtitute a right-angled Triangle, which hath b for an Hypothenuſal, *viz.*

Hyp.	Baſe,	Perp.	
b .	$\frac{ssb - rrb}{ss + rr}$	: $\frac{2rsb}{ss + rr}$	(ΔM.)

Likewiſe theſe three following numbers will conſtitute a right-angled Triangle, having d for an Hypothenuſal, *viz.*

Hyp.	Baſe,	Perp.	
d .	$\frac{ssd - rrd}{ss + rr}$	. $\frac{2rsd}{ss + rr}$	(ΔN.)

Which two Triangles laſt before expreſt, to wit, ΔM and ΔN, are like, for each of them is like to a right-angled Triangle whoſe three ſides are $ss + rr$, $ss - rr$, and $2rs$; Now I ſay, if the Perpendiculars and Baſes of the two right-angled Triangles K and L before expreſt in *Obſervat.* 1. be well viewed; it will be evident, that they are deduced from the two like right-angled Triangles M and N before expreſt in this *Obſervat* 2. which have b and d for Hypothenuſals. For, firſt, the Perpendicular of ΔK is compos'd of the Baſe of ΔM and the Perpendicular of ΔN; ſecondly, the Baſe of ΔK is equal to the difference between the Perpendicular of ΔM and the Baſe of ΔN; thirdly, the Perpendicular of ΔL is equal to the difference between the Baſe of ΔM and the Perpendicular of ΔN; laſtly, the Baſe of ΔL is compos'd of the Perpendicular of ΔM and the Baſe of ΔN.

3. Hence therefore another Canon comes to light, to ſolve as well the preceding *Queſt.* 2. as alſo the following Propoſition, (which is *Prop.* 47. in *pag.* 35. of *Vieta*'s Works,) *viz.*

From two right-angled Triangles given to deduce a third right-angled Triangle, ſuch, that the Square of the Hypothenuſal of the third may be equal to the Squares of the Hypothenuſals of the firſt and ſecond.

This Propoſition may be ſolved by the following

CANON 3.

CANON 3.

First, (by the Canon in *Observat.* 10. *Resolut.* 2. of the preceding *Queſt.* 1.) find out two like right angled Triangles in numbers, ſuch, that their Hypothenuſals may be the ſides of the two Squares given in the foregoing *Queſt.* 2. then, for the Perpendicular of the third right-angled Triangle ſought, take the ſumm of the Baſe of the firſt and Perpendicular of the ſecond; for the Baſe of the third, take the difference between the Perpendicular of the firſt and Baſe of the ſecond; and the Hypothenuſal of the third ſhall be the ſquare Root of the ſumm of the Squares of the Hypothenuſals of the firſt and ſecond.

Or thus:

For the Perpendicular of the third right-angled Triangle ſought, take the difference between the Baſe of the firſt and Perpendicular of the ſecond; for the Baſe of the third, take the ſumm of the Perpendicular of the firſt and Baſe of the ſecond; and the Hypothenuſal of the third ſhall be the ſame as before is expreſt.

Example 1. *in Numbers.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two other Squares.

	Hyp.	*Baſe.*	*Perp.*
The ſides of 4 and 9 the two Squares given are . . .	2 and 3		
Find a firſt right-angled Triangle in numbers whoſe Hypothenuſal ſhall be 2, the ſide of 4 the leſſer of the two Squares given, as, ...	2 ,	$\frac{4}{5}$,	$\frac{3}{5}$
Find likewiſe a ſecond right-angled Triangle like to the firſt, and ſuch, that its Hypothenuſal may be 3 the ſide of the greater Square given, as, ...	3 ,	$\frac{9}{5}$,	$\frac{12}{5}$
Then by uſing the Baſes and Perpendiculars of thoſe two like right-angled Triangles as *Canon* 3. doth direct, this third right-angled Triangle will be found out, whoſe Hypothenuſal is equal to $\sqrt{:4+9}$: that is, $\sqrt{13}$, and conſequently the Baſe and Perpendicular are the ſides of the two Squares ſought,	$\sqrt{13}$,	$\frac{1}{5}$,	$\frac{18}{5}$
Or, according to the latter part of *Canon* 3. the ſides of the two Squares ſought will be found the Baſe and Perpendicular of this third right-angled Triangle whoſe Hypothenuſal is $\sqrt{13}$, that is, $\sqrt{:4+9}$: as before;	$\sqrt{13}$,	$\frac{17}{5}$,	$\frac{6}{5}$

Example 2.

Let it be required to divide 25, which is compoſed of two Squares, 9 and 16, into two other Squares.

	Hyp.	*Baſe,*	*Perp.*
The ſides of 9 and 16 the two given Squares are . . .	3 and 4		
Find a firſt right-angled Triangle in numbers, whoſe Hypothenuſal ſhall be 3 the ſide of the leſſer Square given, as,	3 ,	$\frac{16}{13}$,	$\frac{11}{13}$
Find likewiſe a ſecond right-angled Triangle like to the firſt, and ſuch, that its Hypothenuſal ſhall be 4 the ſide of the greater Square given, as, ...	4 ,	$\frac{48}{13}$,	$\frac{20}{13}$
Then by uſing the Baſes and Perpendiculars of the two like right-angled Triangles laſt found out, according to the direction of *Canon* 3. this third right-angled Triangle will be dicovered, whoſe Hypothenuſal is 5, that is, $\sqrt{:9+16}$: and conſequently the Baſe and Perpendicular are the ſides of the two Squares ſought, ...	5 ,	$\frac{33}{13}$,	$\frac{56}{13}$
Or, according to the latter part of *Canon* 3. the ſides of the two Squares ſought will be found the Baſe and Perpendicular of this third right-angled Triangle, whoſe Hypothenuſal is 5, that is, $\sqrt{:9+16}$: as before, ...	5 ,	$\frac{63}{13}$,	$\frac{16}{13}$

4. If every one of the three ſides of the two right-angled Triangles K and L before expreſt in *Obſervat.* 1. having z for a common Hypothenuſal, Rational or Irrational, be multiplied by $ss+rr$, the Products ſhall be alſo the ſides of two right-angled Triangles like to the two former reſpectively; which Products or ſides ſhall be theſe, *viz.*

Hyp.

Hyp.	$zss+zrr$	Hyp.	$zss+zrr$
Perp.	$2rsd+ssb-rrb$	Perp.	$ssb-rrb \backsim 2rsd$
Baſe,	$ssd-rrd \backsim 2rsb$	Baſe,	$2rsb+ssd-rrd$

Now if the two right-angled Triangles laſt expreſt be well examined, it will appear, that each of them may be deduced from two right-angled Triangles, one of which hath for its Hypothenuſal $ss+rr$, Baſe $ss-rr$, and Perpendicular $2rs$, (or $2rs$ may be called the Baſe, and $ss-rr$ the Perpendicular;) but of the other the Hypothenuſal is $z = \sqrt{:bb+dd}$: Rational or Irrational, the Baſe is b, and the Perpendicular is d, (or d may be called the Baſe, and b the Perpendicular;) I ſay, from theſe two laſt mentioned Triangles each of the two former may be deduced in ſuch manner as is directed in the following *Canon*, which is the ſame with that raiſed by *Vieta* in ſolving *Prop.* 46. in *pag.* 34. of his Works, *viz.*

From two right-angled Triangles given, to form a third right-angled Triangle.

CANON.

For the Hypothenuſal of the third right-angled Triangle, take the Product of the multiplication of the Hypothenuſals of the two right-angled Triangles given: for the Perpendicular, the ſumm of the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt: and for the Baſe, take the difference between the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

Or thus:

For the Hypothenuſal of the third right-angled Triangle, take (as before) the Product of the multiplication of the Hypothenuſals of the firſt and ſecond right-angled Triangles given: for the Perpendicular, the difference between the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Baſe of the ſecond into the Perpendicular of the firſt: laſtly, for the Baſe, take the ſumm of the Product of the Baſes of the firſt and ſecond, and the Product of their Perpendiculars.

An Example in Numbers.	*Hyp.*	*Baſe,*	*Perp.*
Let there be two right-angled Triangles given in numbers, ſuppoſe theſe, ...	5 ; 13 ,	3 ; 5 ,	4 ; 12
Then from thoſe Triangles, theſe two are deduced by the two Canons laſt before expreſt, *viz.*	65 , 65 ,	33 , 63 ,	56 ; 16

Note 1. If the two right-angled Triangles given be unlike, then either of thoſe Canons will form a third right-angled Triangle; but if like, then the firſt only will take place: for when the two right-angled Triangles given are like, then the difference of the Products mentioned in the latter *Canon* are equal to nothing, as will be evident to every diligent Reader.

Note 2. If from any right-angled Triangle taken twice, ſuppoſe from theſe two, ...	H	B	P
	H	B	P
A third right-angled Triangle be deduced according to the firſt Canon, as this, ...	HH	BB ∽ PP	2BP

Then the angle at the Baſe of the third right-angled Triangle ſo deduced, *viz.* the angle oppoſite to the ſide 2BP ſhall be equal to the double of the angle at the Baſe of the firſt right-angled Triangle, *viz.* of the angle oppoſite to the ſide P; or elſe equal to the Complement of the ſaid double angle unto two right-angles, when the ſaid double exceeds a right-angle.

Likewiſe, if from two unlike right-angled Triangles, ſuppoſe from theſe, ...	H	B	P
	h	*b*	*p*
A third right-angled Triangle be deduced according to the firſt Canon, as theſe, ...	H*h*	B*b* ∽ P*p*	B*p*+P*b*

Then the angle at the Baſe of this third right-angled Triangle, *viz.* the angle oppoſite to the ſide B*p*+P*b* ſhall be equal to the ſumm of the angles at the Baſes of the firſt and ſecond right-angled Triangles, *viz.* of the angles oppoſite to the ſides P and *p*; or elſe equal to the Complement of the ſaid ſumm unto two right-angles, when that ſumm exceeds a right-angle.

The converſe of this rare Speculation is demonſtrated by *Anderſonus*, in *Theorem.* 2. of *Vieta*'s myſterious Doctrine of Angular Sections; and likewiſe by *Herigonius* at the latter end of the Firſt Tome of his *Curſus Mathemat.*

QUEST. 3.

To divide a given ſquare number into two ſuch Squares, that one of them may conſiſt within given limits.

Let it be required to divide 16 into two ſuch Squares, that one of them may be greater than 10, but leſs than 11.

Or thus:

A Rational number 4 being given for the Hypothenuſal of a right-angled Triangle, to find the Baſe and Perpendicular in ſuch Rational numbers, that one of them may be greater than $\sqrt{10}$, but leſs than $\sqrt{11}$.

RESOLUTION.

1. For the given Hypothenuſal 4, (which is the ſide of the given Square 16) put . . . . . d
2. For $\sqrt{10}$ the leſſer of the preſcribed limits, put . . . f
3. For $\sqrt{11}$ the greater of the preſcribed limits, put . . . g
4. Let two unequal numbers be repreſented by . . . . s and r
5. Then the ſides about the right-angle of a right-angled Triangle whoſe Hypothenuſal is d, will be found equal to theſe Quantities, (by the Canon in *Obſervat.* 5. *Reſolut.* 2. *Queſt.* 1.) *viz.* . . . . . $\frac{2rsd}{ss+rr}$ and $\frac{ssd \sim rrd}{ss+rr}$

6. But ſince this Queſtion requires that one of thoſe ſides, ſuppoſe $\frac{2rsd}{ss+rr}$, may be greater than f, yet leſs than g, the ſaid numbers s and r cannot be any two unequal numbers; and therefore I ſhall here ſhew a way to chuſe them, ſo as that they may cauſe the ſaid ſide to agree with the ſaid limits: To which end, firſt, a number at pleaſure may be taken for one of the ſaid numbers s and r, as, $1 = r$; and then to ſearch out limits for the chuſing of s, I proceed in this manner, *viz.* I put a inſtead of s while it is unknown, and then ſince $1 = r$, the before-mentioned ſide $\frac{2rsd}{ss+rr}$ will be expreſt thus, $\frac{2da}{aa+1}$, where the number a only is unknown: Now,

7. Let it be ſuppoſed (according to the import of the Queſtion) that . . . . . $\frac{2da}{aa+1} \sqsubset f$
8. Let it alſo be ſuppoſed that . . . . . $\frac{2da}{aa+1} \sqsupset g$
9. Then by multiplying each part of the ſuppoſition in the ſeventh ſtep by $aa+1$, it follows that . . . . $2da \sqsubset faa+f$
10. Therefore by comparing the latter part of the ninth ſtep to the former, ... $faa+f \sqsupset 2da$
11. And by dividing each part in the laſt ſtep by f, it follows, that . . . . . $aa+1 \sqsupset \frac{2da}{f}$
12. And by ſubtracting 1 from each part, ... $aa \sqsupset \frac{2da}{f} - 1$
13. Likewiſe by equal ſubtraction of $\frac{2da}{f}$ it follows, that $aa - \frac{2da}{f} \sqsupset -1$
14. And by adding the Square of half the Coefficient $\frac{2d}{f}$, to each part in the thirteenth ſtep, ... $aa - \frac{2da}{f} + \frac{dd}{ff} \sqsupset \frac{dd}{ff} - 1$
15. And by extracting the ſquare Root out of each part in the fourteenth ſtep, ... $a - \frac{d}{f} \sqsupset \sqrt{:\frac{dd-ff}{ff}:}$
16. Wherefore by adding $\frac{d}{f}$ to each part in the fifteenth ſtep, it follows that . . . . . $a \sqsupset \frac{d}{f} + \sqrt{:\frac{dd-ff}{ff}:}$

Again, 17.

17. Again, becauſe $\frac{d}{f} - a$, (as well as $a - \frac{d}{f}$,) may be the ſide of the Square in the firſt part of the fourteenth ſtep, it thence follows that . . . . . $\frac{d}{f} - a \sqsupset \sqrt{:\frac{dd-ff}{ff}:}$
18. And by adding a to each part in the ſeventeenth ſtep, $\frac{d}{f} \sqsupset a + \sqrt{:\frac{dd-ff}{ff}:}$
19. And by equal ſubtraction of $\sqrt{:\frac{dd-ff}{ff}:}$ it follows that $\frac{d}{f} - \sqrt{:\frac{dd-ff}{ff}:} \sqsupset a$
20. Wherefore by comparing the latter part of the nineteenth ſtep to the firſt, it's evident that . . . . . $a \sqsubset \frac{d}{f} - \sqrt{:\frac{dd-ff}{ff}:}$
21. Again, by ſuppoſition in the eighth ſtep, ... $\frac{2da}{aa+1} \sqsupset g$
22. And conſequently, ... $g \sqsubset \frac{2da}{aa+1}$
23. Whence by arguing in like manner as before, with f, from the ninth ſtep to the ſixteenth, it will appear that . . . . . $a \sqsubset \frac{d}{g} + \sqrt{:\frac{dd-gg}{gg}:}$
24. Again, by arguing with g in like manner as before with f, from the ſeventeenth ſtep to the twentyeth, it will be evident that . . . . . $a \sqsupset \frac{d}{g} - \sqrt{:\frac{dd-gg}{gg}:}$
25. Now becauſe a was put inſtead of s, the ſixteenth and twenty-third ſteps give a Canon for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:dd-ff:}}{f} \quad (2.039, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:dd-gg:}}{g} \quad (1.880, \&c.)$$

26. Again, the twentyeth and twenty-fourth ſteps give another Canon for limiting the number s, (which was repreſented by a in the preceding argumentation,) when $r = 1$, *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:dd-ff:}}{f} \quad (0.490, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:dd-gg:}}{g} \quad (0.531, \&c.)$$

27. Therefore, if 1 be put for r, and there be given (as before in the firſt, ſecond and third ſteps,) $4 = d$; $\sqrt{10} = f$; and $\sqrt{11} = g$; then by *Canon* 1. s may be any number leſs than $2\frac{19}{1000}$, but greater than $1\frac{88}{100}$; and conſequently, if $1 = r$, and $s = 2$, (which is within the laſt mentioned limits of s,) then the ſides of the two Squares ſought (being expounded according to the two Quantities in the fifth ſtep of the Reſolution of this *Queſt.* 3.) ſhall be $\frac{16}{5}$ and $\frac{12}{5}$, *viz.*

$$\frac{16}{5} = \frac{2rsd}{ss+rr}; \text{ and, } \frac{12}{5} = \frac{ssd-rrd}{ss+rr}.$$

Therefore the two Squares ſought are $2\frac{16}{25}$ and $1\frac{44}{25}$, whoſe ſumm is $4\frac{60}{25}$, that is, $16\frac{4}{5}$ and one of thoſe Squares, to wit, $2\frac{16}{25}$, or $10\frac{4}{25}$, is greater than 10, but leſs than 11; as was required.

Again, if $1 = r$; $4 = d$; $\sqrt{10} = f$; $\sqrt{11} = g$; (as before,) then by *Canon* 2. s may be any Fraction greater than $\frac{820}{1000}$, but leſs than $\frac{831}{1000}$: and conſequently, if $1 = r$. and $s = \frac{1}{2}$, (which is within the laſt mentioned limits of s,) then the ſides of the two Squares ſought will be found the ſame as before, *viz.*

$$\frac{16}{5} = \frac{2rsd}{rr+ss}; \text{ and, } \frac{12}{5} = \frac{rrd-ssd}{rr+ss}.$$

Again, if $1 = r$, and $s = \frac{13}{25}$, (which is alſo within the limits of s diſcovered by *Canon* 2.) then the ſides of the two Squares ſought will be found theſe, to wit, $1\frac{202}{397}$ and $\frac{212}{397}$;

whoſe Squares are $\frac{1620000}{157609}$ and $\frac{111744}{157609}$, which added together make 16; and the firſt of thoſe Squares is greater than 10, but leſs than 11; as was required.

Note, That the manner of ſearching out limits in this and divers following Queſtions, is agreeable to the method of reſolving Quadratick Equations in *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1.

QUEST. 4.

(This is the fifth of the fourth Book of Vieta's *Zeteticks; 'tis alſo reſolved by* Bachet *in his Comment upon the twelfth of the fifth Book of* Diophantus; *but I ſhall wave their ways of Reſolution, and deduce one from* Canon 1. *in the fifteenth ſtep of* Reſolut. 2. *of the preceding ſecond Queſtion of this Book.)*

To divide a given number which is compos'd of two Squares, into two other Squares, that one of the Squares ſought may conſiſt within given limits.

Preparation.

Becauſe the following Reſolution of this Queſtion preſuppoſeth each of the preſcribed limits to be greater than the leſſer of the two Squares given, I ſhall in the firſt place ſhew how from the given limits, when they are not qualified as aforeſaid, to inferr others, each of which ſhall be greater than the leſſer of the two Squares given, and then the following Reſolution will ſolve the Queſtion propos'd according to any poſſible limits whatever.

Caſe 1. *When the leſſer of the given limits is equal to the leſſer of the given Squares.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two ſuch other Squares, that one of them may be greater than 4, but leſs than 5: Here inſtead of 4 the leſſer limit, (which is equal to the leſſer Square given,) we may take $4\frac{1}{2}$ or any number between 4 and 5, (5 being the greater limit given:) then ſince $4\frac{1}{2}$ and 5 are each of them greater than 4, (the leſſer Square given,) the following Reſolution will find out two Squares whoſe ſumm ſhall be 13; and one of them ſhall be greater than $4\frac{1}{2}$, but leſs than 5, and conſequently greater than 4, but leſs than 5; as was required.

Caſe 2. *When the leſſer limit is leſs than the leſſer Square given, but the greater limit exceeds the ſame;* viz. *When the leſſer Square given falls between the given limits.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two ſuch other Squares, that one of them may be greater than 3, but leſs than 5: Here inſtead of 3 we may take $4\frac{1}{2}$, or any number between 4 the leſſer Square given, and 5 the greater limit: then ſince $4\frac{1}{2}$ and 5 are each of them greater than 4, (the leſſer Square given,) the following Reſolution will find out two Squares whoſe ſumm ſhall be 13; and one of them ſhall be greater than $4\frac{1}{2}$, but leſs than 5, and conſequently greater than 3, but leſs than 5; as was required.

Caſe 3. *When the greater of the two limits given is equal to the leſſer of the two Squares given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two ſuch other Squares, that one of them may be greater than 3, but leſs than 4: Firſt, ſubtract 3 and 4 ſeverally from 13, ſo each of the Remainders 10 and 9 is greater than 4 the leſſer Square given, and therefore by the following Reſolution two Squares may be found out whoſe ſumm ſhall be 13; and one of them leſs than 10, but greater than 9, and conſequently the other Square ſhall be greater than 3, but leſs than 4; as was required.

Caſe 4. *When each of the two limits given is leſs than the leſſer Square given.*

Let it be required to divide 13, which is compos'd of two Squares, 4 and 9, into two ſuch other Squares, that one of them may be greater than 1, but leſs than 2: Firſt, ſubtract the ſaid limits 1 and 2 ſeverally from 13 the number given to be divided, ſo each of the Remainders 12 and 11 is greater than 4 the leſſer Square given; and therefore by the following Reſolution two Squares may be found out whoſe ſumm ſhall be 13; and one of them leſs than 12, but greater than 11, and conſequently the other Square ſhall be greater than 1, but leſs than 2; as was required.

Now let it be deſired to divide 13, which is compos'd of two Squares, 4 and 9, into two ſuch other Squares, that one of them may be greater than 6, but leſs than 7.

RESO-

RESOLUTION.

1. For 2, the ſide or ſquare Root of 4 the leſſer of the two Squares given, put $\lambda\lambda\lambda\ b$
2. For 3, the ſide of 9 the greater Square given, put . . . . . $\lambda\lambda\lambda\ d$
3. For $\sqrt{6}$, that is, the ſquare Root of the leſſer of the two limits given, put $\lambda\lambda\lambda\ f$
4. For $\sqrt{7}$, that is, the ſquare Root of the greater of the two limits, put $\lambda\lambda\lambda\ g$
5. Let two unequal numbers be repreſented by . . . . . $\lambda\ s$ and r
6. Now if s be greater than r, and be to r in any Reaſon (or Proportion) except that which $d+b$ hath to $d-b$, then by *Canon* 1. in the fifteenth ſtep of *Reſolut.* 2. of the preceding *Queſt.* 2. the ſides of two Squares different from dd and bb, but ſuch, whoſe ſumm is equal to $bb+dd$, ſhall be equal to theſe two following Quantities, (which are expreſt alſo in the thirteenth and fourteenth ſteps of the ſaid *Reſolut.* 2. *Queſt.* 2.) *viz.*
$$\frac{2rsd+ssb-rrb}{ss+rr} \text{ and } \frac{ssd-rrd\backsim 2rsb}{ss+rr}.$$
7. But becauſe it's deſired that one of thoſe two Quantities or ſides laſt above expreſt, ſuppoſe, $\frac{2rsd+ssb-rrb}{ss+rr}$ may be greater than f, but leſs than g, the two unequal numbers s and r muſt be choſen ſo as that they may cauſe the ſaid ſide to agree with the ſaid limits. To which end, firſt, a number at pleaſure may be taken for one of the ſaid numbers s and r, as $1=r$, and then to ſearch out limits for the chuſing of s, I proceed in this manner, *viz.* I put a inſtead of s while it is unknown, and then ſince $1=r$, the before-mentioned ſide $\frac{2rsd+ssb-rrb}{ss+rr}$ will ſtand thus, $\frac{2da+baa-b}{aa+1}$, where the number a only is unknown: Now,
8. Let it be ſuppoſed that . . . . . $\frac{2da+baa-b}{aa+1} \sqsubset f$
9. Let it alſo be ſuppoſed that . . . . . $\frac{2da+baa-b}{aa+1} \sqsupset g$
10. Then by multiplying each part in the eighth ſtep by $aa+1$, it follows that . . . . . $2da+baa-b \sqsubset faa+f$
11. And by adding b to each part in the tenth ſtep, $2da+baa \sqsubset faa+f+b$
12. And by ſubtracting baa from each part in the eleventh ſtep, it follows that . . . . $2da \sqsubset faa-baa+f+b$
13. By ſuppoſition in the firſt and third ſteps, (agreeable to the *Preparation* to the Reſolution of this Queſtion,) f is greater than b, ſuppoſe therefore . . . . . $c=f-b$
14. Then from the twelfth and thirteenth ſteps it follows, that . . . . . $2da \sqsubset caa+f+b$
15. And by ſubtracting $f+b$ from each part in the fourteenth ſtep, it's manifeſt that . . $2da-f-b \sqsubset caa$
16. And by dividing each part of the fifteenth ſtep by c, ... $\frac{2d}{c}a-\frac{f+b}{c} \sqsubset aa$
17. And by ſubtracting $\frac{2d}{c}a$ from each part of the ſixteenth ſtep, ... $-\frac{f+b}{c} \sqsubset aa-\frac{2d}{c}a$
18. And by adding the Square of half the Coefficient $\frac{2d}{c}$ to each part of the ſeventeenth ſtep, it follows that . . . . . $\frac{dd}{cc}-\frac{f+b}{c} \sqsubset aa-\frac{2d}{c}a+\frac{dd}{cc}$
19. And by extracting the ſquare Root out of each part of the eighteenth ſtep, . . . $\sqrt{:\frac{dd-fc-bc}{cc}:} \sqsubset a-\frac{d}{c}$
20. And by adding $\frac{d}{c}$ to each part of the 19*th* ſtep, it follows, that . . . . . $\frac{d}{c}+\sqrt{\frac{dd-fc-bc}{cc}} \sqsubset a$
21. Wherefore by comparing the quantity a in the latter part of the 20*th* ſtep to the ſumm of the quantities in the firſt part, it is found that $a \sqsupset \frac{d}{c}+\sqrt{\frac{dd-fc-bc}{cc}}$

22. Again,

22. Again, because $\frac{d}{c} - a$ (as well as $a - \frac{d}{c}$ in the latter part of the nineteenth ſtep,) may be the ſquare Root of the Square which is the latter part of the eighteenth ſtep, it follows, that $\sqrt{\frac{dd - fc - bc}{cc}} \sqsubset \frac{d}{c} - a$

23. And by adding a to each part of the twenty-ſecond ſtep, ... $a + \sqrt{\frac{dd - fc - bc}{cc}} \sqsubset \frac{d}{c}$

24. Wherefore by ſubtracting $\sqrt{\frac{dd - fc - bc}{cc}}$ from each part of the twenty-third ſtep, it's evident that . . . . . $a \sqsubset \frac{d}{c} - \sqrt{\frac{dd - fc - bc}{cc}}$

25. Again, by ſuppoſition in the ninth ſtep, $\frac{2da + baa - b}{aa + 1} \sqsupset g$

26. Whence by multiplying each part by $aa + 1$, $2da + baa - b \sqsupset gaa + g$

27. And by ſubtracting baa from each part of the twenty-ſixth ſtep, ... $2da - b \sqsupset gaa - baa + g$

28. And by adding b to each part of the twenty-ſeventh ſtep, ... $2da \sqsupset gaa - baa + g + b$

29. By ſuppoſition in the firſt and fourth ſteps, (agreeable to the *Preparation* to the Reſolution of this Queſtion,) g is greater than b; ſuppoſe therefore, ... $n = g - b$

30. Then from the twenty-eighth and twenty-ninth ſteps it follows, that . . . . . $2da \sqsupset naa + g + b$

31. Whence, by arguing in like manner as before from the fourteenth ſtep to the twenty-firſt, it will appear that . . . . . $a \sqsubset \frac{d}{n} + \sqrt{\frac{dd - gn - bn}{nn}}$

32. Again, by arguing in like manner as before from the twenty-ſecond ſtep to the twenty-fourth, it will be evident that . . . . . $a \sqsupset \frac{d}{n} - \sqrt{\frac{dd - gn - bn}{nn}}$

33. Then out of the twenty-firſt and thirty-firſt ſteps, after a, c and n are exchanged for s, $f - b$ and $g - b$, for theſe are equal to thoſe, as appears by the Poſitions in the ſeventh, thirteenth and twenty-ninth ſteps, the following *Canon* 1. ariſeth for limiting the number s, when $r = 1$; *viz.*

CANON 1.

$$s \sqsupset \frac{d + \sqrt{:dd + bb - ff:}}{f - b} \quad (12.562, \&c.)$$

$$s \sqsubset \frac{d + \sqrt{:dd + bb - gg:}}{g - b} \quad (8.439, \&c.)$$

Again, out of the twenty-fourth and thirty-ſecond ſteps, after a, c and n are exchanged for s, $f - b$ and $g - b$, (as before,) another *Canon* ariſeth for limiting the number s, when $r = 1$; *viz.*

CANON 2.

$$s \sqsubset \frac{d - \sqrt{:dd + bb - ff:}}{f - b} \quad (0.788, \&c.)$$

$$s \sqsupset \frac{d - \sqrt{:dd + bb - gg:}}{g - b} \quad (0.852, \&c.)$$

Therefore if 1 be taken for the value of r, and there be given, $2 = b$; $3 = d$; $\sqrt{6} = f$; and $\sqrt{7} = g$; (as before in the firſt, ſecond, third and fourth ſteps,) then by *Canon* 1. above-expreſt, s may be any number between $12\frac{162}{1000}$ and $8\frac{112}{1000}$; and conſequently, if $r = 1$ and $s = 9$, (which value of s is within the limits laſt before-mentioned, then the ſides of the two Squares ſought (being expounded according to the two Quantities in the ſixth ſtep of the Reſolution of this *Queſt.* 4.) ſhall be $\frac{102}{41}$ and $\frac{102}{41}$, *viz.*

$2rsd + \&c.$

$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{102}{41}$, and $\frac{ssd - rrd \sim 2rsb}{ss + rr} = \frac{102}{41}$.

Therefore the two Squares ſought are $\frac{11449}{1681}$ and $\frac{10404}{1681}$, whoſe ſumm is $\frac{21853}{1681}$, that is, 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$; $2 = b$; $3 = d$; $\sqrt{6} = f$; $\sqrt{7} = g$; (as before,) then by *Canon* 2. s may be any Fraction between $\frac{282}{1000}$ and $\frac{852}{1000}$; and conſequently, if $1 = r$ and $\frac{21}{25} = s$, (which value of s is within the laſt mentioned limits,) then the ſides of the two Squares ſought will be found $\frac{1324}{533}$ and $\frac{1326}{533}$, *viz.*

$$\frac{2rsd + ssb - rrb}{ss + rr} = \frac{1391}{533}, \text{ and } \frac{ssd - rrd \sim 2rsb}{ss + rr} = \frac{1326}{533}.$$

Therefore the two Squares ſought are $\frac{1934881}{284089}$ and $\frac{1758276}{284089}$, whoſe ſumm is $\frac{3693157}{284089}$, *viz.* 13; and the firſt of thoſe Squares is greater than 6, but leſs than 7, as was required.

Again, if $1 = r$, and $s = \frac{4}{5}$, (which value of s is alſo within the limits diſcovered by *Canon* 2.) then the ſides of the two Squares ſought being expounded as before, will be found $\frac{102}{41}$ and $\frac{102}{41}$; which are the ſame with thoſe before-found in the Example of *Canon* 1.

Note. If 1 be put equal to r, and the number s be taken by *Canon* 2. then becauſe in this caſe s is leſs than r, the *Algebraical* Rules of + and − in adding, ſubtracting, *&c.* muſt be obſerved to reſolve the aforeſaid literal values of the ſides of the Squares ſought into numbers, as in the two laſt Examples.

QUEST. 5. (Quæſt. 11. Lib. 2. *Diophant.*)

To find two ſquare numbers whoſe difference ſhall be equal to a given number, ſuppoſe 60, (or d.)

RESOLUTION.

1. The given difference, 60, that is, ... d
2. Let ſome number whoſe Square is leſs than the given difference be repreſented by . . . . . b
3. For the ſide of the leſſer Square ſought put . . . . . a
4. And for the ſide of the greater Square ſought put . . . . . $a + b$
5. Therefore the leſſer Square is . . . . . aa
6. And the greater Square is . . . . . $aa + 2ba + bb$
7. And the difference of thoſe Squares is . . . . . $2ba + bb$
8. But the ſaid difference muſt be equal to the given difference d, therefore . . . . . $2ba + bb = d$
9. Which Equation, after due Reduction, makes known the value of the ſide of the leſſer Square, *viz.* . . . . . $a = \frac{d - bb}{2b}$
10. And from the ninth and fourth ſteps, the value of the ſide of the greater Square is alſo diſcovered, *viz.* . . . . $a + b = \frac{d + bb}{2b}$

The two laſt ſteps give the following

CANON 1.

Take any ſquare number leſs than the given difference, and ſubtract it from the ſaid difference; then divide the Remainder by the double of the ſide of the Square firſt taken, and the Quotient ſhall be the ſide of the leſſer of the two Squares ſought; laſtly, this ſide added to the ſide of the Square firſt taken, gives the ſide of the other Square ſought.

So if two Squares be deſired whoſe difference ſhall be 60, I take a ſquare number leſs than 60, as 36; this ſubtracted from that leaves 24, which divided by 12 the double of the ſquare Root of 36, gives the Quotient 2, which ſhall be the ſide of the leſſer Square ſought; and then by adding 6 the ſquare Root of the ſaid 36, to the ſide 2, the ſumm 8 is the ſide of the greater Square ſought; laſtly, the Squares of the ſaid ſides 2 and 8, to wit, 4 and 64 will ſolve the Queſtion, for their difference is 60, as was required.

Obſervations upon Queſt. 5.

1. It is evident by the two laſt ſteps of the preceding *Reſolution*, that the values of the ſides of the two Squares ſought are $\frac{d - bb}{2b}$ and $\frac{d + bb}{2b}$

Now

Now if we ſuppoſe $bc = d$, then thoſe ſides will be converted into theſe, *viz.* . . . . . $\frac{bc + bb}{2b}$ and $\frac{bc - bb}{2b}$

Which laſt mentioned ſides or Quotients, after the common Factor b is caſt away, will be reduced to theſe, to wit, . . . $\frac{1}{2}c + \frac{1}{2}b$ and $\frac{1}{2}c - \frac{1}{2}b$

Hence ariſeth this elegant Canon, often uſed by *Diophantus* to find out two Squares in a given difference, *viz.*

CANON 2.

Take two ſuch unequal numbers that the Product of their multiplication may be equal to the given difference; then half the ſumm and half the difference of thoſe two numbers ſhall be the ſides of the two Squares ſought.

As, for example, if two Squares be deſired whoſe difference ſhall be 60, I take two ſuch numbers (10 and 6) which being mutually multiplied make 60; then half the ſumm of 10 and 6 and half their difference are 8 and 2 the ſides of the two Squares ſought, and conſequently the Squares themſelves are 64 and 4, whoſe difference is 60; as was required.

Again, inſtead of 10 and 6 taken as before, we may take 30 and 2, for the Product of theſe is equal to the given difference 60; then half the ſumm of 30 and 2, and half their difference, give 16 and 14, whoſe Squares 256 and 196 have 60 for their difference; as was required.

After the ſame manner, Fractions being admitted, innumerable pairs of Squares may be found out, ſuch, that the difference of each pair ſhall be equal to one and the ſame number given: For if the given number be divided by a number taken at pleaſure, half the ſumm, and half the difference of the Diviſor and Quotient ſhall be the ſides of two Squares whoſe difference is equal to the given number.

2. But for farther illuſtration of the truth of the preceding *Canon* 2. let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater; then

The Square of $\frac{1}{2}c + \frac{1}{2}b$ is . . . . . $\frac{1}{4}cc + \frac{1}{2}cb + \frac{1}{4}bb$,
The Square of $\frac{1}{2}c - \frac{1}{2}b$ is . . . . . $\frac{1}{4}cc - \frac{1}{2}cb + \frac{1}{4}bb$,
The difference of thoſe Squares is . . . . . $+ cb$.

Whence it is manifeſt, That the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of half the ſumm of the ſaid two numbers, and the leſſer is the Square of half their difference. Wherefore the truth of the foregoing *Canon* 2. doth evidently appear.

3. *Vieta* uſeth the following Canon (which differs but little from the preceding *Canon* 2.) to find out two Squares in a given difference.

CANON 3.

Take two ſuch unequal numbers, that the Product of their multiplication may be equal to a quarter of the given difference of two Squares ſought; then the ſumm and difference of thoſe two numbers firſt taken ſhall be the ſides of the deſired Squares.

As, for example, if it be deſired to find out two Squares whoſe difference ſhall be 60; firſt, I take $\frac{1}{4}$ of the ſaid 60, to wit, 15; then I chuſe two ſuch unequal numbers that the Product of their multiplication may make 15, as 5 and 3; laſtly, the ſumm, and difference of 5 and 3, give 8 and 2 for the ſides of two Squares whoſe difference is 60.

The truth of this *Canon* 3. may be demonſtrated thus; let c and b repreſent two unequal numbers, and ſuppoſe c to be the greater, then

The Square of $c + b$ is . . . . . $cc + 2cb + bb$,
The Square of $c - b$ is . . . . . $cc - 2cb + bb$,
The difference of thoſe Squares is . . . . . $+ 4cb$.

Whence it is manifeſt, That the quadruple of the Product of the multiplication of any two unequal numbers is equal to the difference of two Squares, the greater of which is the Square of the ſumm of thoſe numbers, and the leſſer Square is the Square of the difference of the ſame two numbers. Wherefore the truth of *Canon* 3. is evident.

4. If a Square be equal to two Squares, then (by *prop.* 47. *Elem.* 1. Euclid.) the ſides of thoſe three Squares will conſtitute a right-angled Triangle, *viz.* the greateſt ſide ſhall be the Hypothenuſal, and the other two the ſides about the right-angle; whence it follows, that the Square of one of the ſides about the right-angle is equal to the difference of the Squares of the other two ſides: And therefore if any Rational number be given for

for one of the ſides about the right-angle of a right-angled Triangle, the other ſide about the right-angle and the Hypothenuſal ſhall be given alſo in Rational numbers by the help of any of the three preceding Canons: As, for example, if 4 be given for the Baſe, the Square thereof is 16, then by any of the ſaid Canons find out two Squares whoſe difference may be 16, ſuch are 25 and 9, (and innumerable other pairs of Squares;) therefore their ſquare Roots or ſides, *viz* 5 and 3, ſhall be the deſired Hypothenuſal and Perpendicular. Whence it is evident, that by the like Operation innumerable right-angled Triangles may be found out in Rational numbers, which ſhall have one common Baſe (or Perpendicular) preſcribed.

QUEST. 6.

To find two ſuch numbers, that the Product of their multiplication may be equal to a given number, ſuppoſe d, and that the Square of half the ſumm of the ſaid numbers may be greater than a number given, ſuppoſe b.

RESOLUTION.

1. For one of the numbers ſought put . . . . . a
2. Then by dividing the given Product d by a, the Quotient ſhall be the other number ſought, to wit, ... $\frac{d}{a}$
3. Therefore half the ſumm of thoſe two numbers is . . $\frac{aa + d}{2a}$
4. Now ſuppoſe it be deſired that the Square of the ſaid half ſumm may be greater than the given number b, then it neceſſarily follows, that the ſumm it ſelf muſt be greater than the ſquare Root of b, viz. . . . . $\frac{aa + d}{2a} \sqsubset \sqrt{b}$
5. Therefore from the fourth ſtep, by multiplying each part by $2a$, it follows, that . . . . . $aa + d \sqsubset 2a\sqrt{b}$
6. That is, (becauſe $a\sqrt{4b} = 2a\sqrt{b}$,) . . . . . $aa + d \sqsubset a\sqrt{4b}$
7. And by ſubtracting d from each part of the ſixth ſtep, it follows that . . . . . $aa \sqsubset a\sqrt{4b} - d$
8. And by ſubtracting $a\sqrt{4b}$ from each part of the ſeventh ſtep, $aa - a\sqrt{4b} \sqsubset -d$
9. And by adding a quarter of the Square of the known Coefficient $\sqrt{4b}$, to wit, b, to each part of the eighth ſtep, it follows that . . . . . $aa - a\sqrt{4b} + b \sqsubset b - d$
10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $a - \sqrt{b} \sqsubset \sqrt{:b - d:}$
11. Wherefore by adding $\sqrt{b}$ to each part of the tenth ſtep, $a \sqsubset \sqrt{b} + \sqrt{:b - d:}$
12. Again, becauſe $-a + \sqrt{b}$ (as well as $a - \sqrt{b}$) may be the ſide of the Square $aa - a\sqrt{4b} + b$ in the firſt part of ninth ſtep, it thence follows that . . . . . $-a + \sqrt{b} \sqsubset \sqrt{:b - d:}$
13. And by adding a to each part of the twelfth ſtep, . . $\sqrt{b} \sqsubset a + \sqrt{:b - d:}$
14. And by ſubtracting $\sqrt{:b - d:}$ from each part of the thirteenth ſtep, ... $\sqrt{b} - \sqrt{:b - d:} \sqsubset a$
15. Wherefore from the fourteenth ſtep, by comparing the latter part to the former, ... $a \sqsupset \sqrt{b} - \sqrt{:b - d:}$
16. The eleventh and fifteenth ſteps give limits for the choice of (a) one of the two numbers ſought by this ſixth Queſtion, when it requires that the Square of half the ſumm of the ſame numbers may be greater than a given number; and the premiſſes afford this following

CANON 1.

For one of the numbers ſought take any number greater than $\sqrt{b} + \sqrt{:b - d:}$ or leſs than $\sqrt{b} - \sqrt{:b - d:}$: then divide d the given Product of the multiplication of the two numbers ſought, by the number firſt taken, ſo ſhall the Quotient be the other number ſought.

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$;
Thence it follows, that . . . $\sqrt{b} + \sqrt{:b - d:} = 21.856$, *&c.*
Alſo, ... $\sqrt{b} - \sqrt{:b - d:} = 5.856$, *&c.*

Therefore according to the direction of the preceding Canon, I take for one of the two numbers fought ſome number greater than $21\frac{827}{1000}$, or leſs than $5\frac{826}{1000}$, as the number 2; by this I divide the given number $128 = d$, and the Quotient gives 64 for the other numbers fought; which two numbers, 2 and 64, will ſolve the Queſtion, as will be evident by

The Proof.

The Product of the multiplication of 2 and 64 makes the given number 128 (or d,) and the Square of half the ſumm of 2 and 64, *viz.* the Square of 33 is 1089, which is greater than 192, (or b,) as was required. But to the end there may be a poſſibility of ſolving the Queſtion propoſed, the Canon above-expreſt doth ſhew there is a neceſſity that the number d muſt not exceed the number b.

17. The preceding Reſolution of *Queſt.* 6. preſuppoſeth it to be deſired that the Square of half the ſumm of the two numbers fought may be greater than a number given; but if it were deſired that the ſaid Square might be leſs than a number given, then $\sqsupset$ being uſed inſtead of $\sqsubset$ in the ſaid Reſolution, there would at length ariſe this following *Canon* to ſolve the ſaid *Queſtion* in the latter Caſe.

CANON 2.

For one of the numbers fought take any number leſs than $\sqrt{b + \sqrt{:b^2 - d^2:}}$: but greater than $\sqrt{b - \sqrt{:b^2 - d^2:}}$: then divide d the given Product of the multiplication of the two numbers fought, by the number firſt taken, and the Quotient ſhall be the other number fought.

An Example of this Canon.

Suppoſe (as before) . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . $\sqrt{b + \sqrt{:b^2 - d^2:}} = 21.856$, *&c.*
Alſo, ... $\sqrt{b - \sqrt{:b^2 - d^2:}} = 5.856$, *&c.*

Therefore (according to the latter Canon) I take for one of the two numbers fought ſome number between $5\frac{827}{1000}$ and $21\frac{826}{1000}$, as 16; by this I divide the given number 128 (or d,) and the Quotient gives 8 for the other number fought; which two numbers, 16 and 8, will ſolve the Queſtion when it requires that the Square of half their ſumm may be leſs than the given number 192, as may eaſily be proved: For the Product of the ſaid 16 and 8 makes the given number 128, and the Square of half the ſumm of 16 and 8, *viz.* the Square of 12, is 144, which is leſs than the given number 192; as was required.

QUEST. 7.

To find two ſquare numbers in a given difference, and that one of thoſe Squares may be greater or leſs than a given number.

1. Let it be required to find two ſuch ſquare numbers, that their difference may be equal to a given number, ſuppoſe d; and that the greater Square may exceed a given number, ſuppoſe b.

RESOLUTION.

It is manifeſt by *Canon* 2. of the foregoing *Queſt.* 5. That if two numbers be taken, ſuch, that the Product of their multiplication is equal to the given difference of two Squares fought, then half the ſumm and half the difference of the numbers ſo taken ſhall be the ſides of thoſe Squares: Therefore if two numbers be found out, ſuch, that their Product is equal to the given difference d, and that the Square of half the ſumm of the ſame numbers is greater than the given number b, then that Square ſhall be the greater of the two Squares required, and the Square of half the difference of the ſaid numbers ſhall be the leſſer Square required: But two ſuch numbers may be found out by the firſt Canon of the preceding ſixth Queſtion, and conſequently this ſeventh Queſtion may be ſolved by the following

CANON 1.

Take ſome number greater than $\sqrt{b + \sqrt{:b^2 - d^2:}}$: or leſs than $\sqrt{b - \sqrt{:b^2 - d^2:}}$: then divide d the given difference of the two Squares fought by the number firſt taken, and reſerve the Quotient; laſtly, half the ſumm and half the difference of the ſaid Quotient and number firſt taken ſhall be the ſides of the two Squares fought.

An

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$,
Thence it follows that . . . . $\sqrt{b + \sqrt{:b^2 - d^2:}} = 21.856$, *&c.*
Alſo, ... $\sqrt{b - \sqrt{:b^2 - d^2:}} = 5.856$, *&c.*

Therefore according to the direction of the Canon, I take ſome number greater than $21\frac{827}{1000}$, or leſs than $5\frac{826}{1000}$, as 2; then by this 2 I divide 128, (to wit, d) and the Quotient is 64; laſtly, half the ſumm of the ſaid 2 and 64 is 33, and half their difference is 31, which 33 and 31 are the ſides of two Squares that will ſolve the Queſtion propoſed, as will be evident by

The Proof.

The Squares of 33 and 31 are 1089 and 961; the difference of theſe is equal to the given difference 128, (to wit, d;) and the greater Square 1089 is greater than 192, (or b;) as was required.

2. In like manner, if it were required to find out two Squares whoſe difference ſhall be equal to a given number d, and the greater Square leſs than a given number b; the ſides of the ſaid Squares may be found out by this following

CANON 2.

Take ſome number leſs than $\sqrt{b + \sqrt{:b^2 - d^2:}}$: but greater than $\sqrt{b - \sqrt{:b^2 - d^2:}}$: then divide d the given difference of the Squares fought, by the number ſo taken, and reſerve the Quotient; laſtly, half the ſumm and half the difference of the ſaid Quotient and number firſt taken ſhall be the ſides of the two Squares fought.

An Example in Numbers.

Suppoſe . . . . . $d = 128$, and $b = 192$,
Thence it follows, that . . . . $\sqrt{b + \sqrt{:b^2 - d^2:}} = 21.856$, *&c.*
Alſo, ... $\sqrt{b - \sqrt{:b^2 - d^2:}} = 5.856$, *&c.*

Therefore according to the direction of the laſt preceding Canon, I take ſome number between $5\frac{827}{1000}$ and $21\frac{826}{1000}$, as 16; then by this I divide the given number 128, (to wit, d,) and the Quotient is 8; laſtly, half the ſumm of the ſaid 16 and 8 is 12, but half their difference is 4; which 12 and 4 are the ſides of two Squares that will ſolve the Queſtion, as will be evident by

The Proof.

The Squares of 12 and 4 are 144 and 16; the difference of theſe is equal to the given difference 128, (or d;) and the greater Square 144 is leſs than 192, (or b;) as was required.

3. But if it were required to find out two Squares in a given difference d, and that the leſſer Square might be greater than a given number g; they may be diſcovered by the help of the preceding Canons of this ſeventh Queſtion, in this manner, *viz.*

Let it be required to find two Squares whoſe difference ſhall be 24 (or d,) and that the leſſer Square may be greater than 12, (or g.) Here the ſcope muſt be to find out two ſuch Squares that their difference may be 24, and that the greater Square may exceed 36, that is, $24 + 12$, and then the leſſer Square will conſequently exceed 12. Therefore,

Suppoſe . . . . . $d = 24$, and $g = 12$,
Suppoſe alſo . . . . . $b = 36 = d + g$,
Thence it follows, that . . . $\sqrt{b + \sqrt{:b^2 - d^2:}} = 9.46$, *&c.*
Alſo, ... $\sqrt{b - \sqrt{:b^2 - d^2:}} = 2.53$, *&c.*

Then (according to the firſt Canon of this ſeventh Queſtion) I take ſome number greater than $9\frac{47}{100}$, or leſs than $2\frac{53}{100}$, as 2; by this I divide 24, (to wit, d,) and the Quotient is 12; then half the ſumm of 2 and 12 is 7, and half their difference is 5; which 7 and 5 are the ſides of two Squares 49 and 25, whoſe difference is 24, (to wit, d,) and the leſſer Square 25 is greater than 12, (or g;) as was required. But for the greater evidence, let ff be put for the leſſer Square found out, and hh for the greater, then

By Conſtruction, ... $hh = d + ff$,
Alſo by Conſtruction, ... $hh \sqsubset d + g$ $(= b,)$
Therefore . . . . . $ff \sqsubset g$. Which was to be proved.

4. Laſtly,

4. Lastly, if it were desired to find out two Squares in a given difference d, and that the lesser Square might be less than a given number g; let the summ of those two given numbers, (to wit, $d + g$) be called b (as before,) and then by the latter of the two preceding Canons of this seventh Question find out two Squares that their difference may be equal to the given difference d, and that the greater Square may be less than the summ b, so shall the lesser Square be less than the given number g.

QUEST. 8.

[*This is the twelfth of the second Book of* Diophantus, *and the seventh of the fourth Book of* Vieta's *Zeteticks.*]

Two numbers being given, suppose 192 and 128, to find a third, which added to each of those given may make each summ to be a Square.

RESOLUTION 1.

1. For the number sought put . . . . . a
2. Then the Question requires that each of these two summs may be a square number, *viz.* . . . . . $192 + a = \square$; $128 + a = \square$
3. Now that Duplicate Equality (for so *Diophantus* calls it) may be resolved thus, *viz.* First, subtract $128 + a$ from $192 + a$, and the Remainder 64 is the difference both of the given numbers and likewise of the two Squares sought; then (by the preceding seventh Question) find two such square numbers that their difference may be 64, and that the greater Square may exceed 192 the greater number given; such are the Squares 289 and 225, whose sides are 17 and 15.
4. Then equate $192 + a$ to 289, thus, ... $192 + a = 289$
5. Or equate $128 + a$ to 225, thus, ... $128 + a = 225$
6. Lastly, from either of those Equations in the fourth and fifth steps, the number a sought will be also made known, *viz.* . . . $a = 97$

I say 97 will solve the Question; for if it be added to 192 and 128 severally, the summs 289 and 225 are Squares, as was required: And out of the premisses well examined, respect also being had to the preceding seventh Question, there will arise this following Canon to find out innumerable Answers to the Question proposed.

CANON.

7. Take any number greater than the summ, or less than the difference of the square Roots of the two numbers given; divide the difference of the two numbers given, by the number first taken, and reserve the Quotient; then from the Square of half the summ of that Quotient and the number taken, subtract the greater of the two numbers given; or, from the Square of half the difference between the said Quotient and the number first taken, subtract the lesser number given; so shall either of the Remainders (for they are equal to one another) be the numbers sought.

An Example of the Canon.

8. Let there be two numbers given to find a third, according to *Quest.* 8. as, ... 96 ; 8
9. Their difference is 88
10. Also . . . . . $\sqrt{96} + \sqrt{8} = 12.626$, *&c.*
11. And . . . . . $\sqrt{96} - \sqrt{8} = 6.969$, *&c.*
12. Let a number be taken, either greater than $12\frac{632}{1000}$, or less than $6\frac{962}{1000}$, such is 4
13. Divide 88 in the ninth step, by 4 in the twelfth, and the Quotient is 22
14. The half of the summ of 4 and 22 in the twelfth and thirteenth steps is 13
15. The Square of the said 13 is 169
16. From that Square subtract the greater of the two numbers given in the eighth step, to wit, ... 96
17. So the Remainder is the number sought, to wit, . . . 73

The Proof.

$96 + 73 = 169$, whose $\sqrt{}$ is 13;
$8 + 73 = 81$, whose $\sqrt{}$ is 9.

In

In like manner, if it were desired to find out some number signified by (a) that $10a + 54$ may make a Square, also that $10a + 6$ may make a Square; the preceding Canon will give innumerable values of a, among which 1 will be found a true value; for if $a = 1$, then $10a + 54 = 64$, and $10a + 6 = 16$.

Observations upon Quest. 8.

1. *Diophantus* in resolving this Question makes an entrance into one of his peculiar subtilties, which he calls a Duplicate Equality, (an ingenious Invention variously used by him, as divers knotty Questions in this Book will make manifest;) the principal hinge whereof depends on the finding of two Squares whose difference shall be equal to the difference of two *Algebraick* Quantities, each of which is propos'd to be found equal to some known square number: As, in the preceding Resolution of this *Quest.* 8. two Squares, to wit, 289 and 225 are found out; whose difference 64 is equal to the difference of the two *Algebraick* Quantities $192 + a$ and $128 + a$, each of which, according to the import of the Question, is to be found equal to some square number, and therefore the number a sought must be such as will cause that effect.

2. But that the reason of the Operation in resolving the Duplicate Equality in this eighth Question may clearly appear, two things are to be proved, *viz.*

First, That the greater of the two square numbers found out in the third step of the Resolution must necessarily exceed the greater of the two numbers given in the Question, and the lesser Square exceed the lesser number given. To prove this, first, it is intended that the desired number a should be affirmative, that is, greater than nothing; but if any square number not greater than 192 were set in the place of 289 in the Equation in the fourth step, *viz.* $192 + a = 289$; or any square number not greater than 128, in the place of 225 in the Equation in the fifth step, *viz.* $128 + a = 225$; the value of a would be less than nothing: and therefore the necessity of the before-mentioned qualification of the said Squares is apparent.

Secondly, That when two square numbers are found out, such, that their difference is equal to the difference of the two numbers given, and that the greater Square exceeds the greater of those given numbers; then consequently the lesser Square shall exceed the lesser number, and these two last mentioned excesses or differences shall be equal to one another. The truth of this consequence will be evident by the following

THEOREM.

If two square numbers, suppose dd the greater and ff the lesser, have the same difference as two other numbers, suppose b the greater and c the lesser, and that the greater Square dd exceeds the greater number b; then the lesser Square ff shall exceed the lesser number c, and the former excess $dd - b$ shall be equal to the latter $ff - c$. For,

By supposition . . . . . $dd - ff = b - c \sqsupset 0$
Therefore by adding ff to each part, it follows, that . . . $dd = ff + b - c$
But by supposition . . . . . $dd \sqsupset b$
Wherefore, by subtracting b from each part of the Equation in the last step but one, that which the *Theorem* affirms is manifest; *viz.* . . . . . $dd - b = ff - c$

And consequently either of those equal excesses or differences (in the last Equation) shall be a number to solve the Question proposed; for it is manifest that if the former excess $dd - b$ be added to b, and the latter excess $ff - c$ to c, the summs will be Squares, to wit, dd and ff.

Another manner of resolving the foregoing Quest. 8. *which is here repeated*, viz.

To find a number which added severally to 128 and 192, may make the summs to be Squares.

RESOLUTION 2.

1. For the number sought put $aa - 128$ whereby part of the Question is satisfied; for if $aa - 128$ be added to 128, it makes a Square, to wit, aa; let therefore the number sought be feigned to be . . . . . $aa - 128$

2. But

2. But the Queſtion requires alſo, that if the number ſought be added to 192, the ſumm may be a Square; add therefore $aa - 128$ to 192, ſo the ſumm $aa - 128 + 192$, that is, $aa + 64$ muſt be equal to a ſquare number, *viz.* . . . . . $aa + 64 = \square$

3. It remains therefore to equate the ſaid $aa + 64$ to a Square, whoſe ſide (to the end the value of aa may be greater than 128, as the number $aa - 128$ aſſumed in the firſt ſtep requires) may be feigned to be $a +$ any Abſolute number leſs than $2\frac{142}{1000}$, or $-a +$ any Abſolute number greater than $25\frac{128}{1000}$, (which limits are diſcovered by the following third way of reſolving this Queſtion;) let therefore the ſaid ſide be feigned to be $a + 2$, and then the Square of this ſide being equated to $aa + 64$ as the ſecond ſtep requires, this Equation ariſeth, *viz.*

$$aa + 4a + 4 = aa + 64.$$

4. Which Equation, after due Reduction, makes known the value of a, viz. . . . . $a = 15$

Therefore by the firſt and fourth ſteps the number ſought will be found 97, which will ſolve the Queſtion; for if 97 be added to 128 and 192 ſeverally, the ſumms are Squares, to wit, 225 and 289; and the limits in the third ſtep for feigning the ſide of one of the Squares ſought do ſhew that the Queſtion is capable of innumerable Anſwers.

A third manner of reſolving the preceding Queſt. 8. *which is here repeated*, viz.

To find a number, which added firſt to a given number (f,) and then to a greater given number (b,) may make the ſumms to be Squares.

RESOLUTION 3.

1. For the difference of the two given numbers b and f put d, viz. ſuppoſe . . . . . $d = b - f$.

2. And for the number ſought put $aa - f$, (a repreſenting a number unknown,) whereby the firſt part of the Queſtion is ſatisfied; for if $aa - f$ be added to f it makes a Square, to wit, aa; let therefore the number ſought be feigned to be . . . . . $aa - f$

3. But the Queſtion requires alſo, that if the number ſought be added to b it may make a Square; add therefore $aa - f$ to b, and it makes $aa + b - f$, that is, $aa + d$, (for d was put equal to $b - f$,) which muſt be equal to a Square, *viz.* . . . . . $aa + d = \square$

4. It remains then to equate $aa + d$ to ſome Square, whoſe ſide may be feigned to be either $a + e$, or $-a + u$, (which numbers, a, e and u are all yet unknown;) Firſt, let the ſide of the ſaid Square be feigned to be $a + e$, ſo its Square being equated to $aa + d$, this Equation ariſeth, *viz.*

$$aa + 2ae + ee = aa + d.$$

5. Which Equation, after it is duly reduced to find what a is equal to, gives . . . . . $a = \frac{d - ee}{2e}$

6. But by the ſecond ſtep . . . . . $a \sqsubset \sqrt{f}$

7. Therefore by the fifth and ſixth ſteps . . . . . $\frac{d - ee}{2e} \sqsubset \sqrt{f}$

8. And by multiplying each part of the ſeventh ſtep by $2e$, it follows, that . . . . . $d - ee \sqsubset 2e\sqrt{f}$, or $e\sqrt{4f}$

9. And by adding ee to each part of the eighth ſtep, it gives . . . . . $d \sqsubset ee + e\sqrt{4f}$

10. And by adding f, that is, $\frac{1}{4}$ of the Square of the known Coefficient $\sqrt{4f}$ in the ninth ſtep, to each part thereof, it follows, that . . . . . $d + f \sqsubset ee + e\sqrt{4f} + f$

11. And by extracting the ſquare Root out of each part of the tenth ſtep, ... $\sqrt{:d + f:} \sqsubset e + \sqrt{f}$

12. And by ſubtracting $\sqrt{f}$ from each part of the eleventh ſtep, ... $\sqrt{:d + f:} - \sqrt{f} \sqsubset e$

13. And by comparing the latter part of the twelfth ſtep to the firſt part, it's manifeſt that . . . . . $e \sqsupset \sqrt{:d + f:} - \sqrt{f}$

14. But by the firſt ſtep, ... $b = d + f$

15. And conſequently . . . . . $\sqrt{b} = \sqrt{:d + f:}$

16. Where-

16. Wherefore, by ſetting $\sqrt{b}$ in the place of $\sqrt{:d + f:}$ in the thirteenth ſtep, it's evident that . . . . . $e \sqsupset \sqrt{b} - \sqrt{f}$

17. And becauſe by the fifth ſtep, ... $ee \sqsupset d$

18. Therefore . . . . . $e \sqsupset \sqrt{d}$

19. Again, foraſmuch as the ſide of the Square mentioned in the fourth ſtep may be feigned to be $-a + u$, let the Square of $-a + u$ be equated to $aa + d$, as the third ſtep requires, ſo this Equation ariſeth, *viz.*

$$aa + d = aa - 2ua + uu.$$

20. Which Equation, after due Reduction, to find out what a is equal to, gives . . . . . $a = \frac{uu - d}{2u}$

21. But by the ſecond ſtep, ... $a \sqsubset \sqrt{f}$

22. Therefore from the twentyeth and twenty-firſt ſteps, ... $\frac{uu - d}{2u} \sqsubset \sqrt{f}$

23. And by arguing to find out limits for u, in like manner as before for e from the ſeventh to the ſixteenth ſtep *inclusivè*, it will at length appear, that . . . . . $u \sqsubset \sqrt{b} + \sqrt{f}$

24. And becauſe by the twentyeth ſtep . . . . . $uu \sqsubset d$

25. Therefore . . . . . $u \sqsubset \sqrt{d}$

26. Now ſuppoſe . . . . . $192 = b$

27. And conſequently . . . . . $128 = f$, $64 = b - f = d$

28. And from the ſixteenth and twenty-ſixth ſteps it follows that . . . . . $e \sqsupset 2\frac{142}{1000}$, &c. $(\sqrt{b} - \sqrt{f})$

29. And from the twenty-third and twenty-ſixth ſteps, ... $u \sqsubset 25\frac{128}{1000}$, &c. $(\sqrt{b} + \sqrt{f})$

30. Likewiſe from the eighteenth and twenty-ſixth ſteps, ... $e \sqsupset 8\ (\sqrt{d})$

31. And from the twenty-fifth, and twenty-ſeventh ſteps . . . . . $u \sqsubset 8$

But of the limits found out in the four laſt preceding ſteps, the two former only are neceſſary for chuſing the numbers e and u, for the two latter not being ſo ſtrict as the two former are uſeleſs. Then after the number e or u is duely choſen according to the ſaid limits in the twenty-eighth and twenty-ninth ſteps, the number a will be diſcovered either by the fifth, or by the twentieth ſtep; and laſtly, the number ſought by the ſecond and twenty-ſixth ſteps. All which will be farther illuſtrated by the following Canon and Examples.

CANON.

32. Take any number leſs than the difference, or greater than the ſumm of the ſquare Roots of the two numbers given in the Queſtion: then divide the difference between the Square of the number firſt taken, and the difference of the two numbers given, by the double of the number taken; and from the Square of the Quotient ſubtract the leſſer of the two numbers given, ſo the Remainder ſhall be the number ſought.

Example 1.

Let there be two numbers given to find a third, according to *Queſt.* 8. as, ... $b = 192$, and $f = 128$

Thence it follows, that . . . . . $b - f = 64 = d$

Alſo, ... $\sqrt{b} - \sqrt{f} = 2.542$, &c.

And . . . . . $\sqrt{b} + \sqrt{f} = 25.170$, &c.

Now according to the *Canon*, take ſome number leſs than $2\frac{142}{1000}$, and call it e, as . . . . . $e = 1$

Thence it follows, that . . . . . $\frac{d - ee}{2e} = 31\frac{1}{2} = a$

And laſtly, ... $aa - f = 864\frac{1}{4}$ ſought.

The Proof.

$192 + 864\frac{1}{4} = 1056\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $32\frac{1}{2}$.
$128 + 864\frac{1}{4} = 992\frac{1}{4}$, whoſe $\sqrt{}$ extracted is $31\frac{1}{2}$.

Example 2.

Example 2.

Again, the same things being given as in *Example* 1. take some number greater than $25\frac{121}{1000}$, and call it n, as . . $n = 32$

Thence it follows that . . . . . $\frac{nn-d}{2n} = 15 = a$

And lastly, ... $aa-f = 97$ sought.

The Proof.

$192 + 97 = 289$, whose $\surd$ is 17.
$128 + 97 = 225$, whose $\surd$ is 15.

Note. In divers of *Diophantus*'s Questions, where Algebraick Quantities are to be equated to Squares, there is great use of finding out Limits, (after the manner delivered in the last preceding *Resolution,*) to direct how to feign the sides of the said Squares, so, as that their values in numbers may be greater than nothing; and therefore for the more ample Illustration of that Method I have framed the five Questions next following, in the Resolutions whereof, the industrious Learner will meet with no difficulty, if he be well exercis'd in the manner of resolving Quadratick Equations according to *Sect.* 5, 7, 9. *Chap.* 15. *Book* 1. as also in the *Observations* upon the first Question of this third Book.

QUEST. 9.

To find a number, call it a, that shall be less than 3, and cause $aa + 12$ to be a square number.

RESOLUTION.

1. Put Letters for the given numbers, *viz.* . . . . . $d = 12$, $f = 3$
2. Then the Question requires that $aa + d$ may be equal to a Square, but its Side (or Root) must be so feigned that the value of a may be less than f, and greater than nothing; to which end the said Side may be feigned to be either $a + e$, or $-a + n$; (which a, e and n do represent numbers yet unknown;) First therefore supposing the said Side to be $a + e$, the Square thereof is $aa + 2ae + ee$, which must be equated to $aa + d$ above-mentioned; hence this following Equation ariseth, *viz.*

$$aa + 2ae + ee = aa + d.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{d-ee}{2e}$
4. And since the Question requires that a may be less than 3, *viz.* . . . . . $a \sqsupset f$
5. Therefore from the third and fourth steps . . . . $\frac{d-ee}{2e} \sqsupset f$
6. And by multiplying each part of the fifth step by $2e$, it follows that . . . . . $d - ee \sqsupset 2fe$
7. And by adding ee to each part of the sixth step, . . $d \sqsupset ee + 2fe$
8. And by adding the Square of half the Coefficient $2f$ to each part of the seventh step, it gives . . . . . $d + ff \sqsupset ee + 2fe + ff$
9. And by extracting the square Root out of each part of the eighth step, ... $\surd:\overline{d+ff}: \sqsupset e+f$
10. And by subtracting f from each part of the ninth step, $\surd:\overline{d+ff}: - f \sqsupset e$
11. Wherefore from the tenth step, by comparing the latter part to the first, ... $e \sqsubset \surd:\overline{d+ff}: - f$
12. And since by the third step $ee \sqsupset d$, therefore, . . $e \sqsupset \surd d$
13. Again, for as much as the Side of the Square mentioned in the second step may be feigned to be $-a + n$, let the Square of $-a + n$ be equated to $aa + d$ as the Question requires, so this Equation ariseth, *viz.*

$$aa + d = aa - 2na + nn.$$

14. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{nn-d}{2n}$

15. And

15. And because the Question requires a to be less than 3, *viz.* . . . . . $a \sqsupset f$
16. Therefore from the fourteenth and fifteenth steps it follows that . . . . . $\frac{nn-d}{2n} \sqsupset f$
17. And by continuing the Process to find out Limits for n, in like manner as before for e from the fifth step to the eleventh inclusive, it will at length appear that . . . . $n \sqsupset \surd:\overline{d+ff}: + f$
18. And since by the sixteenth step, $nn \sqsubset d$, therefore . $n \sqsubset \surd d$

From the first, eleventh, twelfth, seventeenth, eighteenth, third, and fourteenth steps the following Canon is deduced, by the help whereof innumerable Answers may be found out to the Question proposed.

CANON.

19. Take any number (e) between $\surd:\overline{d+ff}: - f$ and $\surd d$, that is, between $1\frac{182}{1000}$, *&c.* and $3\frac{464}{1000}$, *&c.* Or any number (n) between $\surd d$ and $\surd:\overline{d+ff}: + f$, that is, between $3\frac{464}{1000}$, *&c.* and $7\frac{182}{1000}$, *&c.* Then divide the difference between the Square of the number taken and d, or 12, by the double of the number taken, so the Quotient shall be the number a sought.

Example 1.

Let there be two numbers given in such manner as before is supposed in this *Quest.* 9. *viz.* . . . . . $d = 12$, and $f = 3$

Then according to the first limits in the Canon take some number between $1\frac{181}{1000}$ and $3\frac{464}{1000}$ as 2, and call this e, *viz.* suppose . . . . . $e = 2$

And then by the latter part of the Canon, ... $\frac{d-ee}{2e} = 2 = a$ sought.

Which number 2, to wit, a, will solve the Question, for it is less than 3; and $aa + 12$, that is, 16, is a Square, as was required.

Example 2.

Again, the same things being given as in *Example* 1. take some number between $3\frac{465}{1000}$ and $7\frac{182}{1000}$, (according to the latter limits in the Canon,) as 4, and call this n, *viz.* suppose . . $n = 4$

And then you will find . . . . . $\frac{nn-d}{2n} = \frac{1}{2} = a$

Which Fraction $\frac{1}{2}$, to wit, a, will solve the Question, for it is less than 3; and $aa + 12$, that is, $12\frac{1}{4}$, is a Square as was required.

QUEST. 10.

To find out a number, call it a, that $aa - 60$ may be greater than $5a$, but less than $8a$.

RESOLUTION.

1. Put Letters for the given numbers, as . . . . $b = 60$, $c = 5$, $d = 8$
2. Then the Question requires that $aa - b$ may be greater than ca, yet less than da; first then let us suppose $aa - b \sqsubset ca$
3. Thence it follows, by adding b to each part, that . . $aa \sqsubset ca + b$
4. And by subtracting ca from each part in the third step $aa - ca \sqsubset b$
5. And by adding the Square of half the known Coefficient c to each part of the fourth step, it follows that $aa - ca + \frac{1}{4}cc \sqsubset b + \frac{1}{4}cc$
6. And by extracting the square Root out of each part of the fifth step, it gives . . . . . $a - \frac{1}{2}c \sqsubset \surd:\overline{b+\frac{1}{4}cc}:$
7. Wherefore by adding $\frac{1}{2}c$ to each part of the sixth step, $a \sqsubset \frac{1}{2}c + \surd:\overline{b+\frac{1}{4}cc}:$
8. Again, let us suppose, as the Question also requires, that $aa - b \sqsupset da$
9. Whence by arguing in like manner as before from the second step to the seventh inclusive, saving that instead of $\sqsubset$ there, $\sqsupset$ is to be used in this latter argumentation, it will at length appear that . . . . . $a \sqsupset \frac{1}{2}d + \surd:\overline{b+\frac{1}{4}dd}:$

10. Thus, (by the ſeventh and eighth ſteps) limits are diſcovered, within which any number may be taken for the value of a the number ſought, *viz.*

$$a \sqsubset \tfrac{1}{2}c \dashv \sqrt{:b \dashv \tfrac{1}{4}cc:} \quad (\ 10\tfrac{6194102}{10000000},\ \&c.\ )$$
$$a \sqsupset \tfrac{1}{2}d \dashv \sqrt{:b \dashv \tfrac{1}{4}dd:} \quad (\ 12\tfrac{7177978}{10000000},\ \&c.\ )$$

As, for example, if $a = 12$, which is within the ſaid Limits, then $aa - 60 = 84$; alſo $5a = 60$, and $8a = 96$: But 84 (that is, $aa - 60$) is greater than 60, (that is, $5a$,) and leſs than 96, (that is, $8a$;) and therefore the number 12, (that is, a,) doth manifeſtly ſolve the Queſtion propoſed.

QUEST. 11.

To find out a number, call it a, that ſhall be greater than $10\frac{6194102}{10000000}$, but leſs than $12\frac{7177978}{10000000}$, and cauſe $aa - 60$ to be equal to ſome ſquare number.

RESOLUTION.

1. Put Letters for the given numbers, as, ... $\left\{ \begin{array}{l} b = 60, \\ f = 10\frac{6194102}{10000000}, \\ d = 12\frac{7177978}{10000000}. \end{array} \right.$

2. Then, (according to the import of the Queſtion,) $aa - b$ muſt be equal to ſome Square, but the ſide thereof muſt be ſo feigned that the value of a may be greater than f, but leſs than d; to which purpoſe, the ſaid ſide may be feigned to be $a - e$, or $e - a$, (which a and e do repreſent numbers unknown,) and then the Square of the ſaid $a - e$, or $e - a$ being equated to $aa - b$ above mentioned, gives this Equation, *viz.*

$$aa - b = aa - 2ae \dashv ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $\left\{ a = \frac{ee \dashv b}{2e} \right.$

4. But according to the Queſtion, ... $\rangle\ a \sqsubset f$

5. Therefore from the third and fourth ſteps, ... $\left\{ \frac{ee \dashv b}{2e} \sqsubset f \right.$

6. And by multiplying each part of the fifth ſtep by $2e$, it follows that . . . . . $\left\{ ee \dashv b \sqsubset 2fe \right.$

7. And by ſubtracting b from each part of the ſixth ſtep, . . $\rangle\ ee \sqsubset 2fe - b$

8. And by equal ſubtraction of $2fe$ from each part of the ſeventh ſtep, ... $\left\{ ee - 2fe \sqsubset -b \right.$

9. And by adding ff, that is, the Square of half the known Coefficient $2f$ in the eighth ſtep, to each part, it follows that $\left\{ ee - 2fe \dashv ff \sqsubset ff - b \right.$

10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $\left\{ e - f \sqsubset \sqrt{:ff - b:} \right.$

11. Wherefore by adding f to each part of the tenth ſtep, it's evident that . . . . . $\left\{ e \sqsubset f \dashv \sqrt{:ff - b:} \right.$

12. Again, becauſe $f - e$, (as well as $e - f$,) may be the ſide of the Square $ee - 2fe \dashv ff$ in the firſt part of the ninth ſtep, it thence follows, that . . . . . $\left\{ f - e \sqsubset \sqrt{:ff - b:} \right.$

13. And by adding e to each part of the twelfth ſtep, . . . $\rangle\ f \sqsubset e \dashv \sqrt{:ff - b:}$

14. And by ſubtracting $\sqrt{:ff - b:}$ from each part of the thirteenth ſtep, ... $\left\{ f - \sqrt{:ff - b:} \sqsubset e \right.$

15. Wherefore from the fourteenth ſtep, by comparing the latter part to the former, 'tis manifeſt that . . . . . $\left\{ e \sqsupset f - \sqrt{:ff - b:} \right.$

16. Again, becauſe the Queſtion requires . . . . . $\rangle\ a \sqsupset d$

17. It follows from the third and ſixteenth ſteps, that . . . $\left\{ \frac{ee \dashv b}{2e} \sqsupset d \right.$

18. Whence by arguing in like manner as before from the fifth ſtep to the fifteenth *incluſive*, ſaving that d is to be uſed here, inſtead of f there, and $\sqsupset$ inſtead of $\sqsubset$, it will at length appear that . . . . . $\left\{ \begin{array}{l} e \sqsupset d \dashv \sqrt{:dd - b:} \\ e \sqsubset d - \sqrt{:dd - b:} \end{array} \right.$

From the eleventh, eighteenth, fifteenth, firſt and third ſteps the following Canon ariſeth, which will find out innumerable Anſwers to the Queſtion propoſed.

CANON.

CANON.

19. Take any number (e) greater than $f \dashv \sqrt{:ff - b:}$: but leſs than $d \dashv \sqrt{:dd - b:}$: (that is, any number between $17\frac{9333}{10000}$, *&c.* and $22\frac{8047}{10000}$, *&c.*) or any number greater than $d - \sqrt{:dd - b:}$: but leſs than $f - \sqrt{:ff - b:}$: (that is, any number between $2\frac{6311}{10000}$, *&c.* and $3\frac{3455}{10000}$, *&c.*) Then $\frac{ee \dashv b}{2e}$ ſhall be equal to (a) the number ſought.

Examples.

Firſt, for the number e take 22 which is within the former limits in the Canon; then $\frac{ee \dashv b}{2e}$ gives $12\frac{4}{11}$ for the number a ſought by the Queſtion: For if from the Square of $12\frac{4}{11}$, to wit, $\frac{18496}{121}$, you ſubtract the given number 60, (or b,) the Remainder $\frac{11236}{121}$ is a Square whoſe ſide is $\frac{106}{11}$; and the ſaid $12\frac{4}{11}$ (that is, a,) is greater than $10\frac{61}{100}$, *&c.* but leſs than $12\frac{75}{100}$, *&c.* as the Queſtion requires.

Again, for the number e take 3 which is within the latter limits in the Canon; then $\frac{ee \dashv b}{2e}$ gives $11\frac{1}{2}$ (the number a,) which will likewiſe ſolve the Queſtion propoſed: For if from the Square of $11\frac{1}{2}$ you ſubtract 60, there will remain a Square, to wit, $\frac{289}{4}$, whoſe ſide is $\frac{17}{2}$; and the ſaid $11\frac{1}{2}$ (that is, a) is greater than $10\frac{61}{100}$, *&c.* but leſs than $12\frac{75}{100}$, *&c.* as was required.

QUEST. 12.

To find out a number, call it a, that ſhall be greater than $2\frac{1}{2}$, (a number given,) and cauſe $aa \dashv 4a \dashv 2$ to be equal to ſome ſquare number.

RESOLUTION.

1. Put letters for the given numbers, as, ... $\left\{ \begin{array}{l} b = 4, \\ d = 2\frac{1}{2}, \\ f = 2. \end{array} \right.$

2. Then the Queſtion requires that $aa \dashv ba \dashv f$ may make a ſquare number, but its ſide muſt be ſo feigned that the value of a may be greater than d. Now to cauſe thoſe effects, the ſaid ſide may be feigned to be $a - e$, or $e - a$, (which e and a do repreſent numbers yet unknown,) and then the Square of $a - e$ or $e - a$, that is, $aa - 2ae \dashv ee$, being equated to $aa \dashv ba \dashv f$, gives this Equation, *viz.*

$$aa \dashv ba \dashv f = aa - 2ae \dashv ee.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $\left\{ a = \frac{ee - f}{2e \dashv b} \right.$

4. And becauſe the Queſtion requires . . . . . $\rangle\ a \sqsubset d$

5. It follows from the third and fourth ſteps, that . . . $\left\{ \frac{ee - f}{2e \dashv b} \sqsubset d \right.$

6. And by multiplying each part of the fifth ſtep, by $2e \dashv b$, $\rangle\ ee - f \sqsubset 2de \dashv db$

7. And by adding f to each part of the ſixth ſtep, . . . $\rangle\ ee \sqsubset 2de \dashv db \dashv f$

8. And by ſubtracting $2de$ from each part of the ſeventh ſtep, ... $\left\{ ee - 2de \sqsubset db \dashv f \right.$

9. And by adding the Square of half the known Coefficient $2d$ in the eighth ſtep, to each part, it's manifeſt that $\left\{ ee - 2de \dashv dd \sqsubset db \dashv f \dashv dd \right.$

10. And by extracting the ſquare Root out of each part of the ninth ſtep, ... $\left\{ e - d \sqsubset \sqrt{:db \dashv f \dashv dd:} \right.$

11. Wherefore by adding d to each part of the tenth ſtep, it's evident that . . . . . $\left\{ e \sqsubset \sqrt{:db \dashv f \dashv dd:} \dashv d \right.$

12. And conſequently by reſolving the latter part of the eleventh ſtep into numbers, according to the Poſitions in the firſt, ... $\left\{ e \sqsubset 6\frac{112}{1000},\ \&c. \right.$

13. The third ſtep alſo ſhews that $ee \sqsubset f$, and conſequently . . . . . $\left\{ e \sqsubset \sqrt{f},\ (1\frac{414}{1000},\ \&c.) \right.$

But this latter limit for the chuſing of e is uſeleſs, for if e be greater than $6\frac{7\,7\,2}{1\,0\,0\,0}$, &c. as appears by the twelfth ſtep, it is evidently greater than $1\frac{4\,1\,4}{1\,0\,0\,0}$, &c.

14. Laſtly, from the eleventh, twelfth, third and firſt ſteps the following *Canon* ariſeth, which will find innumerable Anſwers to the Queſtion propoſed.

CANON 1.

Take any number greater than $\sqrt{:\overline{db}+f+\overline{dd}:}+d$, (*viz.* greater than $6\frac{7\,7\,2}{1\,0\,0\,0}$, &c.) and call the number taken e. Then $\frac{ee-f}{2e+b}$ ſhall be equal to the number a ſought.

15. But if it were deſired to find a number a that might be leſs than $2\frac{1}{2}$, and greater than nothing, and make $aa+4a+2$ to be a ſquare number, then the ſame Poſitions and Proceſs being made as before, ſaving that $\sqsupset$ is to be uſed inſtead of $\sqsubset$ from the fourth ſtep to the twelfth incluſive, at length there would ariſe this following

CANON 2.

Take any number (e) greater than $\sqrt{f}$, but leſs than $\sqrt{:\overline{db}+f+\overline{dd}:}+d$; (*viz.* any number between $1\frac{4\,1\,4}{1\,0\,0\,0}$, &c. and $6\frac{7\,7\,2}{1\,0\,0\,0}$, &c.) Then $\frac{ee-f}{2e+b}$ will give the number a ſought.

An Example of the firſt Canon.

For the number e take 8 which exceeds $6\frac{7\,7\,2}{1\,0\,0\,0}$, &c. as the firſt Canon doth direct. Then $\frac{ee-f}{2e+b}$ gives $3\frac{1}{10}$ for the number a ſought; for 'tis greater than $2\frac{1}{2}$ (or d,) and $aa+4a+2$ makes a Square, to wit, $\frac{24\,0\,1}{1\,0\,0}$, whoſe ſide is $\frac{49}{10}$, as was required.

Note, That $a+u$ might be feigned to be the ſide of the Square mentioned in the ſecond ſtep, and thence limits would be diſcovered to chuſe the number u, by which the number a would conſequently be made known; but I leave the ſearch of theſe latter limits as an exerciſe for the Learner.

QUEST. 13.

To find out a number, call it a; that ſhall be greater than 1, but leſs than 4, and make $121+45a-9aa$ to be a ſquare number.

RESOLUTION.

1. Firſt put Conſonants to repreſent the numbers given in the Queſtion, as, ... $\begin{cases} b = 1 \\ d = 4 \\ f = 11 \\ ff = 121 \\ g = 45 \\ h = 9 \end{cases}$

2. Then the Queſtion requires that $ff+ga-haa$ may make a ſquare number, whoſe ſide muſt be ſo feigned that the value of a may be greater than b, but leſs than d: To which purpoſe the ſaid ſide may be feigned to be $f+ea$, or $f-ua$; (where a, e, u do repreſent numbers unknown:) Firſt then let the ſaid ſide be feigned $f+ea$, and let its Square $ff+2fea+eeaa$ be equated to $ff+ga-haa$ above-mentioned, ſo this following Equation ariſeth, *viz.*

$$ff+2fea+eeaa = ff+ga-haa.$$

3. Which Equation, after due Reduction to find out the value of a, gives . . . . . $a = \frac{g-2fe}{h+ee}$

4. And becauſe the Queſtion requires . . . . . $a \sqsubset b$

5. It follows from the third and fourth ſteps, that $\frac{g-2fe}{h+ee} \sqsubset b$

6. And by multiplying each part in the fifth ſtep, by the Denominator $h+ee$, it follows, that . . $g-2fe \sqsubset bh+bee$

7. And by ſubtracting bh from each part in the ſixth ſtep, ... $g-2fe-bh \sqsubset bee$

8. And

8. And by adding $2fe$ to each part in the ſeventh ſtep, $g-bh \sqsubset bee+2fe$

9. And by dividing every quantity in the eighth ſtep by b, that bee may be freed from its Coefficient b, it follows that . . . . . $\frac{g-bh}{b} \sqsubset ee+\frac{2f}{b}e$

10. And by adding $\frac{ff}{bb}$, that is, the Square of half the Coefficient $\frac{2f}{b}$ to each part of the ninth ſtep, it gives . . . . . $\frac{ff+bg-bbh}{bb} \sqsubset ee+\frac{2f}{b}e+\frac{ff}{bb}$

11. And by extracting the ſquare Root out of each part of the tenth ſtep, ... $\sqrt{\frac{ff+bg-bbh}{bb}} \sqsubset e+\frac{f}{b}$

12. And by ſubtracting $\frac{f}{b}$ from each part of the eleventh ſtep, ... $\sqrt{:\frac{ff+bg-bbh}{bb}:}-\frac{f}{b} \sqsubset e$

13. Therefore from the twelfth ſtep, by comparing the latter part to the firſt, it's manifeſt that . . $e \sqsupset \sqrt{:\frac{ff+bg-bbh}{bb}:}-\frac{f}{b}$

14. Again, becauſe the Queſtion requires . . . $a \sqsupset d$

15. And by the third ſtep, ... $a = \frac{g-2fe}{h+ee}$

16. It follows from the fourteenth and fifteenth ſteps, that . . . . . $\frac{g-2fe}{h+ee} \sqsupset d$

17. Whence by arguing in like manner as before from the fifth ſtep to the thirteenth incluſive, it will at length appear that . . . . . $e \sqsubset \sqrt{:\frac{ff+dg-ddh}{dd}:}-\frac{f}{d}$

18. Again, let the ſide of the Square mentioned in the ſecond ſtep be feigned to be $f-ua$, and then the Square of $f-ua$ being equated to $ff+ga-haa$, (as the Queſtion requires) this following Equation ariſeth, *viz.*

$$ff-2fua+uaaa = ff+ga-haa.$$

19. Which Equation gives this value of a, viz. . . $a = \frac{g+2fu}{h+uu}$

20. But the value of a laſt mentioned muſt (as the Queſtion requires) be greater than b, and leſs than d; and if the Proceſs be continued from the laſt preceding ſtep, to find out limits for u in like manner as before for e from the third ſtep to the ſeventeenth incluſive, it will at length appear that

$$u \sqsupset \frac{f}{b}+\sqrt{\frac{ff+bg-bbh}{bb}},$$

$$u \sqsubset \frac{f}{d}+\sqrt{\frac{ff+dg-ddh}{dd}}.$$

Now after any number is taken for the value of e within the limits in the thirteenth and ſeventeenth ſteps, the number a required by the Queſtion will be diſcovered by the third and firſt ſteps. Or, after any number is taken for the value of u within the limits in the preceding twentieth ſtep, the number a ſought will be made known by the nineteenth and firſt ſteps. All which will be made manifeſt by the following Canon and Examples.

CANON.

21. Take any number leſs than $\sqrt{:\frac{ff+bg-bbh}{bb}:}-\frac{f}{b}$, but greater than $\sqrt{:\frac{ff+dg-ddh}{dd}:}-\frac{f}{d}$, (that is, any number between $1\frac{1\,2\,9}{1\,0\,0\,0}$, &c. and $\frac{1\,8\,2}{1\,0\,0\,0}$, &c.) and call the number taken e; then $\frac{g-2fe}{h+ee}$ ſhall be the number a ſought. Or take any number leſs than $\frac{f}{b}+\sqrt{\frac{ff+bg-bbh}{bb}}$, but greater than $\frac{f}{d}+\sqrt{\frac{ff+dg-ddh}{dd}}$, (that is, any number between $23\frac{1\,2\,9}{1\,0\,0\,0}$, &c. and $5\frac{8\,8\,2}{1\,0\,0\,0}$, &c.) and call the number taken u; then $\frac{g+2fu}{h+uu}$ will give the number a ſought.

Examples.

Examples.

Suppoſe $e = 1$, which is within the firſt limits in the Canon, then $\frac{g - 2fe}{h + ee} = \frac{23}{10} = a$ the number ſought: For $\frac{23}{10}$ is greater than 1, but leſs than 4; and if $a = \frac{23}{10}$, then $121 + 45a - 9aa = \frac{126482}{100}$, which is a Square, (for its ſide is $\frac{1333}{10}$) as the Queſtion requires.

Again, ſuppoſe $e = \frac{1}{2}$, (which is likewiſe within the firſt limits,) then $\frac{g - 2fe}{h + ee} = \frac{16}{15} = a$ the number ſought: For $\frac{16}{15}$ is greater than 1, but leſs than 4; and if $a = \frac{16}{15}$, then alſo $121 + 45a - 9aa$ makes a Square, to wit, $\frac{121121}{225}$, whoſe ſide is $\frac{1189}{15}$.

Again, ſuppoſe $u = 18$, which is within the latter limits in the Canon, then $\frac{g + 2fu}{h + uu} = \frac{42}{37} = a$ the number ſought: For if $a = \frac{42}{37}$, then $121 + 45a - 9aa$ makes a Square, to wit, $\frac{2256121}{1369}$, whoſe ſide is $\frac{1525}{37}$; and $\frac{42}{37}$ (or a) is greater than 1, but leſs than 4, as the Queſtion requires.

QUEST. 14. (Quæſt. 13. Lib. 2. *Diophant.*)

To find a number, that if it be ſubtracted firſt from 192, and then from 64, each Remainder may be a Square.

RESOLUTION.

1. For the number ſought put . . . . . } a
2. Which number muſt be ſuch, that each of theſe Quantities (or Remainders) may make a Square, *viz.* . . . } $192 - a = \square$, $64 - a = \square$
3. Now to reſolve that Duplicate Equality; firſt, (by *Canon* 2. *Queſt.* 7. of this *Book* 3.) find out two ſuch ſquare numbers that their difference may be equal to 128, that is, the difference of the two given numbers 192 and 64, or the difference between the two Algebraick Quantities $192 - a$ and $64 - a$, and that the greater Square may be leſs than 192, (the greater of the two numbers given in the Queſtion;) but two ſuch Squares are 144 and 16.
4. Then from either of theſe Equations, ... } $192 - a = 144$, $64 - a = 16$
5. One and the ſame value of a, that is, the number ſought will be diſcovered, *viz.* . . . . . } . . . $a = 48$
7. I ſay 48 will ſolve the Queſtion, as will be evident by

The Proof.

$192 - 48 = 144$, $64 - 48 = 16$, } which are Squares, as was required.

The premiſſes give this following

CANON.

8. Firſt, (by the ſecond Canon of the ſeventh Queſtion of this third Book,) find out two ſquare numbers in the ſame difference with the numbers given, and that the greater Square may be leſs than the greater number given; whence conſequently, (as will appear by the following *Theorem*,) the leſſer Square ſhall be leſs than the leſſer number given: Then from the greater number given ſubtract the greater Square, or from the leſſer number ſubtract the leſſer Square, ſo ſhall either of thoſe Remainders (for they are equal to one another) be the number ſought.

But the certainty of this Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by this following

THEOREM.

9. If two Square numbers have the ſame difference as two other numbers, and that the greater number exceeds the greater Square, then the leſſer number ſhall exceed the leſſer Square, and the exceſs of the greater number above the greater Square ſhall be equal to the exceſs of the leſſer number above the leſſer Square. To make this manifeſt, let dd and gg repreſent two ſquare numbers, whereof dd is the greater; alſo b the greater, and c the leſſer of two other numbers; Then, (according to the import of the *Theorem*,)

10. Suppoſe . . . . . } $b - c = dd - gg \sqsubset 0$
11. And . . . . . } $b \sqsubset dd$
12. Then by adding c to each part of the Equation in the tenth ſtep, it follows that . . . . . } $b = c + dd - gg$
13. Therefore by ſubtracting dd from each part of the laſt Equation, (which ſubtraction appears to be poſſible by the eleventh and tenth ſteps,) that which the Theorem affirms is manifeſt, *viz.* . . . . . } $b - dd = c - gg$
14. And conſequently either of thoſe two equal Exceſſes or Remainders which make the laſt Equation, ſhall be a number that will ſolve the Queſtion propoſed; for it is manifeſt, that if the firſt Exceſs $b - dd$ be ſubtracted from b, and the latter Exceſs $c - gg$ from c, the Remainders will be Squares, to wit, dd and gg.

To ſolve the foregoing 14th Queſtion *after another manner*, viz.

To find a number, that if it be ſubtracted firſt from 9, and then from 21, each Remainder may be a Square.

RESOLUTION.

1. It is evident, that if $9 - aa$ be ſubtracted from 9, there will remain a Square, to wit, aa, therefore for the number ſought put } $9 - aa$
2. And then if $9 - aa$ be ſubtracted from 21, the Remainder muſt likewiſe make a Square, therefore . . . . . } $aa + 12 = \square$
3. It remains to equate $aa + 12$ to ſome Square, whoſe ſide muſt be ſo feigned that the value of a may be leſs than 3, (for by the firſt ſtep $aa \sqsupset 9$, and conſequently $a \sqsupset 3$;) But to cauſe that effect, the ſaid ſide may be feigned to be either $a +$ any abſolute number between $1\frac{182}{1000}$, *&c.* and $3\frac{464}{1000}$, *&c.* or elſe $-a +$ any abſolute number between $3\frac{464}{1000}$, *&c.* and $7\frac{182}{1000}$, *&c.* (which limits are found out by the ninth Queſtion of this third Book;) let therefore the ſaid ſide be feigned $a + a$, and then by equating the Square of $a + 2$ to $aa + 12$ before-mentioned in the ſecond ſtep, this Equation ariſeth, *viz.*

$$aa + 4a + 4 = aa + 12.$$

4. Which Equation gives . . . . . } $a = 2$.

Therefore from the fourth and firſt ſteps the number ſought is 5; for if it be ſubtracted from 9 and 21 ſeverally, it leaves the Squares 4 and 16.

From this latter Reſolution of *Queſt.* 14. (reſpect being had to the foregoing ninth Queſtion,) a Canon may be deduced to find out innumerable Anſwers to the ſaid fourteenth Queſtion; but I leave it to the Learners exerciſe.

QUEST. 15. (Quæſt. 14. Lib. 2. *Diophant.*)

To find a number, from which if 27 and 15 (two numbers given) be ſeverally ſubtracted, each Remainder may be a Square.

RESOLUTION 1.

1. For the number ſought put . . . . . } a
2. Which number muſt be ſuch that each of theſe Quantities or Remainders may make a Square, *viz.* . . . . . } $a - 27 = \square$, $a - 15 = \square$
3. Now to reſolve that Duplicate Equality, find out (by *Canon* 2. of the preceding *Queſt.* 5.) two ſquare numbers whoſe difference may be 12, that is, the difference of the two given numbers 27 and 15; but here is no need of limiting either of the ſaid Squares: Suppoſe then the ſaid Squares are found 16 and 4,
4. Then from either of theſe Equations, ... } $a - 27 = 4$, $a - 15 = 16$
5. The number a ſought will be diſcovered, *viz.* . . . . . } . . . $a = 31$

I ſay 31 will ſolve the Queſtion; for if from 31 you ſubtract 27 and 15 ſeverally, the Remainders are Squares, to wit, 4 and 16.

From the premiſſes there ariſeth this following

CANON.

CANON.

6. First, (by the preceding fifth Queſtion,) find two ſquare numbers that ſhall have the ſame difference as the two numbers given; then add the leſſer Square to the greater number, or the greater Square to the leſſer number; ſo ſhall either of thoſe ſumms (for they are equal to one another) be the number ſought.

The truth of which Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by the following

THEOREM.

7. If two ſquare numbers, ſuppoſe *dd* the greater and *gg* the leſſer, have the ſame difference as two other numbers, ſuppoſe *b* the greater and *c* the leſſer; then the ſumm of the leſſer Square and the greater number ſhall be equal to the ſumm of the greater Square and the leſſer number: For,

By ſuppoſition, ... $b - c = dd - gg$
And by adding gg to each part, ... $b - c + gg = dd$
Wherefore by adding c to each part of the laſt Equation, that which the Theorem affirms is manifeſt, *viz.* . . . . $b + gg = c + dd$

8. And conſequently either of thoſe two equal ſumms in the laſt Equation ſhall be a number to ſolve the Queſtion propoſed: For it is evident, that if b be ſubtracted from the firſt ſumm $b + gg$, and c from the latter ſumm $c + dd$, the Remainders are Squares, to wit, gg and dd.

To ſolve the foregoing Queſt. 15. *after another manner*, viz.

To find a number, from which if 27 and 15 be ſeverally ſubtracted, each Remainder may be a Square.

RESOLUTION 2.

1. It is evident that if 27 be ſubtracted from $aa + 27$, the Remainder will be a Square, to wit, aa; therefore for the number ſought put . . . . . $aa + 27$
2. But then 15 being ſubtracted from the ſaid $aa + 27$, the Remainder muſt likewiſe be equal to a Square, therefore . . $aa + 12 = \square$
3. It remains to equate $aa + 12$ to ſome Square, whoſe ſide may be feigned either $a +$ any abſolute number leſs than $\sqrt{12}$, or $3\frac{464}{1000}$, *&c.* or elſe $-a +$ any abſolute number greater than the ſaid $3\frac{464}{1000}$, *&c.* (which limits are found out in like manner as in the foregoing *Queſt.* 9.) Let therefore the ſaid ſide be feigned $a + 3$, and then by equating the Square of $a + 3$, that is, $aa + 6a + 9$ to $aa + 12$, the value of a will thence be found $\frac{1}{2}$, and conſequently the number ſought, (which in the firſt ſtep was put $aa + 27$) ſhall be $27\frac{1}{4}$, which will ſolve the Queſtion: For if from $27\frac{1}{4}$ the given numbers 27 and 15 be ſeverally ſubtracted, the Remainders will be Squares, to wit, $\frac{1}{4}$ and $\frac{42}{4}$.

But if this ſecond manner of reſolving *Queſt.* 15. be formed by Literal *Algebra*, (like to the third manner of reſolving the preceding *Queſt.* 8.) there will ariſe this

CANON.

Take any number greater than the ſumm, or leſs than the difference of the ſquare Roots of the two numbers given; then divide the difference between the Square of the number taken and the difference of the given numbers by the double of the number taken; laſtly, to the Square of that Quotient add the greater of the numbers given, ſo ſhall the ſumm be the number ſought.

A third way of ſolving the preceding Queſt. 15.

Let the Poſitions in the firſt and ſecond ſteps of the preceding *Reſolut.* 2. be reſumed; then ſince $aa + 12$ muſt be equal to a Square, 'tis evident that 12 is the difference between that Square and aa; therefore by the preceding fifth Queſtion find two Squares whoſe difference may be 12; ſuch are 16 and 4, the leſſer of which ſhall be the value of aa; therefore $aa + 27$ which was put for the number ſought will be found 31, as before in the firſt Reſolution of this Queſtion.

QUEST. 16.

QUEST. 16.

To find a number, that if 12 be added to it, and 8 ſubtracted from the ſame, as well the Summ as the Remainder may be a Square.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then each of theſe Quantities muſt make a Square, *viz.* . . $a + 12 = \square$; $a - 8 = \square$
3. Now to reſolve that Duplicate Equality, firſt, ſubtract $a - 8$ from $a + 12$, and the Remainder is 20; this is equal as well to the ſumm of the given numbers as to the difference of the two Squares ſought: Then (by the ſecond *Canon* of the fifth *Queſtion* of this third *Book*) find two Squares whoſe difference ſhall be 20; ſuch are 36 and 16.
4. Then from either of theſe Equations . . . . . $a + 12 = 36$; $a - 8 = 16$
5. The number a ſought will be made known, *viz.* . . . . $a = 24$

Which number found out, to wit, 24, will ſolve the Queſtion; for if it be increaſed with 12, and leſſened by 8, the Summ and Remainder are Squares, to wit, 36 and 16.

The ſubſtance of the Reſolution is contain'd in this following

CANON.

6. First, (by the ſecond *Canon* of the fifth *Queſtion* aforegoing) find out two ſquare numbers whoſe difference ſhall be equal to the ſumm of the two numbers given; then ſubtract the number given to be added (whether it be the greater or the leſſer of thoſe given) from the greater Square, or add the number given to be ſubtracted to the leſſer Square; ſo as well the Remainder as the Summ (for they are equal to one another) ſhall be the number ſought.

The certainty of this Canon, and conſequently of the Reſolution of the Duplicate Equality in the Queſtion, will be evident by this following

THEOREM.

7. If there be two ſquare numbers, ſuppoſe *dd* the greater, and *gg* the leſſer, whoſe difference is equal to the ſumm of two other numbers, *b* and *c*; then the exceſs of the greater Square above either of the two numbers, ſhall be equal to the ſumm of the leſſer Square and the other number.
8. For, by ſuppoſition . . . . . $dd - gg = b + c$
9. Whence by adding gg to each part, it follows that . . . $dd = gg + b + c$
10. Therefore by ſubtracting b from each part of the laſt Equation, this ariſeth, *viz.* . . . . . $dd - b = gg + c$
11. Likewiſe, by ſubtracting c from each part of the Equation in the ninth ſtep, there remains . . . . . $dd - c = gg + b$

Wherefore from the two laſt Equations, the truth of the Theorem, and conſequently of the Canon is manifeſt.

QUEST. 17.

To find a number, that if it be added to, and ſubtracted from a given ſquare number, ſuppoſe 4, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Which number muſt be ſuch, that if it be added to and ſubtracted from 4, as well the Summ as the Remainder may make a Square, *viz.* . . . . . $4 + a = \square$; $4 - a = \square$
3. To reſolve that Duplicate Equality, firſt, (after the manner of *Example* 3. *Canon* 1. *Reſolut.* 2. *Queſt.* 2. of this *Book*,) divide 8 the double of the given Square 4 into two unequal Squares, $\frac{256}{25}$ and $\frac{4}{25}$.

4. Then from either of these Equations, ... $\left\{\begin{array}{l} 4+a = \frac{196}{25} \\ 4-a = \frac{4}{25} \end{array}\right.$

5. The number sought will be discovered, *viz.* . . . . . $a = \frac{24}{25}$

Which number $\frac{24}{25}$ will solve the Question proposed; for if it be added to, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{196}{25}$ and $\frac{4}{25}$, whose sides are $\frac{14}{5}$ and $\frac{2}{5}$. By the like Operation (the substance whereof is contained in the following Canon,) you may find out innumerable Answers to this 17th Question.

CANON.

6. Divide the double of the given Square into two unequal Squares, (by the preceding *Quest.* 2.) then from the greater of the two Squares found out subtract the given Square, or from this subtract the lesser of those two, so shall either of the Remainders, (for they are equal to one another) be the number sought.

But the certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in this 17th Question, will be evident by the following

THEOREM.

7. If two square numbers, suppose *cc* the greater, and *dd* the lesser, be equal to the double of a square number, as $bb+bb$, or $2bb$; then the excess of the greater of those unequal Squares above one of the equal Squares shall be equal to the excess of the other of the equal Squares above the lesser of the unequal Squares.

8. For by supposition . . . . . $cc+dd = bb+bb$
9. Also by supposition . . . . . $cc \sqsubset dd$
10. Therefore . . . . . $cc \sqsubset bb$
11. And . . . . . $dd \sqsupset bb$
12. And by subtraction of *bb* from each part of the Equation in the eighth step, ... $cc+dd-bb = bb$
13. Wherefore by subtracting *dd* from each part of the last Equation, (which subtraction the 10th and 11th steps do shew to be possible,) that which the Theorem asserts is manifest, *viz.* $cc-bb = bb-dd$
14. And consequently either of the Excesses (or Remainders) which make the last preceding Equation shall be the number *a* sought: For if the first Excess $cc-bb$ be added to *bb*, and the latter Excess $bb-dd$ subtracted from *bb*, the Summ and Remainder are Squares, to wit, *cc* and *dd*.

QUEST. 18.

To find a number, that if a given Square 9 be added to that number, and from another given Square 4 the same number sought be subtracted, the Summ and Remainder may be Squares.

RESOLUTION.

1. For the number sought put . . . . . a
2. Then the Question requires that . . . . . $\left\{\begin{array}{l} 9+a = \square \\ 4-a = \square \end{array}\right.$
3. To resolve that Duplicate Equality, first (by the preceding *Quest.* 4.) divide 13 the summ of the given Squares 9 and 4 into two such other Squares that one of these found may exceed 9 the Square given to be added; but two such Squares are $\frac{324}{25}$ and $\frac{1}{25}$, whose summ is 13, and the greater of them exceeds 9.
4. Then from either of these Equations . . . . . $\left\{\begin{array}{l} 9+a = \frac{324}{25} \\ 4-a = \frac{1}{25} \end{array}\right.$
5. The number (*a*) sought is discovered, *viz.* . . . . . $a = \frac{22}{25}$

Which number found out, to wit, $\frac{22}{25}$, will solve the Question; for if it be added to 9, and subtracted from 4, the Summ and Remainder are Squares, to wit, $\frac{324}{25}$ and $\frac{1}{25}$, whose sides are $\frac{18}{5}$ and $\frac{1}{5}$. By the like Operation (the substance whereof is expresst in the following Canon,) you may find out innumerable Answers to this 18th Question, because (by *Quest.* 4. of this *Book*) the summ of two Squares may be divided into as many pairs of Squares as you please, such, that one of each pair shall consist within given limits.

CANON.

CANON.

7. First, (by *Quest.* 4. of this *Book*,) divide the summ of the two Squares given into two such Squares, that the greater of these found out may exceed the Square given to be added; then from the greater of the two Squares found out subtract the Square given to be added; or, from the other Square given subtract the other Square found out; so shall either of the Remainders (for they are equal to one another) be the number sought.

The certainty of this Canon, and consequently of the Resolution of the Duplicate Equality in the Question proposed will be manifest by this following

THEOREM.

8. If the summ of two square numbers, suppose *bb* the greater, and *cc* the lesser, be found equal to the summ of two other unequal Squares, *dd* and *ff*, and that the greater of the two former exceeds either of the two latter, then the other of the two latter shall exceed the lesser of the two former, and one excess shall be equal to the other. For,

9. By supposition . . . . . $bb+cc = dd+ff$
10. By supposition also . . . . . $bb \sqsubset dd$
11. Therefore . . . . . $cc \sqsupset ff$
12. And by subtracting *dd* from each part of the Equation in the ninth step, this ariseth, *viz.* . . . . . $bb+cc-dd = ff$
13. Wherefore by subtracting *cc* from each part of the last Equation, (which subtraction the tenth and eleventh steps do shew to be possible) that which the Theorem affirms is manifest, *viz.* . . . . . $bb-dd = ff-cc$
14. And consequently the truth of the Canon and Resolution of the Duplicate Equality in this 18th Question is evident.

QUEST. 19.

To find a square number, that if it be increased or lessened by its side, may make a Square.

RESOLUTION.

1. For the side of the Square sought put . . . . . a
2. Therefore the Square it self is . . . . . aa
3. Then the Question requires . . . . . $\left\{\begin{array}{l} aa+a = \square \\ aa-a = \square \end{array}\right.$
4. Which Duplicate Equality differs but little from that in the foregoing seventeenth Question, and may be resolved thus: First, (after the manner of *Example* 3. *Canon* 1. *Resolut.* 2. *Quest.* 2. of this *Book*,) divide 2, which is compos'd of two Squares, 1 and 1, suppos'd to be prefixt to *aa* and *aa* in the Duplicate Equality in the third step, into two unequal Squares, $\frac{49}{25}$ and $\frac{1}{25}$; then multiply each of these by *aa*, and equate the greater Product $\frac{49}{25}aa$ to $aa+a$, or the lesser Product $\frac{1}{25}aa$ to $aa-a$, so from either of those Equations one and the same value of *a* will be discovered, *viz.* $a = \frac{24}{24}$, which is the side of the Square sought; for if $\frac{24}{24}$ be added to and subtracted from its Square $\frac{576}{576}$, the Summ and Remainder are Squares, to wit, $\frac{1225}{576}$ and $\frac{625}{576}$, whose sides are $\frac{35}{24}$ and $\frac{25}{24}$. It is also easie to be perceived that this Question is capable of innumerable Answers.

QUEST. 20.

To find two numbers in a given Reason, suppose the greater to the lesser as 2 to 1; and that the Square of the summ of the two numbers being added to each of them may make square numbers.

RESOLUTION.

1. For the lesser number sought put . . . . . a
2. Then the greater, to the end it may be to the lesser as 2 to 1, shall be . . . . . $2a$
3. Therefore the summ of the two numbers sought is . . . . . $3a$
4. And the Square of their summ is . . . . . $9aa$

5. To which Square if the two numbers $2a$ and a be severally added, each ſumm muſt be a Square, therefore . . . . $\left\{\begin{array}{l} 9aa+2a=\square \\ 9aa+a=\square \end{array}\right.$

6. Which Duplicate Equality, (according to *Diophantus*'s Method, before explain'd and demonſtrated in the Obſervations upon the firſt manner of ſolving the eighth Queſtion of this Book,) may be reſolved thus; *viz.* Firſt, ſubtract $9aa+a$ from $9aa+2a$, and the Remainder a is the difference of the two Squares that are to be equated to thoſe Algebraick Quantities; then find two Squares whoſe difference may be equal to the ſaid difference a, but with this Caution, that in each of thoſe Squares there may be found $9aa$, to the end that when the greater Square is equated to $9aa+2a$, or the leſſer to $9aa+a$, the ſaid $9aa$, after due Reduction, may vaniſh, and an Equation remain between ſome number of a and ſome known Rational number, whence the value of a will be expreſſible by ſome known number either Affirmative or Negative. Now to find out two ſuch Squares, let two numbers be taken, ſuch, that (according to *Canon* 2. *Queſt.* 5. of this *Book*,) the Product of their Multiplication may make a, and that the half of their ſumm may conſiſt of $3a+$ ſome abſolute number, and the half of their difference of $3a-$ ſome abſolute number, (for then it will follow that as well in the Square of the ſaid half ſumm, as in the Square of the ſaid half difference there will be found $9aa$;) whence we may inferr, that the ſumm of the ſaid two numbers muſt conſiſt of $a+$ ſome abſolute number, and their difference of $6a-$ ſome abſolute number; to which purpoſe, divide 1 by 6, (1 becauſe the difference is $1a$, and 6 becauſe it is the double of the ſquare Root of 9 which is prefixt to aa,) ſo the Quotient is $\frac{1}{6}$, and conſequently $6a$ multiplied by $\frac{1}{6}$ makes a; whence 'tis evident that the only two numbers fit for the purpoſe aforeſaid are $6a$ and $\frac{1}{6}$, whoſe half ſumm is $3a+\frac{1}{12}$, and half difference $3a-\frac{1}{12}$, (or $-3a+\frac{1}{12}$,) therefore the Squares of the ſaid half ſumm and half difference are $9aa+\frac{1}{2}a+\frac{1}{144}$ and $9aa-\frac{1}{2}a+\frac{1}{144}$; now let the greater of thoſe Squares be equated to $9aa+2a$, and the leſſer to $9aa+a$, ſo theſe two following Equations will ariſe,

$$\textit{viz.}\ \left\{\begin{array}{l} 9aa+2a=9aa+\frac{1}{2}a+\frac{1}{144}; \\ 9aa+a=9aa-\frac{1}{2}a+\frac{1}{144}. \end{array}\right.$$

7. Then from either of thoſe Equations, after due Reduction, the value of a will be found to be $\frac{1}{216}$, and conſequently, (from the firſt and ſecond ſteps,) the numbers ſought are $\frac{1}{216}$ and $\frac{3}{216}$, which will ſolve the Queſtion propoſed: For firſt, the greater hath ſuch proportion to the leſſer as 2 to 1; and if the Square of the ſumm of $\frac{1}{216}$ and $\frac{2}{216}$ be added to them ſeverally, the two ſumms will be Squares, to wit $\frac{49}{51984}$ and $\frac{121}{51984}$, whoſe ſides are $\frac{7}{228}$ and $\frac{11}{228}$.

8. In like manner, to reſolve this Duplicate Equality, *viz.*

$$\text{If}\ \left\{\begin{array}{l} 4aa+3a-1=\square \\ 4aa-a-1=\square \end{array}\right\}\ \text{what is}\ a=?$$

First, I find the difference of thoſe two Algebraick Quantities to be $4a$; then I ſearch out two Quantities that being mutually multiplied may make $4a$, and that as well in half their ſumm as in half their difference there may be found $2a$, (that is, the ſquare Root of $4aa$;) ſo by working as before is directed, I find $4a$ and 1 to be the only two Quantities agreeing with thoſe conditions: Then the Square of half the ſumm of $4a$ and 1, *viz.* the Square of $2a+\frac{1}{2}$ being equated to $4aa+3a-1$ will give $a=\frac{1}{4}$; or, the Square of half the difference of $4a$ and 1, *viz.* the Square of $2a-\frac{1}{2}$ being equated to $4aa-a-1$, gives $a=\frac{1}{4}$, as before.

QUEST. 21.

To find two numbers in a given Reaſon, ſuppoſe the greater to the leſs as 3 to 2, and that the ſumm of the numbers being added to each of their Squares, may make Squares.

RESOLUTION.

1. For the leſſer number put . . . . . $2a$
2. Then the greater (to the end that both numbers may be in the Reaſon preſcribed,) ſhall be . . . . . $3a$
3. Therefore their ſumm is . . . . . $5a$

4. Which

4. Which ſumm added to the Square of each of the ſaid two numbers $2a$ and $3a$, muſt (as the Queſtion requires) make a Square, therefore . . . . . $\left\{\begin{array}{l} 4aa+5a=\square \\ 9aa+5a=\square \end{array}\right.$

5. Now in order to reſolve that Duplicate equality, it muſt firſt be reduced to another, wherein there may be equal ſquare numbers prefixt to aa, which may be done thus; Divide 9 the greater of the two Squares that are prefixt to aa by the leſſer 4, and then by the Quotient $\frac{9}{4}$ multiply that Algebraick Quantity where the ſaid Diviſor 4 is prefixt to aa, viz. $4aa+5a$ by $\frac{9}{4}$, and it produceth $9aa+\frac{45}{4}a$ to be equated to a Square; ſo now inſtead of the Duplicate equality in the fourth ſtep, this ariſeth,

$$\textit{viz.}\ \left\{\begin{array}{l} 9aa+\frac{45}{4}a=\square \\ 9aa+5a=\square. \end{array}\right.$$

6. In which Duplicate equality laſt above expreſt, there are equal Squares, to wit, 9 and 9 prefixt to aa, and therefore it may be reſolved after the manner before ſhewn in the 20th Queſtion, and when the value of a is diſcovered in this latter duplicate equality, it will neceſſarily conſtitute the former duplicate equality in the fourth ſtep; for as a Square multiplied by a Square produceth a Square, ſo converſly, a Square divided by a Square gives the Quotient a Square. In order then to reſolve the ſaid duplicate equality in the fifth ſtep, ſubtract $9aa+5a$ from $9aa+\frac{45}{4}a$, and the Remainder is $\frac{45}{4}a$, this muſt be eſteem'd the Product made by the mutual multiplication of two quantities to be taken with ſuch Caution, that as well in half their ſumm as in half their difference there may be found $3a$, (becauſe the ſquare Root of $9aa$ in each of the two quantities to be equated to a Square is $3a$;) ſo by conſidering well that Caution, and what hath been ſaid to the like purpoſe in the ſixth ſtep of the Reſolution of the foregoing 20th Queſtion, you will find that $6a$ and $\frac{45}{24}$ are the only two numbers, that being mutually multiplied make $\frac{45}{4}a$, and have $3a$ as well in half their ſumm as in half their difference: Therefore let the Square of half the ſumm of the ſaid $6a$ and $\frac{45}{24}$, *viz.* the Square $9aa+\frac{45}{24}a+\frac{2025}{2304}$ be equated to $9aa+\frac{45}{4}a$; or, let the Square of half the difference of the ſaid $6a$ and $\frac{45}{24}$, *viz.* the Square $9aa-\frac{45}{24}a+\frac{2025}{2304}$, be equated to $9aa+5a$; ſo from either of thoſe Equations, one and the ſame value of a will be diſcovered, *viz.* $a=\frac{135}{3744}$, and conſequently $2a$ and $3a$, which in the firſt and ſecond ſteps were put for the numbers ſought, will be diſcovered to be $\frac{135}{1872}$ and $\frac{135}{1248}$, which will ſolve the Queſtion: For, firſt, the greater is in proportion to the leſſer as 3 to 2; and if their ſumm be added to their Squares ſeverally, the two ſumms made by ſuch addition will be Squares, to wit, $\frac{4005225}{3504384}$ and $\frac{3375225}{1557504}$, whoſe ſides are $\frac{2115}{1872}$ and $\frac{1575}{1248}$.

7. But the Duplicate equality in the fourth ſtep may be reduced to another wherein there ſhall be equal ſquare numbers prefixt to aa by this following Operation, which differs from that in the ſixth ſtep, *viz.* becauſe 9 times 4 makes the ſame Product as 4 times 9, and becauſe a Square multiplied by a Square produceth a Square, let $4aa+5a$ (in the fourth ſtep) be multiplied by 9, and $9aa+5a$ by 4; ſo there will neceſſarily be found $9aa$ in each Product, and this following duplicate equality comes now to be reſolved inſtead of that in the fourth ſtep,

$$\textit{viz.}\ \left\{\begin{array}{l} 36aa+45a=\square \\ 36aa+20a=\square. \end{array}\right.$$

8. Laſtly, this Duplicate equality having equal ſquare numbers prefixt to aa, may be reſolved like that in the preceding fifth ſtep, and at length the value of a will be found $\frac{135}{3744}$, as before.

QUEST. 22.

To find two ſuch ſquare numbers, that if to the Product of their multiplication a given number (d) be added, the ſumm may be a Square.

RESOLUTION.

1. For one of the Squares ſought take any known ſquare number which may be repreſented by . . . . . bb
2. And for the other Square ſought put . . . . . aa
3. Then the Product of their multiplication is . . . . . $bbaa$
4. To which Product the given number d being added, the ſumm is $bbaa+d$

5. Which

5. Which ſumm muſt be equal to a Square, the ſide whereof may be feigned to be ba — any known number greater than $\sqrt{d}$, ſuppoſe $ba - c$; then the Square of $ba - c$, that is, $bbaa - 2bca + cc$ being equated to $bbaa + d$, this Equation ariſeth, *viz.*

$$bbaa + d = bbaa - 2bca + cc.$$

6. Whence, after due Reduction, ... $\rightarrow a = \frac{cc - d}{2bc}$

From the premiſſes ariſeth this following

CANON.

For one of the Squares ſought take any ſquare number; then from any ſquare number ſubtract the given number, and divide the Remainder by the double of the Product made by the multiplication of the ſides of thoſe two Squares; ſo the Quotient ſhall be the ſide of the other Square ſought.

An Examp'e in Numbers.

Let the number given be . . . . . $\rightarrow 12 = d$
For one of the Squares ſought take any ſquare number, as . . . $\rightarrow 4 = bb$
Take alſo ſome other ſquare number greater than 12, (or d,) as $\rightarrow 36 = cc$
Then (by the Canon) the ſide of the other Square ſought ſhall be $\rightarrow 1 = \frac{cc - d}{2bc}$

I ſay, 4 and 1 are two Squares, which will ſolve the Queſtion when the number given is 12; for if to 4, the Product of 4 and 1, you add 12, the ſumm makes a Square, to wit, 16. By the like Operation you may find out innumerable Anſwers to the Queſtion without varying the given number; and 'tis eaſie alſo to find out other Canons to ſolve the ſame.

QUEST. 23. (Quæſt. 15. Lib. 2. *Diophant.*)

To divide a given number (b) into two parts, and to find a ſquare number, which if it be increaſed with each of thoſe parts, may make a Square.

RESOLUTION.

1. Take two ſuch numbers, that the ſumm of their Squares may be leſs than the giuen number b, ſuppoſe theſe, . . } c and d
2. Then for the ſide of the Square ſought put . . . . } a
3. The Square thereof is . . . . . } aa
4. To the ſide a add ſeverally c and d, and aſſume the ſumms to be the ſides of two Squares, ſo the firſt ſide will be } $a + c$
5. And the other ſide . . . . . } $a + d$
6. The Square of $a + c$ (in the fourth ſtep) is . . . . } $aa + 2ca + cc$
7. The Square of $a + d$ (in the fifth ſtep) is . . . . } $aa + 2da + dd$
8. Then for one of the deſired parts of (b) put $2ca + cc$, (for it's evident, that if the Square aa in the third ſtep be increaſed with $2ca + cc$, it makes the Square $aa + 2ca + cc$ in the ſixth ſtep,) . . . . . } $2ca + cc$
9. And for the other part of b put $2da + dd$, for this added to aa makes a Square, to wit, that in the ſeventh ſtep . . } $2da + dd$
10. But the ſumm of the parts in the eighth and ninth ſteps muſt be equal to b, therefore . . . . . } $2ca + cc + 2da + dd = b$
11. Which Equation, after due Reduction, gives . . . $\rightarrow a = \frac{b - cc - dd}{2c + 2d}$

The premiſſes well examined, afford this following

CANON.

12. Take two numbers, with this Caution, that the ſumm of their Squares may be leſs than the number given to be divided; then ſubtract the ſumm of thoſe Squares from the given number, and divide the Remainder by the double ſumm of the numbers taken, ſo the Quotient ſhall be the ſide of the Square ſought; then multiply the double of the ſaid ſide ſeverally by the numbers firſt taken, and to the Products add ſeverally the reſpective Squares of the numbers taken; ſo the ſumms made by thoſe additions ſhall be the deſired parts of the number given.

An

An Example in Numbers.

Let the number given to be divided be . . . . . $\rightarrow 33 = b$
Let two numbers be taken, ſuch, that the ſumm of their Squares may be leſs than 33, as . . . . . } $2 = c$ / $3 = d$
Then (by the Canon) the ſide of the Square ſought, (which ſide is repreſented by a in the Reſolution,) ſhall be . . . } $2 = \frac{b - cc - dd}{2c + 2d}$
Alſo, one of the deſired parts of 33 is . . . . . $\rightarrow 12 = 2ca + cc$
And the other is . . . . . $\rightarrow 21 = 2da + dd$

The Proof.

$4 = aa$ the Square ſought;
$12 + 21 = 33$ the number given;
$4 + 12 = 16$, $4 + 21 = 25$ } which are Squares; as was required.

QUEST. 24.

To find two ſuch numbers, that their ſumm may make a Square: Alſo, that each number being added to the Square of the other number, may make a Square.

RESOLUTION.

1. For the ſumm of the two numbers ſought aſſume ſome Square, as $\rightarrow ee$
2. And for the firſt of the two deſired numbers put . . . . . $\rightarrow a$
3. Therefore the other ſhall be . . . . . $\rightarrow ee - a$
4. Which added to the Square of the firſt number a, makes the ſumm $\rightarrow aa - a + ee$
5. Which ſumm laſt expreſt, the Queſtion requires to be a Square, and ſuch it will be, if we ſuppoſe $e = \frac{1}{2}$; for then the ſaid $aa - a + ee$ (in the fourth ſtep) will be equal to $aa - a + \frac{1}{4}$, which is the Square of $a - \frac{1}{2}$, or $\frac{1}{2} - a$: Now therefore let the Reſolution be renewed thus,
6. For the ſumm of the two numbers ſought, (inſtead of ee) put $\rightarrow \frac{1}{4}$
7. And for the firſt number put (as before) . . . . . $\rightarrow a$
8. Therefore the other ſhall be . . . . . $\rightarrow \frac{1}{4} - a$
9. Which latter number added to the Square of the firſt, makes the ſumm $\rightarrow aa - a + \frac{1}{4}$

But the ſumm laſt expreſt is manifeſtly a Square, whoſe ſide is $a - \frac{1}{2}$, or $\frac{1}{2} - a$; therefore two of the conditions in the Queſtion are ſatisfied.

10. Again, if to the Square of the ſecond number $\frac{1}{4} - a$, *viz.* to $aa - \frac{1}{2}a + \frac{1}{16}$, the firſt number a be added, the ſumm is $aa + \frac{1}{2}a + \frac{1}{16}$, which the Queſtion likewiſe requires to be a Square, and ſo it is, for 'tis the Square of $a + \frac{1}{4}$; but if the laſt mentioned ſumm had not happened to have been a Square, then a Square might have been feigned equal to it, according to the method in divers preceding Queſtions of this *Book* 3.

The premiſſes diſcover this following

THEOREM.

11. If the Fraction $\frac{1}{4}$ be divided into any two parts, each part increaſed with the Square of the other part ſhall make a Square.

By the help therefore of this Theorem, innumerable Anſwers to the Queſtion propoſed may be found out.

An Example.

Let two Fractions be taken whoſe ſumm makes $\frac{1}{4}$, as $\frac{1}{6}$ and $\frac{1}{12}$; I ſay theſe will ſolve the Queſtion: For firſt, their ſumm is a Square; ſecondly, the firſt Fraction $\frac{1}{6}$ increaſed with $\frac{1}{144}$, (the Square of the latter Fraction $\frac{1}{12}$) makes the Square $\frac{25}{144}$; likewiſe the latter Fraction $\frac{1}{12}$ increaſed with $\frac{1}{36}$, the Square of the firſt Fraction $\frac{1}{6}$) makes a Square, to wit, $\frac{1}{9}$.

Moreover, by the help of the ſaid Theorem, this following Queſtion may be ſolved, *viz.*

12. To find two numbers in a given Reaſon, ſuppoſe the greater to the leſſer as 3 to 2; and that each number being added to the Square of the other number, may make a Square.

Divide the Fraction $\frac{1}{4}$ into two ſuch parts, that the greater may be to the leſſer as 3 to 2, ſo you will find $\frac{3}{20}$ and $\frac{1}{10}$, which will ſolve the Queſtion: For firſt, by

by Conſtruction they are in the Reaſon preſcribed; ſecondly, $\frac{3}{10}$ with $\frac{1}{100}$ (the Square of $\frac{1}{10}$) makes the Square $\frac{4}{25}$; and laſtly, $\frac{1}{10}$ with $\frac{2}{400}$ (the Square of $\frac{1}{20}$) makes the Square $\frac{42}{400}$.

13. Another manner of ſolving the Queſtion laſt propoſed may be this, *viz.*

For the two numbers ſought in the given Reaſon of 3 to 2, put } $3a$ and $2a$

Then, ſince the Queſtion requires, that each number being added to the Square of the other number may make a Square, this Duplicate equality ariſeth, *viz.* . . . . . } $9aa + 2a = \square$; $4aa + 3a = \square$

Which Duplicate equality may be reſolved like that in the foregoing twenty-firſt Queſtion; ſo the value of a will be found $\frac{161}{10080}$, and conſequently $3a$ and $2a$ give $\frac{120}{10080}$ and $\frac{77}{10080}$ for the numbers ſought: For firſt, they are in proportion as 3 to 2; ſecondly, if to the Square of the firſt you add the ſecond number, it makes the Square $\frac{25812249}{101606400}$, whoſe ſide is $\frac{5077}{10080}$; laſtly, if to the Square of the ſecond number you add the firſt, it makes the Square $\frac{11412249}{101606400}$, whoſe ſide is $\frac{3383}{10080}$.

QUEST. 25. (Quæſt. 29. Lib. 2. *Diophant.*)

To find two ſuch ſquare numbers, that each being increaſed with the Product of their multiplication may make a Square.

RESOLUTION.

1. Firſt, (by the preceding *Queſt.* 5.) find out two Squares that may differ by unity, ſuch are $\frac{9}{16}$ and $\frac{25}{16}$, and take the leſſer for one of the Squares ſought, as . . . . . } $\frac{9}{16}$
2. Then for the other Square ſought aſſume . . . . . aa
3. Therefore the Product of their multiplication is . . . . . $\frac{9}{16}aa$

Which Product $\frac{9}{16}aa$ being added to the ſecond Square aa doth manifeſtly make a Square, to wit, $\frac{25}{16}aa$; but if the ſaid Product $\frac{9}{16}aa$ be added to the firſt Square $\frac{9}{16}$ it muſt alſo make a Square, therefore $\frac{9}{16}aa + \frac{9}{16}$ not being a Square, muſt be equated to a Square, the ſide whereof may be variouſly feigned, let it be $\frac{3}{4}a - 1$; then the Square of $\frac{3}{4}a - 1$ being equated to $\frac{9}{16}aa + \frac{9}{16}$ gives $a = \frac{24}{7}$, and conſequently $aa = \frac{576}{49}$ is the ſecond Square ſought. I ſay, $\frac{9}{16}$ and $\frac{576}{49}$ are two Squares, which will ſolve the Queſtion, as will appear by

The Proof.

The two Squares found out are	$\frac{9}{16}$	and $\frac{576}{49}$
The Product of their multiplication is	$\frac{9}{16} \cdot \frac{576}{49} = \frac{324}{49}$	
Which Product added ſeverally to $\frac{9}{16}$ and $\frac{576}{49}$, makes theſe Squares, ...	$\frac{1521}{784}$	and $\frac{18225}{784}$
The ſides of which laſt mentioned Squares are	$\frac{39}{28}$	and $\frac{135}{28}$

After the ſame manner you may eaſily find out two ſuch Squares, that if the Product of their multiplication be ſubtracted from them ſeverally, the Remainders may be Squares.

QUEST. 26. (Quæſt. 35. Lib. 2. *Diophant.*)

To find three ſuch numbers, that the Square of every one of them being added to the ſumm of the three numbers, may make a Square.

RESOLUTION.

1. If to the Product of the multiplication of any two unequal numbers the Square of half their difference be added, the ſumm ſhall be a Square, to wit, the Square of half the ſumm of the two numbers multiplied: Therefore by the help of this Theorem numbers proper for the Reſolution of the Queſtion propoſed may be taken, *viz.* Take ſome number at pleaſure, as 12, and divide it thrice into two ſuch numbers that the Product of the two numbers of each pair may make 12; ſuch are theſe three pairs of numbers, *viz.* 1, 12. | 2, 6. | 3, 4; then take the half-difference of the two numbers of each pair, ſo you will find the three half-differences to be theſe, $\frac{11}{2}$, 2 and $\frac{1}{2}$. Now by the Theorem above-mentioned, if to the Product 12 the Squares of the ſaid three half-differences be ſeverally added, every one of the ſumms will be a Square; therefore

2. For

2. For the ſumm of the three numbers ſought put . . . . . $12aa$
3. And for the firſt number . . . . . $\frac{11}{2}a$
4. And for the ſecond number . . . . . $2a$
5. And for the third number . . . . . $\frac{1}{2}a$
6. Therefore the ſumm of thoſe three numbers is . . . . . $8a$
7. Which ſumm muſt be equal to $12aa$ in the ſecond ſtep, therefore . $8a = 12aa$
8. Which Equation, after due Reduction, gives . . . . . $a = \frac{2}{3}$
9. Therefore from the eighth, third, fourth and fifth ſteps the three numbers ſought are theſe, *viz.* . . . . . } $\frac{11}{3}$, $\frac{4}{3}$, $\frac{1}{3}$

10. Which three numbers will ſolve the Queſtion, for if their Squares be ſeverally added to their ſumm, the three ſumms will be Squares, to wit, $\frac{169}{9}$, $\frac{64}{9}$ and $\frac{49}{9}$. It is alſo evident, that the Queſtion may be extended to four, five, or as many numbers as ſhall be deſired, by this following

CANON.

11. Take ſome number at pleaſure, and divide it into two numbers as many times as there be numbers deſired by the Queſtion, but ſo, as that the Product of the two numbers of each pair may make the number firſt taken; then divide the ſumm of the half-differences of the two numbers of each pair by the number taken, and multiply the Quotient by the ſaid half-differences ſeverally; ſo the Products ſhall be the numbers ſought.

The certainty of this Canon depends upon the Theorem aſſumed in the Reſolution, which Theorem may be demonſtrated thus;

Let two numbers be aſſumed, as, ...	a the greater, e the leſſer.
The Product of their multiplication is	ae
The half-difference of the ſaid two numbers is	$\frac{1}{2}a - \frac{1}{2}e$
The Square of the ſaid half-difference is	$\frac{1}{4}aa - \frac{1}{2}ae + \frac{1}{4}ee$
To which Square if ae the Product above-mentioned be added, the ſumm is	$\frac{1}{4}aa + \frac{1}{2}ae + \frac{1}{4}ee$

Which ſumm is the Square of $\frac{1}{2}a + \frac{1}{2}e$, to wit, half the ſumm of the two numbers a and e firſt taken, as the Theorem affirmed.

QUEST. 27. (pars Quæſt. 9. Lib. 3. *Diophant.*)

To find three Squares in Arithmetical proportion, and ſuch, that the half of their ſumm may exceed the greateſt of the three Squares.

RESOLUTION.

1. For the leaſt of the three Squares ſought put . . . . . aa
2. And to the end the mean Square may exceed the leaſt, let the ſide of the mean Square be $a + 1$, therefore the mean Square is } $aa + 2a + 1$
3. Therefore the exceſs of the mean Square above the leaſt is . . . $2a + 1$
4. And by adding the ſaid exceſs to the mean Square, the ſumm muſt be equal to the greateſt, which ſumm is . . . . . } $aa + 4a + 2$
5. Therefore $aa + 4a + 2$ muſt be equated to ſome Square, with this condition, that $\frac{1}{2}aa + 3a + \frac{1}{2}$, which is the half-ſumm of the three Squares in the firſt, ſecond and fourth ſteps, muſt (according to the Queſtion) exceed the greateſt of the ſaid three Squares; ſuppoſing therefore $\frac{1}{2}aa + 3a + \frac{1}{2} \sqsubset aa + 4a + 2$, it will thence follow (by arguing after the manner of ſearching out limits in divers of the foregoing Queſtions) that $a \sqsubset 1 + \sqrt{2}$, that is, $a \sqsubset 2.414$, *&c.*

Therefore $aa + 4a + 2$ (in the fourth ſtep) muſt be ſo equated to a Square, that the value of a may be greater than $2\frac{414}{1000}$, *&c.* Now to cauſe that effect, innumerable values of a may be found out by the firſt Canon of the twelfth Queſtion of this Book. Suppoſe therefore a be found equal to $\frac{11}{10}$, (as in the Example of that Canon) then from the firſt, ſecond and fourth ſteps, theſe three Squares will be diſcovered, to wit, $\frac{261}{100}$, $\frac{169}{100}$ and $\frac{241}{100}$, which will ſolve the Queſtion.

And becauſe a Square multiplied by a Square produces a Square, by multiplying ſeverally the ſaid three Squares found out by the common Denominator 100, the Products

961, 1681 and 2401 will be Squares, whose sides are 31, 41 and 49; also the difference of the first and second is equal to the difference of the second and third, (each difference being 720;) therefore the said three Squares are in Arithmetical proportion, and the half of their summ is manifestly greater than every one of them: Therefore all the conditions in the Question are satisfied.

QUEST. 28. (Quæst. 9. Lib. 3. *Diophant.*)

To find three numbers in Arithmetical proportion, and such, that the summ of every two of them may make a Square.

RESOLUTION.

1. By the preceding twenty-seventh Question find three Squares in Arithmetical proportion, and such, that the half of their summ may exceed every one of them, (the reason of which condition will be evident by the following eighth, ninth and tenth steps,) such are these three Squares, whose sides are 31, 41 and 49, . . . — 961 ; 1681 ; 2401
2. For the three numbers sought put . . . . . — a ; e ; y
3. Then equate the summ of the first and second numbers to the least of the three Squares before found, *viz.* suppose . . — $a+e = 961$
4. Likewise equate the summ of the first and third numbers to the mean Square, and it makes . . . . . — $a+y = 1681$
5. Equate also the summ of the second and third numbers to the greatest Square, and it gives . . . . . — $e+y = 2401$
6. The summ of the three last Equations is . . . . — $2a+2e+2y = 5043$
7. The half of the said summ is . . . . . — $a+e+y = 2521\frac{1}{2}$
8. Then by subtracting the third Equation from the seventh, there remains . . . . . — $y = 1560\frac{1}{2}$
9. And by subtracting the fourth Equation from the seventh, there remains . . . . . — $e = 840\frac{1}{2}$
10. And by subtracting the fifth Equation from the seventh, there remains . . . . . — $a = 120\frac{1}{2}$

I say these three numbers, $120\frac{1}{2}$, $840\frac{1}{2}$, $1560\frac{1}{2}$ will solve the Question; for the difference between the first and second, to wit, 720, is equal to the difference between the second and third; therefore they are in Arithmetical proportion, and the summ of every two of them makes a Square,

viz.
$120\frac{1}{2} + 840\frac{1}{2} = 961$, whose $\surd$ is 31;
$120\frac{1}{2} + 1560\frac{1}{2} = 1681$, whose $\surd$ is 41;
$840\frac{1}{2} + 1560\frac{1}{2} = 2401$, whose $\surd$ is 49.

QUEST. 29. (Quæst. 12. Lib. 3. *Diophant.*)

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

This Question may be resolved divers ways; I shall here shew three of my own, and if the curious Reader desire to see more variety, he may consult Vieta's *Zetet. 7. lib. 5. also* Bachet's *Comment, and the Solution of* Slusius *in* pag. *177. of his* Mesolabum, *printed in* 1668.

RESOLUTION 1.

1. First, by *Quest.* 22. of this *Book*) I seek two such Squares, that if to the Product of their multiplication the given number 12 be added, the summ shall be a Square; such are these Squares 1 and 4, which may be taken for two of the three numbers sought, ... — 1 and 4
2. Then for the third number sought I put . . . . . — a
3. Now (according to the Question) the Product of the multiplication of the first and third numbers being increased with 12 must make a Square; also, the Product of the second and third numbers together with 12 must make a Square; it remains therefore to resolve this Duplicate equality,

viz. $\begin{cases} 1a + 12 = \square \\ 4a + 12 = \square \end{cases}$

4. But before the said Duplicate equality can be resolved, it must be reduced to another that shall have equal numbers of a, to which purpose I multiply the first of the two quantities to be equated, to wit, $1a+12$, by 4, (which is prefixt to the latter of those two quantities) and it produceth $4a+48$ to be equated to a Square, so this Duplicate equality ariseth, (instead of the former,)

viz. $\begin{cases} 4a + 48 = \square \\ 4a + 12 = \square \end{cases}$

5. Now to resolve this latter Duplicate equality, (and consequently the former,) I proceed according to the first manner of solving the preceding *Quest.* 8. *viz.* First, the difference between $4a+48$ and $4a+12$ is 36; then I seek two such Squares that their difference may be 36, and that the greater of them may exceed 48; but two such Squares are 100 and 64, (found out by *Canon* 1. of the preceding *Quest.* 7.)
6. Then from either of these Equations, ... — $\begin{cases} 4a+48 = 100 \\ 4a+12 = 64 \end{cases}$
7. The third number sought is discovered, *viz.* . . . . — $a = 13$

I say, the numbers 1, 4 and 13 will solve the Question proposed; for if 12 be added to the Product of the first and second, likewise to the Product of the first and third, and lastly to the Product of the second and third, the three summs will be Squares, to wit, 16, 25 and 64. The premisses shew how to solve the Question by innumerable Answers.

Another way of resolving Quest. 29. *which is here repeated*, viz.

To find three such numbers, that if to the Product of the multiplication of every two of them, a number given, suppose 12, be added, the three summs shall be Squares.

RESOLUTION 2.

1. For the given number 12 put . . . . . — b
2. For the three numbers sought put . . . . . — a, e, u
3. Then supposing b to be the difference of two Squares, search out those Squares (by the fifth Question of this Book) and let cc represent the greater, and xx the lesser; therefore . . . . — $cc - b = xx$
4. Now the Question requires, that if b be added to the Product of the multiplication of a and e (the first and second numbers sought) the summ must be a Square, therefore by supposing that Square to be cc above found, this Equation arises, *viz.* . . . . — $ae + b = cc$
5. Therefore from the last Equation by equal subtraction of b, . — $ae = cc - b$
6. And because by Construction in the third step . . . . . — $xx = cc - b$
7. Therefore from the fifth and sixth steps . . . . . — $ae = xx$
8. Therefore by dividing each part of the last Equation by a, it gives . . . . . — $e = \frac{xx}{a}$
9. Again, (by the fifth Question) find two other Squares whose difference shall be equal to b, suppose dd the greater Square, and zz the lesser, therefore . . . . . — $dd - b = zz$
10. Then the Product of the multiplication of the first and third numbers sought, with b added, makes $au+b$, which must be equal to some Square, let it be dd before found, therefore . . — $au + b = dd$
11. Therefore by equal subtraction of b from the last Equation, . — $au = dd - b$
12. And because by Construction in the ninth step . . . . — $zz = dd - b$
13. Therefore from the two last Equations . . . . . — $au = zz$
14. Therefore by dividing each part of the last Equation by a, it gives — $u = \frac{zz}{a}$
15. Now since by the second, eighth, fifth, fourteenth and eleventh steps, the positions for the three numbers sought are a, $\frac{xx}{a}$, (or $\frac{cc-b}{a}$) $\frac{zz}{a}$, (or $\frac{dd-b}{a}$) it is evident,

evident, that if b be added to the Product of the multiplication of the firſt and ſecond numbers, the ſumm is a Square, to wit, cc. Likewiſe, if b be added to the Product of the firſt and third numbers the ſumm is a Square, to wit, dd; but if b be added to the Product of the ſecond and third numbers the ſumm muſt alſo be a Square; therefore $\frac{xxzz}{aa} + b$ muſt be equal to a Square whoſe ſide we may feign to be $\frac{xz}{a} + t$, or $\frac{xz}{a} - t$, and conſequently a will be found equal to $\frac{2xzt}{bb+tt}$; But x, z, b and t are known numbers, therefore a the firſt number ſought is known alſo; and from the eighth and fourteenth ſteps the ſecond and third numbers will be diſcovered.

From this ſecond Reſolution of *Queſt.* 29. it will not be difficult to deduce the following

CANON.

Firſt, ſuppoſing the given number (b) to be the difference of two Squares, find out (by the ſecond Canon of the foregoing fifth Queſtion) two pair of Squares in that difference, and let the ſide of the leſſer Square of the one pair be called (x,) and the ſide of the leſſer Square of the other pair, (z;) then take ſome ſquare number whoſe ſide may be called (t,) and let the difference between (tt and b) be called (g;) then divide the double of the ſolid Product of the three ſides x, z, t, viz. $2xzt$, by (g,) and the Quotient ſhall be one of the three numbers ſought; laſtly, multiply ſeverally the ſides x and z by g, and divide the firſt Product by $2zt$, and the latter by $2xt$, ſo the Quotients ſhall be the two other numbers ſought. Compare this Canon with the two following Examples.

$b = 12$	12	⊱ given in the Queſtion.
$x = 2$	2	} found out according to the directions in the Canon.
$z = \frac{11}{10}$	$\frac{1}{2}$	
$t = \frac{1}{2}$	5	
$g = \frac{19}{4}$	13	
$\frac{2xzt}{g} = \frac{4}{5}$	$\frac{10}{13}$	} found out by the Canon to ſolve the Queſtion.
$\frac{gx}{2xt} = 5$	$\frac{26}{5}$	
$\frac{gz}{2xt} = \frac{162}{80}$	$\frac{13}{40}$	

I ſay the number given being 12, the Queſtion may be ſolved by theſe three numbers $\frac{4}{5}, 5, \frac{162}{80}$; likewiſe by theſe, $\frac{10}{13}, \frac{26}{5}, \frac{13}{40}$, as may eaſily be proved.

A third way of reſolving Queſt. 29. *which is here repeated*, viz.

To find three ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 12, be added, the three ſumms ſhall be Squares.

RESOLUTION 3.

1. For the given number 12 put . . . . . ⊱ b
2. For the three numbers ſought put . . . . . ⊱ a, e, u
3. Then, according to the Queſtion, $ae + b$ muſt be equal to a Square, let it be ſome known Square cc, therefore . . . . . } $ae + b = cc$
4. Therefore from the laſt Equation by equal ſubtraction of b, . . ⊱ $ae = cc - b$
5. And by dividing each part of the laſt Equation by a, it gives . ⊱ $e = \frac{cc-b}{a}$
6. Thus, the firſt of the three numbers ſought being a, and the ſecond $\frac{cc-b}{a}$, it is evident, that if b be added to the Product of their multiplication, the ſumm is a Square, to wit, cc.
7. Again, according to the conditions in the Queſtion, $au + b$ muſt be equal to a Square, let it be ſome known Square dd, therefore } $au + b = dd$
8. Therefore by equal ſubtraction of b, ... ⊱ $au = dd - b$

9. And

9. And by dividing each part of the laſt Equation by a, ... ⊱ $u = \frac{dd-b}{a}$

10. Thus, the firſt of the three numbers ſought being a, and the third $\frac{dd-b}{a}$, it is evident, that if b be added to the Product of their multiplication the ſumm is a Square, to wit, dd. But the Queſtion requires alſo, that if to the Product of the ſecond and third numbers the given number b be added, the ſumm muſt be a Square, therefore let the ſecond and third numbers, to wit, $\frac{cc-b}{a}$ and $\frac{dd-b}{a}$ be mutually multiplied, and to the Product add b, ſo the ſumm will be $\frac{baa+ddcc+bb-bdd-bcc}{aa}$, which muſt be equal to a Square, the ſide whereof may be feigned to be either $\frac{dc+b}{a}$ or $\frac{dc-b}{a}$; firſt let the ſide be $\frac{dc+b}{a}$, then its Square being equated to the ſumm above-mentioned, after due Reduction of that Equation it will appear that $a = c + d$.

11. Therefore, the Equation laſt expreſt, to wit, $a = c + d$ being compared with the Quantities in the ſecond, fifth and ninth ſteps, the three numbers ſought, to wit, a, e, u will be found equal to theſe known Quantities, *viz.*

$$c + d \quad , \quad \frac{cc-b}{c+d} \quad , \quad \frac{dd-b}{c+d}.$$

12. Again, for as much as the ſide of the Square to be equated to $\frac{baa+ddcc+bb-bdd-bcc}{aa}$ (the ſumm above-mentioned in the tenth ſtep) may be feigned to be $\frac{dc-b}{a}$, (as well as $\frac{dc+b}{a}$,) let the Square of $\frac{dc-b}{a}$ be equated to the ſaid ſumm, then by proceeding as before, three other numbers capable of ſolving the Queſtion will be found equal to theſe, *viz.*

$$c - d \quad , \quad \frac{cc-b}{c-d} \quad , \quad \frac{dd-b}{c-d}.$$

From the premiſſes two excellent Canons are deducible to ſolve the forgoing *Queſt.* 29.

CANON 1.

13. Subtract the given number from two Squares ſeverally, then divide each of the Remainders by the ſumm of the ſides of the ſame Squares, ſo ſhall the two Quotients and the ſaid ſumm of the ſides be three numbers which will ſolve the Queſtion.

For example, let the given number be 12, ſubtract it from the Squares 36 and 64, the Remainders are 24 and 52, which being ſeverally divided by 14, (the ſumm of 6 and 8, which are the ſides of the ſaid Squares 36 and 64,) the Quotients $1\frac{5}{7}$ and $3\frac{5}{7}$, with the ſaid 14, are three numbers to ſolve the Queſtion, as will be evident by

The Proof.

$14 \times 1\frac{5}{7}, + 12$	$= 36$	} Which are Squares, as was required.
$14 \times 3\frac{5}{7}, + 12$	$= 64$	
$1\frac{5}{7} \times 3\frac{5}{7}, + 12$	$= \frac{900}{49}$	

CANON 2.

14. Subtract the given number from two Squares ſeverally, then divide each of the Remainders by the difference of the ſides of the ſame Squares, ſo ſhall the two Quotients and the ſaid difference be three numbers, which will ſolve the Queſtion.

For example, let the given number be 12, ſubtract it from the Squares 36 and 64, and divide each of the Remainders 24 and 52, by 2, (which is the difference of 6 and 8, the ſides of the ſaid Squares 36 and 64,) ſo the Quotients 12 and 26, with the ſaid 2, are three numbers that will ſolve the Queſtion, as will appear by

The Proof.

$2 \times 12, + 12$	$= 36$	} Which are Squares, as was required.
$2 \times 26, + 12$	$= 64$	
$12 \times 26, + 12$	$= 324$	

Bachet

Bachet in his Comment upon *Queſt.* 12. *Lib.* 3. *Dioph.* (which is the ſame with the preceding 29th Queſtion) delivers two Canons, one of which is the ſame with *Canon* 2. above expreſt, and the other is this following

CANON 3.

15. Subtract the given number from two Squares ſeverally, divide the Remainders ſeverally by the difference of the ſides of the ſame Squares; then ſhall the two Quotients and their double ſumm leſſened by the aforeſaid difference be three numbers which will ſolve the Queſtion propounded.

For example, let the given number be 12, ſubtract it from two Squares, ſuppoſe 36 and 64, the Remainders 24 and 52 being divided ſeverally by 2, (the difference of the ſides of the ſaid Squares 36 and 64) give the Quotients 12 and 26, which are two of the three numbers ſought; then from 76 (the double ſumm of the ſaid Quotients) ſubtracting 2 (the before-mentioned difference,) the Remainder 74 ſhall be the third number ſought. By this Operation it is evident that the two firſt numbers are the ſame with thoſe found out by *Canon* 2. but the third numbers are different: I ſay the three numbers 12, 26 and 74 will ſolve the Queſtion, as may eaſily be proved.

But to manifeſt the certainty of *Canon* 3. both its Operation and Demonſtration may be ſymbolically expreſt in this manner, *viz.*

Operation.

16. Take two unequal numbers, as . . . . . { d the greater, c the leſſer.
17. Their difference is . . . . . $d-c$
18. Take any number leſs than the Square of c, as, . . . b
19. Subtract the number b from the Squares of c and d ſeverally, ſo the Remainders are . . . . . $cc-b$ and $dd-b$
20. Divide each of thoſe Remainders by $d-c$, and the Quotients are . . . . . $\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$
21. The double ſumm of thoſe Quotients is . . . . . $\frac{2cc+2dd-4b}{d-c}$
22. From that double ſumm ſubtract $d-c$, and the Remainder is . . . . . $\frac{cc+dd+2cd-4b}{d-c}$
23. Thus, the three numbers found out by *Canon* 3. laſt afore-going to ſolve *Queſt.* 29. are equal to theſe, *viz.*

$$\frac{cc-b}{d-c}, \quad \frac{dd-b}{d-c}; \quad \frac{cc+dd+2cd-4b}{d-c}.$$

Now I ſay, if every two of thoſe three numbers be mutually multiplied, and to the Products ſeverally the number b be added, the three ſumms ſhall be Squares.

Demonſtration.

24. By *Canon* 1. *Reſolut.* 3. *Queſt.* 29. if the Product of the multiplication of the two firſt numbers, to wit, of $\frac{cc-b}{d-c}$ and $\frac{dd-b}{d-c}$ be increaſed with b, the ſumm will be a Square; it remains to prove, that if the Product of the multiplication of the firſt and third numbers be increaſed with b, the ſumm ſhall alſo be a Square; likewiſe that the Product of the multiplication of the ſecond and third numbers increaſed with the number b, makes a Square. But if the number b be added to the Product of the multiplication of the firſt and third numbers, the ſumm is $\frac{cccc+ccdd+2cccd-4ccb-4cdb+4bb}{dd+cc+2cd}$, which is a Square, whoſe ſide is $\frac{cc+cd-2b}{d-c}$; And if b be added to the Product of the multiplication of the ſecond and third numbers, the ſumm is . . . . . $\frac{dddd+ddcc+2dddc-4ddb-4cdb+4bb}{dd+cc+2cd}$, which is a Square whoſe ſide is $\frac{dd+cd-2b}{d-c}$. Therefore the truth of *Canon* 3. is evident.

By the help of the ſecond and third Canons laſt before expreſt, *Bachet* extends the preceding *Queſt.* 29. to four numbers, as I ſhall ſhew in the next Queſtion.

QUEST. 30.

QUEST. 30. (*Bachet* in Quæſt. 12. Lib. 3. *Diophant.*)

To find four ſuch numbers, that if to the Product of the multiplication of every two of them, a number given, ſuppoſe 3, be added, the four ſumms ſhall be Squares.

RESOLUTION.

1. The number given in the Queſtion is 3
2. Let two Squares be feigned from $a+$ two ſuch known numbers that their difference may be a Square, and that each of them may exceed the given number 3. To which end let the ſide of one of thoſe Squares be feigned $a+2$, and let the other ſide be $a+6$, for the difference of 2 and 6 is 4, (a ſquare number;) then will the Squares of the ſaid ſides $a+2$ and $a+6$ be theſe,
   viz. { $aa+4a+4$, (the Square of $a+2$.)
   { $aa+12a+36$, (the Square of $a+6$.)
3. From each of thoſe Squares ſubtract the given number 3, ſo the Remainders will be theſe,
   viz. { $aa+4a+1$
   { $aa+12a+33$.
4. Divide the ſaid Remainders ſeverally by 4, (the difference of the aforeſaid ſides $a+2$ and $a+6$) and let the Quotients be aſſumed for two of the four numbers ſought,
   viz. { $\frac{1}{4}aa+a+\frac{1}{4}$, (the firſt number.)
   { $\frac{1}{4}aa+3a+8\frac{1}{4}$, (the ſecond number.)
5. The double ſumm of the ſaid Quotients (aſſumed in the laſt ſtep for the firſt and ſecond numbers ſought) is $aa+8a+17$, from which ſubtract 4, (the difference before-mentioned) and let the Remainder be aſſumed for the third number,
   viz. $aa+8a+13$, (the third number.)
6. If the Conſtruction hitherto be compared with the third Canon in the third Reſolution of the foregoing 29th Queſtion, it will thence be evident, that if to the Product made by the multiplication of every two of thoſe three numbers before aſſumed in the fourth and fifth ſteps, there be added the given number 3, the three ſumms will be Squares.
7. For the fourth number ſought aſſume the difference of the ſides $a+2$ and $a+6$ before-mentioned, to wit, { 4, (the fourth number.)
8. Then by comparing the Conſtruction in the ſecond, third, fourth and ſeventh ſteps with *Canon* 2. in the third Reſolution of the foregoing twenty-ninth Queſtion, it will be manifeſt, that if to the Product made by the multiplication of every two of theſe three numbers, to wit, the firſt, ſecond and fourth numbers before aſſumed in the fourth and ſeventh ſteps, there be added the given number 3, the three ſumms will be Squares.
9. It remains to make the Product of the multiplication of the third and fourth numbers, the given number 3 being added, equal to a Square; but the third number $aa+8a+13$, multiplied by the fourth number 4, gives the Product $4aa+32a+52$, to which adding 3, the ſumm is $4aa+32a+55$, which muſt be equal to a Square, the ſide whereof may be feigned to be $2a-$ any known number whoſe Square exceeds 55; therefore let the ſaid ſide be $2a-10$, then the Square thereof being equated to $4aa+32a+55$, the value of a will thence be found equal to $\frac{5}{8}$, by the help whereof, recourſe being had to the fourth, fifth and ſeventh ſteps, the four numbers ſought will be found theſe, *viz.* $\frac{345}{256}$, $\frac{2617}{256}$, $\frac{1177}{64}$ and 4, which will ſolve the Queſtion. For if the firſt be multiplied by every one of the other three, and the Products be ſeverally increaſed with the given number 3, the three ſumms will be theſe Squares, to wit, $\frac{2481441}{65536}$, $\frac{3422521}{16384}$, $\frac{441}{64}$, whoſe ſides are $\frac{945}{256}$, $\frac{185}{128}$ and $\frac{21}{8}$. Alſo, if the ſecond be multiplied by the third and fourth numbers ſeverally, and each Product be increaſed with 3, theſe two Squares will ariſe, to wit, $\frac{3112961}{16384}$ and $\frac{32004}{64}$, whoſe ſides are $\frac{1769}{128}$ and $\frac{89}{8}$; laſtly, the Product of the third number multiplied by the fourth being added to 3 makes the Square $\frac{12321}{16}$, whoſe ſide is $\frac{111}{4}$.

QUEST. 31. (Quæſt. 15. Lib. 3. *Diophant.*)

To find three ſuch numbers, that if the Product of the multiplication of every two of them be leſſened by the third, the three Remainders ſhall be Squares.

RESOLUTION.

1. For the firſt number put . . . . . a

2. And

2. And for the ſecond number put $a + $ ſome known ſquare number, as . . . . . $a + 4$
3. Then the Product of the multiplication of the firſt and ſecond numbers is . . . . . $aa + 4a$
4. From which Product if $4a$ be ſubtracted, it is evident the Remainder will be a Square, to wit, aa, therefore for the third number we may put $4a$, (and ſo one of the conditions in the Queſtion will be ſatisfied) . . . . . $4a$
5. Then (from the ſecond and fourth ſteps,) the Product of the multiplication of the ſecond and third number is $4aa + 16a$, from which the firſt number a being ſubtracted, there remains $4aa + 15a$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $4aa + 15a = \square$
6. Alſo (from the firſt and fourth ſteps,) the Product of the multiplication of the firſt and third numbers is $4aa$, from which ſubtracting the ſecond number $a + 4$, there remains $4aa - a - 4$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $4aa - a - 4 = \square$
7. So in the two laſt preceding ſteps we are faln upon a Duplicate equality, which (upon the grounds before demonſtrated) may be reſolved thus, *viz.* Firſt, the difference of the two Algebraical quantities to be equated to two Squares, (by ſubtracting the leſſer from the greater) is manifeſtly $16a + 4$; then two Squares muſt be found, ſuch, that their difference may be $16a + 4$, and that $4aa$ may be in each of thoſe Squares: Therefore (agreeable to *Canon* 2. of *Queſt.* 5. of this *Book*,) two numbers are to be taken, that being mutually multiplied will produce $16a + 4$; moreover, that $2a$ may be found as well in the half-ſumm as in the half-difference of the numbers taken; but the only two numbers that will agree with thoſe conditions are $4a + 1$ and 4, whoſe half-ſumm is $2a + \frac{1}{2}$, and their half-difference is $2a - \frac{1}{2}$.
8. Then by equating the Square of the ſaid $2a + \frac{1}{2}$ to $4aa + 15a$ (in the fifth ſtep,) or the Square of $2a - \frac{1}{2}$ to $4aa - a - 4$ (in the ſixth ſtep,) from either of thoſe Equations the value of a will be found $\frac{1}{4}$, which is the firſt of the three numbers ſought, and conſequently from the ſecond and fourth ſteps, the ſecond and third numbers are $2\frac{1}{4}$ and 5, which three numbers will ſolve the Queſtion; for if from the Product of every two of them the other be ſubtracted, the three Remainders are Squares, to wit, $2\frac{1}{16}$, 25 and 1.

QUEST. 32.

To find three numbers, ſuch, that if to the Square of every one of them the ſumm of the other two be added, the three ſumms may be Squares.

RESOLUTION.

1. Take any number of $a + $ ſome known number, as $a + 1$, for the ſide of a Square, then the Square of $a + 1$ is $aa + 2a + 1$; now if for the firſt number ſought we put a, for the ſecond $2a$, and for the third 1, then it is evident that the Square of the firſt number, together with the ſumm of the ſecond and third, makes a Square, to wit, $aa + 2a + 1$, whereby one of the conditions in the Queſtion is ſatisfied; therefore,

   For the three numbers ſought put . . . . . a, $2a$, 1
2. Then (according to the import of the Queſtion,) the Square of the ſecond number, together with the ſumm of the firſt and third muſt make a Square, therefore . . . . . $4aa + a + 1 = \square$
3. Likewiſe the Square of the third number, together with the ſumm of the firſt and ſecond muſt make a Square, therefore . . . . . $\cdot + 3a + 1 = \square$
4. So we are faln upon a Duplicate equality, which differs from any of the preceding Forms, but (upon the ſame foundation by which thoſe have been reſolved) this may be reſolved thus; Firſt, ſuppoſing the former of the two quantities to be equated to a Square to exceed the latter, (for here we may indifferently take either of them for the greater,) their difference, by ſubtracting $3a + 1$ from $4aa + a + 1$, is manifeſtly $4aa =$

$4aa - 2a$; this difference, (as in moſt of the Duplicate equalities hitherto) muſt be eſteem'd the Product made by the mutual multiplication of two quantities, or factors; but here theſe two factors muſt be ſuch, that as well in half their ſumm as in half their difference there may be found 1, that is, the ſide of the known Square, (or abſolute number) in each of the two quantities to be equated, to the end that when the Square of the ſaid half-ſumm is equated to $4aa + a + 1$, or the Square of the ſaid half-difference to $3a + 1$, the ſquare number 1, by due Reduction of either of thoſe Equations, may vaniſh. Now to find out two factors qualified as aforeſaid, firſt, take 2, the double of the ſide of the known Square 1 in each of the two quantities to be equated, with — prefixt, (becauſe — is prefixt to $2a$ in the difference $4aa - 2a$ above-mentioned,) for part of the firſt of the two deſired factors; then divide $2a$, (which is part of the ſaid difference,) by 2 the double of the ſide of the ſaid known Square 1, and take the Quotient a for the latter factor; then divide $4aa$, (the other part of the ſaid difference $4aa - 2a$) by the ſaid latter factor a, and the Quotient $4a$ connected with -2 firſt taken, ſhall be the compleat firſt factor: So two factors or quantities to agree with the conditions above mentioned are found to be $4a - 2$ and a; for the Product of their multiplication is $4aa - 2a$, and 1 is found both in half their ſumm and half their difference: Then by equating the Square of half the ſumm of the ſaid factors $4a - 2$ and a, *viz.* the Square of $\frac{5}{2}a - 1$, to $4aa + a + 1$; or by equating the Square of half the difference of the ſame factors, *viz* the Square of $\frac{3}{2}a - 1$, to $3a + 1$, the value of a will be diſcovered, *viz.*

5. From either of theſe Equations, after due Reduction, ... $\frac{25}{16}aa - 5a + 1 = 4aa + a + 1$; $\frac{9}{4}aa - 3a + 1 = 3a + 1$
6. The ſame value of a will be diſcovered, *viz.* $a = \frac{4}{3}$

Therefore by the ſixth and firſt ſteps, theſe three numbers are found out, to wit, $\frac{8}{3}$, $1\frac{1}{3}$ and 1, which will ſolve the Queſtion propoſed: For $2\frac{4}{9}$ the Square of the firſt number, together with $1\frac{7}{9}$ the ſumm of the ſecond and third, makes a Square, to wit, $2\frac{11}{9}$; alſo $2\frac{16}{9}$ the Square of the ſecond, together with $2\frac{1}{9}$ the ſumm of the firſt and third, makes a Square, to wit, $2\frac{2}{9}$; and laſtly, 1 the Square of the third, with $2\frac{4}{9}$ the ſumm of the firſt and ſecond makes a Square, to wit, 9. By what hath been ſaid in the firſt ſtep of the Reſolution this Queſtion is capable of innumerable Anſwers.

QUEST. 33.

To find a number leſs than 2 a number given, and ſuch, that if it be multiplied by two given numbers ſeverally, ſuppoſe by 8 and 6, and if to each of the Products a given ſquare number, ſuppoſe 4, be added, the ſumms may be ſquare numbers.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then if that poſition be proſecuted according to the conditions in the Queſtion, this Duplicate equality will ariſe, to wit, ... $8a + 4 = \square$; $6a + 4 = \square$
3. Which kind of Duplicate equality *Diophantus* uſeth in divers Queſtions, and becauſe the Reſolution thereof is a very ſubtil invention, I have framed this Queſtion purpoſely to explain it.

Firſt, obſerve well theſe three numbers, ... $8a + 4$, $6a + 4$ and 4.

Then ſeek what proportion the exceſs of the greateſt of thoſe three numbers above the mean hath to the exceſs of the mean above the leaſt; ſo you will find that the former exceſs is to the latter as 1 to 3. For the exceſs of $8a + 4$ above $6a + 4$ is $2a$, and the exceſs of $6a + 4$ above 4 is $6a$; but $2a$ is to $6a$ as 1 to 3. Therefore the former exceſs is to the latter as 1 to 3, and conſequently the former exceſs is one third part of the latter.

4. Now the principal ſcope in reſolving the ſaid Duplicate equality is to find out two ſquare numbers with this condition, that the exceſs of the greater above the leſs may have ſuch proportion to the exceſs of the leſſer above 4 (the Square given in the Queſtion) as 1 to 3; to wit, as the difference of the numbers 8 and 6, which are prefixt to a in the Duplicate equality, is to 6 the leſſer number prefixt: For when two ſuch Squares are found out, then if the greater be equated to $8a + 4$, or the leſſer to $6a + 4$,

one and the ſame value of a will come forth. But to find out the ſaid two Squares I proceed thus:

5. The leaſt of the three Squares above mentioned, to wit, that given in the Queſtion, by the help whereof the other two are to be found out, is 4
6. And to the end the mean Square may exceed the leaſt, let the ſide of the mean Square be feigned $e+2$, (2 being the ſide of the given Square 4;) therefore the mean Square it ſelf is $ee+4e+4$
7. Therefore the exceſs of the mean Square above the leaſt is manifeſtly . . . . . $ee+4e$
8. But by what hath been ſaid before, the exceſs of the greateſt Square above the mean muſt be $\frac{1}{3}$ part of the exceſs of the mean Square above the leaſt; therefore (from the laſt ſtep) the exceſs of the greateſt Square above the mean ſhall be . . . . . $\frac{1}{3}ee+\frac{4}{3}e$
9. Therefore by adding the laſt mentioned exceſs, to wit, $\frac{1}{3}ee+\frac{4}{3}e$ to the mean Square in the ſixth ſtep, the ſumm will be the greateſt of the ſaid three Squares, to wit, ... $\frac{4}{3}ee+\frac{16}{3}e+4$
10. Which $\frac{4}{3}ee+\frac{16}{3}e+4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out; *viz.* Foraſmuch as the two greateſt of the three Squares above mentioned muſt be ſuch, that when the greateſt is equated to $8a+4$, or the mean to $6a+4$, the value of a may be leſs than 2, (according to the conditions in the Queſtion;) therefore ſuch a ſquare number muſt be found out equal to the ſaid $\frac{4}{3}ee+\frac{16}{3}e+4$, that when 4 is ſubtracted from the ſaid ſquare number, $\frac{1}{8}$ part of the Remainder may be leſs than 2. Therefore from $\frac{4}{3}ee+\frac{16}{3}e+4$ ſubtract 4, and the Remainder is $\frac{4}{3}ee+\frac{16}{3}e$, whereof $\frac{1}{8}$ is $\frac{1}{6}ee+\frac{2}{3}e$, which muſt be leſs than 2; therefore
11. Suppoſe . . . . . $\frac{1}{6}ee+\frac{2}{3}e \sqsupset 2$
12. Thence, by multiplying all by 6, it follows, that . . . . $ee+4e \sqsupset 12$
13. And by adding the Square of half the Coefficient 4 to each part, there ariſeth . . . . . $ee+4e+4 \sqsupset 16$
14. And by extracting the ſquare Root out of each part of the laſt ſtep, ... $e+2 \sqsupset 4$
15. Therefore by equal ſubtraction of 2, it is manifeſt that . . . . $e \sqsupset 2$
16. Thus we have found that $\frac{4}{3}ee+\frac{16}{3}e+4$ muſt be equated to a Square, with this condition, that the value of e may be leſs than 2. Now to cauſe that effect, the ſide of the ſaid Square may be feigned $-2+$ any number of e greater than $3\frac{2}{1000}e$, therefore let the ſaid ſide be feigned $3\frac{1}{3}e-2$, then the Square of $3\frac{1}{3}e-2$ being equated to the ſaid $\frac{4}{3}ee+\frac{16}{3}e+4$, the value of e will thence be found $\frac{24}{11}$.
17. Now if . . . . . $e = \frac{24}{11}$
18. Then conſequently the Square of $3\frac{1}{3}e-2$, that is, the greater of the two Squares ſought, will be . . . . . $\frac{2304}{121}$
19. And the Square of $e+2$ (which in the ſixth ſtep was put for the ſide of the leſſer of the two Squares ſought,) will be $\frac{1849}{121}$
20. Which two Squares, to wit, $\frac{2304}{121}$ and $\frac{1849}{121}$, (whoſe ſides are $\frac{48}{11}$ and $\frac{43}{11}$,) together with 4, (the ſquare number given in the Queſtion,) are ſuch, that the exceſs of the greateſt above the mean is $\frac{1}{3}$ part of the exceſs of the mean above the leaſt, (according to the ſcope deſigned in the fourth ſtep.) Now if the greater Square $\frac{2304}{121}$ be equated to $8a+4$, or the leſſer Square $\frac{1849}{121}$ to $6a+4$, from either of thoſe Equations the value of a, to wit, the number ſought by the Queſtion will be found $\frac{529}{242}$: For firſt, it is leſs than 2, alſo eight times that number, together with 4, makes the Square $\frac{2304}{121}$; and ſix times the ſame number $\frac{529}{242}$, together with 4, makes the Square $\frac{1849}{121}$. It is alſo evident by the ſixteenth ſtep, that as many numbers as one will may be found out to ſolve the Queſtion propoſed.
21. But for the greater evidence of the infallibility of the method of reſolving this Duplicate equality, I ſhall demonſtrate the ſame in manner following, *viz.*

Suppoſe

Suppoſe
- $r = 8$ — two Multiplicators given in the Queſtion,
- $s = 6$ — two Multiplicators given in the Queſtion,
- $c = 4$ — a ſquare number given in the Queſtion,
- $d = \frac{529}{242}$ — two ſquare numbers found out according to the direction
- $f = \frac{1849}{121}$ — in the fourth ſtep of the Reſolution,
- $a =$ — the number ſought.

22. Suppoſe alſo, according to the Conſtruction in the Reſolution, that the exceſs of d above f hath ſuch proportion to the exceſs of f above c, as the exceſs of r above s hath to s, *viz.* as, $d-f \,.\, f-c :: r-s \,.\, s$.
23. Then according to the Conſtruction in the twentieth ſtep of the Reſolution, let theſe two Equations be inſtituted, *viz.* $ra+c = d$, $sa+c = f$.
24. Now ſince the Concluſion of the Reſolution (in the ſaid twentieth ſtep) takes it for granted, that one and the ſame value of a, (to wit, the number ſought) will be given by either of thoſe two Equations, we muſt prove that theſe two Quotients are equal to one another, *viz.* $\frac{d-c}{r} = \frac{f-c}{s}$.

Demonſtration.

25. Foraſmuch as by Conſtruction in the 22d ſtep, ... $d-f \,.\, f-c :: r-s \,.\, s$
26. Therefore by Compoſition of Reaſon, ... $d-c \,.\, f-c :: r \,.\, s$
27. Therefore alternately . . . . . $d-c \,.\, r :: f-c \,.\, s$
28. But if four numbers be Proportionals, the Reaſon of the firſt to the ſecond is equal to the Reaſon of the third to the fourth, therefore . . . . . $\frac{d-c}{r} = \frac{f-c}{s}$

Which was to be demonſtrated.

Obſervat. 1. *upon* Queſt. 33.

In the Duplicate-equality uſed in the preceding *Queſt.* 33. both the numbers of a are affirmative; but if they were both negative, or one of them affirmative and the other negative, the Reſolution would differ very little from the former, as will appear by the two following Queſtions.

QUEST. 1.

1. Let it be required to find out the number ſignified by a in this Duplicate equality, *viz.* . . . . . $4-2a = \square$, $4-3a = \square$

RESOLUTION.

2. Firſt, theſe three numbers are to be conſidered . . . . . $4, 4-2a, 4-3a$

Then becauſe the exceſs of 4 above $4-2a$, hath ſuch proportion to the exceſs of $4-2a$ above $4-3a$, as 2 to 1, let 4 be conſidered as the greateſt of three Squares, and find the other two, with this condition, that the exceſs of 4 above the mean may be the double of the exceſs of the mean above the leaſt; to which end,

3. Let the greateſt of the ſaid three Squares be 4
4. And to the end the mean Square may be leſs than the greateſt, let the ſide of the mean Square be $2-e$, therefore the mean Square ſhall be . . . . . $ee-4e+4$
5. Therefore the exceſs of the greateſt Square above the mean is $4e-ee$
6. Therefore, according to the condition preſcribed in the ſecond ſtep, the half of the exceſs in the fifth ſtep ſhall be the exceſs of the mean Square above the leaſt, to wit, ... $2e-\frac{1}{2}ee$
7. Which laſt exceſs, to wit, $2e-\frac{1}{2}ee$ being ſubtracted from the mean Square in the fourth ſtep, the Remainder ſhall be equal to the leaſt Square, to wit, ... $\frac{1}{2}ee-6e+4$
8. Therefore $\frac{1}{2}ee-6e+4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out, *viz.* Foraſmuch as the leaſt of the three Squares above mentioned muſt be ſuch, that when it is equated to $4-3a$ (in the firſt ſtep,) the value of a may be greater than nothing, it is evident the ſaid leaſt Square muſt neceſſarily be leſs than 4; Suppoſing therefore $\frac{1}{2}ee-6e+4 \sqsupset 4$ from this ſuppoſition,

ſition, (by arguing in like manner as before from the eleventh ſtep to the ſixteenth,) we ſhall find $e \sqsupset 4\frac{1}{2}$; therefore $\frac{1}{2}ee - 6e + 4$ muſt be equated to a Square, ſo, as the value of e may be leſs than 4. Now to cauſe that effect, the ſide of the ſaid Square may be feigned to be $2 -$ any number of e greater than $\frac{1}{2}e$, let therefore the ſaid ſide be feigned $2 - 2e$, then its Square being equated to $\frac{1}{2}ee - 6e + 4$ will give $e = \frac{4}{5}$, and conſequently the mean and leaſt Squares ſought will be $\frac{36}{25}$ and $\frac{4}{25}$, the former of which being equated to $4 - 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the value of a, or the number ſought, will be found $\frac{22}{25}$; which will ſolve *Queſt.* 1. before propounded, as will be evident by

The Proof.

If $\frac{22}{25}$ (or a) be multiplied by 2 and 3 ſeverally, and if the Products be ſeverally ſubtracted from 4, the two Remainders will be $\frac{16}{25}$ and $\frac{4}{25}$, which are Squares, as was required.

QUEST. 2.

1. Let it be required to find out the number ſignified by a, in this Duplicate equality, *viz.* . . . . . $\left\{\begin{array}{l} 4 + 2a = \square \\ 4 - 3a = \square \end{array}\right.$

RESOLUTION.

2. Firſt, theſe three numbers are to be conſidered, . . $\rangle$ $4 + 2a$, 4, and $4 - 3a$

Then, becauſe the exceſs of $4 + 2a$ above 4, hath ſuch proportion to the exceſs of 4 above $4 - 3a$, as 2 to 3, let 4 be aſſumed to be the mean of three Squares, and find out the other two, with this condition, that the exceſs of the greateſt above the mean may be $\frac{2}{3}$ of the exceſs of the mean above the leaſt; to which end,

3. Let the mean Square be	4
4. And to the end the leaſt of the three Squares may be leſs than the mean, let the ſide of the leaſt Square be $2 - e$, therefore the leaſt Square ſhall be . .	$ee - 4e + 4$
5. Therefore the exceſs of the mean Square above the leaſt is	$-ee + 4e$
6. Therefore according to the condition preſcribed in the ſecond ſtep, $\frac{2}{3}$ of the laſt mentioned exceſs ſhall be equal to the exceſs of the greateſt Square above the mean, to wit,	$-\frac{2}{3}ee + \frac{8}{3}e$
7. Which laſt exceſs added to the mean Square 4, will give the greateſt Square, to wit, ...	$-\frac{2}{3}ee + \frac{8}{3}e + 4$

Therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, but the value of e muſt be ſubject to a Determination thus found out; Foraſmuch as the greateſt of the three Squares required muſt be ſuch, that when it is equated to $4 + 2a$ (in the firſt ſtep) the value of a may be greater than nothing, it is evident that the ſaid greateſt Square muſt be greater than 4.

Suppoſe therefore $-\frac{2}{3}ee + \frac{8}{3}e + 4 \sqsubset 4$, thence it will follow, that $e \sqsupset 4$; therefore the ſaid $-\frac{2}{3}ee + \frac{8}{3}e + 4$ muſt be equated to a Square, with this caution, that the value of e may be leſs than 4. Now to find out ſuch a Square, the value of e may be $\frac{2}{7}$, (found out by the Canon of *Queſt.* 13. of this Book, *mutatis mutandis*,) whence the greateſt and leaſt Squares ſought will be $\frac{144}{49}$ and $\frac{4}{49}$; the former of which being equated to $4 + 2a$, or the latter to $4 - 3a$, from either of thoſe Equations the number a will be found $\frac{44}{49}$, which will ſolve the Queſtion, as will be manifeſt by

The Proof.

If $\frac{44}{49}$ (or a) be multiplied firſt by 2, and then by 3; alſo, if the firſt Product be added to 4, and the latter Product be ſubtracted from 4, the Sums and Remainder will be $\frac{144}{49}$ and $\frac{4}{49}$, which are Squares, as was required.

Obſervat. 2. *upon the preceding* Queſt. 33.

By the ſame artifice that hath been uſed in ſolving the ſaid *Queſt.* 33. this following Duplicate equality may be reſolved, *viz.*

1. Let it be required to find a number, call it a, that ſhall make . . . . . $\left\{\begin{array}{l} 10a + 9 = \square \\ 5a + 4 = \square \end{array}\right.$

2. Firſt,

2. Firſt, by multiplying $5a + 4$ by 9, the Product is $45a + 36$; likewiſe $10a + 9$ multiplied by 4 produceth $40a + 36$; ſo the Duplicate equality propounded is reduced to this,

$$viz. \left\{\begin{array}{l} 45a + 36 = \square \\ 40a + 36 = \square \end{array}\right.$$

3. Which latter Duplicate equality being of the ſame kind with that in the foregoing *Queſt.* 33. may be ſolved by innumerable Anſwers; but for the greater evidence, the ſearch may be made as before, *viz.*

4. Let theſe three numbers be conſidered, ... $\rangle$ $45a + 36$, $40a + 36$, 36

Then becauſe the exceſs of the greateſt of thoſe three numbers above the mean, hath ſuch proportion to the exceſs of the mean above the leaſt as 1 to 8, let 36 be aſſumed for the leaſt of three Squares, and ſearch out the other two, with this caution, That the exceſs of the greateſt above the mean may be $\frac{1}{8}$ of the exceſs of the mean above the leaſt, to which end,

5. Let the leaſt of thoſe three Squares be	36
6. And to the end the mean Square may be greater than the leaſt, let the ſide of the mean Square be $e + 6$, therefore the mean Square it ſelf ſhall be	$ee + 12e + 36$
7. Therefore the exceſs of the mean Square above the leaſt is	$ee + 12e$
8. Therefore $\frac{1}{8}$ of that exceſs (which according to the Caution given as above, muſt be the exceſs of the greateſt Square above the mean) ſhall be	$\frac{1}{8}ee + \frac{1}{2}e$
9. Which laſt exceſs, to wit, $\frac{1}{8}ee + \frac{1}{2}e$, being added to the mean Square in the ſixth ſtep, the ſumm is equal to the greateſt Square, to wit, ...	$\frac{9}{8}ee + 2\frac{1}{2}e + 36$

10. Therefore $\frac{9}{8}ee + 2\frac{1}{2}e + 36$ muſt be equated to a Square, which ſquare number when 'tis found out muſt be equated to $45a + 36$, and therefore the ſaid Square muſt be greater than 36; but from any affirmative value of e whatſoever, the ſaid $\frac{9}{8}ee + 2\frac{1}{2}e + 36$ will be manifeſtly greater than 36. Therefore here being no need of any limit for the value of e, the ſide of the ſaid Square may be variouſly feigned; let then the ſaid ſide be $\frac{3}{2}e - 6$, the Square thereof will be $\frac{9}{4}ee - 18e + 36$, which being equated to $\frac{9}{8}ee + 2\frac{1}{2}e + 36$, the value of e will thence be found 28; and conſequently (from the ninth and ſixth ſteps) the greateſt and mean Squares ſought will be 1296 and 1156, whoſe ſides are 36 and 34: Then,

11. From this Equation, ...	$45a + 36 = 1296$
12. Or from this Equation, ...	$40a + 36 = 1156$
13. The number a will be diſcovered, to wit, ...	$a = 28$

I ſay the number 28 will ſolve the Queſtion, as will be evident by

The Proof.

If 28 (or a) be multiplied by 10 and 5 ſeverally; alſo, if to the former Product 9 be added, and to the latter Product 4, the two ſumms will be 289 and 144, which are Squares, as was required.

Obſervat. 3. *upon* Queſt. 33.

1. In the Duplicate equalities uſed in the preceding *Queſt.* 33. and the two Obſervations thereon, the two Algebraick Quantities given to be equated to two Squares, do conſiſt of two unequal numbers of a and of two known ſquare numbers, either equal or unequal; which kind of Duplicate equality you have ſeen exactly reſolved by Rational numbers: But neither *Diophantus*, nor any Author that I have met with, delivers a Rule to reſolve in all caſes a Duplicate equality conſiſting of unequal numbers of a, and of abſolute numbers which are not Squares. Monſieur *Bachet* indeed (in his Comment upon *Queſt.* 45. *Book* 4. of *Diophantus*) ſhews how to reſolve the laſt mentioned Duplicate equality in two Caſes, which I ſhall here explain.

2. The firſt Caſe is, when in a Duplicate equality of the kind laſt mentioned, the difference of the two Algebraick Quantities propos'd to be equated is ſuch, that if it be multiplied or divided by ſome known number, and if the Product or Quotient be ſubtracted from the leſſer of the two Quantities, the Remainder is a ſquare number. As, for example,

3. Let

3. Let it be required to find out the number ſignified by a in this Duplicate equality, *viz.* . . . . . $\left\{\begin{array}{l} 3a+13 = \square \\ a+7 = \square \end{array}\right.$

4. Becauſe $2a+6$, the difference of the two Algebraick Quantities propoſed, being divided by 2 gives the Quotient $a+3$, which ſubtracted from $a+7$ leaves a Square, to wit, 4; the Duplicate equality propos'd is explicable by the method uſed in the preceding *Queſt.* 33. as I ſhall here make manifeſt.

Firſt, let theſe three numbers be conſidered, to wit, . . $\rightarrow$ $3a+13$, $a+7$, 4

Then, by continuing the ſearch in ſuch manner as hath been ſhewn in ſolving the foregoing *Queſt.* 33. you may find out (among innumerable values of a) $a = 29$, which will ſolve the Duplicate equality propoſed; for if 29 be multiplied by 3, and 13 be added to that Product, the ſumm makes a Square, to wit, 100; alſo, if 7 be added to 29, it makes the Square 36.

5. The ſecond Caſe is, when the difference of the two Algebraick Quantities which are to be equated to two Squares is ſuch, that if it be multiplied or divided by ſome known number, and the Product or Quotient be ſubtracted from the leſſer of thoſe two Algebraick Quantities, there remains a negative known number, which taken affirmatively hath ſuch proportion to the ſaid Multiplicator or Diviſor, as a ſquare number to a ſquare number. For example,

6. Let it be required to find a number, ſuppoſe it to be a, that will make . . . . . $\left\{\begin{array}{l} 6a+25 = \square \\ 2a+3 = \square \end{array}\right.$

7. Becauſe $4a+22$, the difference of thoſe two Algebraick Quantities, being divided by 2 gives the Quotient $2a+11$, which ſubtracted from the leſſer of the ſaid two Quantities, leaves -8; and the number 8 to the Diviſor 2, is as 4 to 1, that is, as a ſquare number to a ſquare number; the Duplicate equality propounded may be reſolved thus, *viz.*

8. Let theſe three numbers be conſidered, to wit, . . $\rightarrow$ $6a+25$, $2a+3$, and -8

Then foraſmuch as $4a+22$, to wit, the difference of the two greater of thoſe three numbers, is the double of $2a+11$ which is the difference of the two leſſer of the ſaid three numbers, ſeek two Squares, that the exceſs of the greater above the leſſer may be the double of the exceſs of the leſſer above -8; to which end you may proceed thus, *viz.*

9. For the leſſer of the two Squares put . . . . . $\rightarrow$ aa
10. Then the exceſs of aa above -8 is . . . . . $\rightarrow$ $aa+8$
11. The double whereof is . . . . . $\rightarrow$ $2aa+16$
12. Which added to the leſſer Square makes a ſumm equal to the greater, to wit, ... $3aa+16$

Now the ſaid $3aa+16$ muſt be equated to a Square, with this caution, That the ſquare number found out muſt exceed 25, to the end that when the ſaid Square is equated to $6a+25$, the value of a may be greater than nothing: But to cauſe that effect, innumerable Squares may be found out, (by the method before-delivered in divers Queſtions of this Book,) ſuch are 64 and 16; for whether you equate 64 to $6a+25$, or 16 to $2a+3$, one and the ſame value of a, to wit, $6\frac{1}{2}$, will be diſcovered to ſolve the Duplicate equality propoſed in the ſixth ſtep.

QUEST. 34. (Quæſt. 17. Lib. 3. *Diophant.*)

To find three ſuch numbers, that the Product of the multiplication of every two of them, with the ſumm of the ſame two numbers, may make a Square.

RESOLUTION.

1. Foraſmuch as the Product of the multiplication of two Squares whoſe ſides differ by unity, being added to the ſumm of the ſaid Squares, will make a Square, (by the *Theorem* at the end of the Reſolution of this Queſtion,) let there be put for the firſt and ſecond numbers . . . . . 4 and 9

So is one of the conditions in the Queſtion ſatisfied; for 4 multiplied by 9 produceth 36, to which if 13 (the ſumm of 4 and 9) be added, the ſumm is a Square, to wit, 49.

2. Let the third number be . . . . . $\rightarrow$ a

3. Then

3. Then the Product of the multiplication of the ſecond and third numbers is $9a$, to which adding their ſumm $a+9$, it makes $10a+9$, which (according to the Queſtion) muſt be equal to a Square, *viz.* . . . . . $10a+9 = \square$

4. Likewiſe the Product of the multiplication of the firſt and third numbers is $4a$, which with their ſumm $a+4$ makes $5a+4$, which (according to the Queſtion) muſt be equal to a Square, *viz.* $5a+4 = \square$

5. So in the two laſt ſteps we are faln into a Duplicate equality, which may be ſolved by innumerable values of a, as hath been ſhewn in the ſecond Obſervation upon the foregoing *Queſt.* 33. of this ~~Chapt.~~ Book. For example, take that value of a there found, to wit, 28 ($= a$) for the third number ſought by this Queſtion; I ſay 4, 9 and 28 will ſolve this 34th Queſtion, as will be evident by

The Proof.

The three numbers found out are . . . . . 4, 9, 28.

Now according to the Queſtion,

I. $4 \times 9, +9+4 = 49$
II. $9 \times 28, +9+28 = 289$
III. $4 \times 28, +4+28 = 144$

Which are Squares, as was required.

The firſt ſtep of the Reſolution of this 34th Queſtion is grounded upon this

THEOREM.

6. If two numbers differ by unity, the Product made by the multiplication of their Squares, together with the ſumm of their Squares ſhall be a Square.

The truth of this Theorem may be demonſtrated thus,

7. Let there be two numbers which differ by 1 (or unity,) as . . . . . a, and $a+1$
8. Then their Squares are . . . . . aa and $aa+2a+1$
9. The ſumm of thoſe Squares is . . . . . $\rightarrow$ $2aa+2a+1$
10. The Product of the multiplication of the ſaid Squares is $\rightarrow$ $aaaa+2aaa+aa$
11. The ſumm of the ſaid Summ and Product in the ninth and tenth ſteps is . . . . . $aaaa+2aaa+3aa+2a+1$
12. Which Aggregate is a Square whoſe ſide is . . . $\rightarrow$ $aa+a+1$

As will eaſily appear by multiplying the ſaid ſide by it ſelf. Therefore the truth of the Theorem is manifeſt.

QUEST. 35. (Quæſt. 18. Lib. 3. *Diophant.*)

This Queſtion is the ſame with the foregoing 34th, *which is here repeated, and ſolved after another manner.*

To find three ſuch numbers, that the Product of the multiplication of every two of them, being added to the ſumm of the ſame two numbers, may make a Square.

RESOLUTION.

1. Let the firſt number be . . . . . $\rightarrow$ a
2. And let the ſecond number be . . . . . $\rightarrow$ 3
3. Then the Product of their multiplication added to their ſumm, makes . . . . . $4a+3$
4. Which $4a+3$ muſt (according to the Queſtion) be equal to ſome Square, let it be 25, therefore . . . . . $4a+3 = 25$
5. Therefore from that Equation, ... $\rightarrow$ . . $a = 5\frac{1}{2}$
6. So we have found two numbers, to wit, $5\frac{1}{2}$ and 3, which will ſatisfie one of the conditions in the Queſtion, for the Product of their multiplication with their ſumm makes 25, which is a Square. It remains to find a third number, which muſt be ſuch, that the Product of the ſecond and third numbers being added to their ſumm may make a Square; alſo, that the Product of the firſt and third numbers being added to their ſumm, may make a Square: Now to find out the ſaid third number, *Diophantus* begins again, thus,
7. Let the firſt number be (as before it was found) . . . $\rightarrow$ $5\frac{1}{2}$

8. And

8. And let the second number be (as before it was assumed,) . . . 3
9. Then for the third number put . . . . . a
10. And since (according to the Question) the Product of the multiplication of the second and third numbers, with their summ, must make a Square; therefore, from the eighth and ninth steps, $4a + 3 = \square$
11. Also the Product of the first and third numbers with their summ must be a Square; therefore, from the seventh and ninth steps, $6\frac{1}{2}a + 5\frac{1}{2} = \square$
12. So in the two last steps we are faln into a Duplicate equality, but 'tis not resolvable by any of the preceding Rules of *Diophantus*; he frames therefore the Positions a-new, wherein his scope is to find such numbers of a in the two Algebraick Quantities to be equated to two Squares, that shall be in proportion one to the other as a square number to a square number, and then he shews how to resolve this new kind of Duplicate equality, which hath not hitherto happened. First, if we examine whence 4 and $6\frac{1}{2}$ (to wit, the numbers prefixt before a in the tenth and eleventh steps) do proceed, we shall find that they arise from the addition of unity to each of the numbers 3 and $5\frac{1}{2}$ first found; (for by multiplying a into those numbers severally, and by adding a to each Product, there ariseth $4a$ and $6\frac{1}{2}a$ above expresst.) Therefore the next search must be to find two such numbers, that being severally increased with unity, the one summ may be to the other as a square number to a square number: And because (by the Theorem in the following first Observation upon this Question,) if we add unity to each of two numbers whereof the greater exceeds the quadruple of the lesser by 3, the two summs will be in the Reason of a Square to a Square; therefore,

13. Let the first of the three numbers sought be . . . . . a
14. Then (by the said Theorem) the second number shall be . . $4a + 3$
15. Now according to the Question, the Product of the first and second numbers together with their summ must be a Square, therefore from the two last preceding steps . . . . . $4aa + 8a + 3 = \square$
16. The side of which Square may be feigned $2a -$ any known number whose Square is greater than 3, let therefore the said side be $2a - 3$, then its Square being equated to $4aa + 8a + 3$, the value of a will be found $\frac{1}{10}$ for the first number, and consequently $\frac{42}{10}$ ($= 4a + 3$) shall be the second number.

So we have found two numbers which will answer the first part of the Question, and moreover they are fit to raise a Duplicate equality that will be explicable by a Rational number: Therefore now an effectual Resolution may be formed thus;

17. Let the first number be . . . . . $\frac{1}{10}$
18. And the second number . . . . . $\frac{42}{10}$
19. And let the third number be . . . . . a
20. Then according to the Question, the Product of the multiplication of the second and third numbers, together with their summ, must be equal to a Square, therefore from the 18th and 19th steps, $\frac{42}{10}a + \frac{42}{10} = \square$
21. Also according to the Question, the Product of the first and third numbers, with their summ, must be a Square; therefore, from the 17th and 19th steps, ... $\frac{1}{10}a + \frac{1}{10} = \square$
22. Now because the numbers drawn into a, in the Duplicate equality expresst in the two last preceding steps, are (by Construction) in the Reason of a Square to a Square, for $\frac{42}{10} . \frac{1}{10} :: 4 . 1$, and consequently (by *Prop.* 19. *Elem.* 7. Euclid.) $\frac{42}{10} \times 1 = \frac{1}{10} \times 4$. Therefore by multiplying the Algebraick quantity in the twentieth step by 1, and that in the twenty-first step by 4, the numbers of a in the Products will be equal to one another, for the first Product will be $\frac{42}{10}a + \frac{42}{10}$, and the latter $\frac{42}{10}a + \frac{4}{10}$; hence a new Duplicate equality is formed,

$$viz. \begin{cases} \frac{42}{10}a + \frac{42}{10} = \square \\ \frac{42}{10}a + \frac{4}{10} = \square \end{cases}$$

23. Which Duplicate equality being of the same kind with that explained in the preceding eighth Question, may be solved by innumerable Answers.

But I shall exhibit only one Answer for an Example. First then, because the difference of the two Algebraick quantities in the Duplicate equality last before expresst is 3, let two Squares be found out (by *Canon* 1. *Quest.* 7. of this *Book*,) whose difference shall be

be 3, and that the greater Square may exceed $\frac{42}{10}$; such are the Squares $\frac{784}{100}$ and $\frac{484}{100}$, whose sides are $\frac{28}{10}$ and $\frac{22}{10}$: Then,

24. From either of these Equations, ... $\begin{cases} \frac{42}{10}a + \frac{42}{10} = \frac{784}{100} \\ \frac{42}{10}a + \frac{42}{10} = \frac{484}{100} \end{cases}$
25. The same value of a will be discovered for the third number sought, *viz.* . . . . . $a = \frac{7}{10}$

Thus three numbers are found out, to wit, $\frac{1}{10}$, $\frac{42}{10}$ and $\frac{7}{10}$, which will solve the Question, as will be evident by

The Proof.

$$\left.\begin{array}{l} \frac{1}{10} \times \frac{42}{10}, + \frac{1}{10} + \frac{42}{10} = \frac{576}{100} \\ \frac{42}{10} \times \frac{7}{10}, + \frac{42}{10} + \frac{7}{10} = \frac{784}{100} \\ \frac{1}{10} \times \frac{7}{10}, + \frac{1}{10} + \frac{7}{10} = \frac{121}{100} \end{array}\right\} \text{Which are Squares, as the Question requires.}$$

Observations upon Quest. 35.

1. If the Resolution of this Question be well examined, it will appear, that the forming of the Duplicate equality in the twentieth and twenty-first steps, where the numbers prefixt to a have such Reason to one another as a square number to a square number; agreeable to the Scope before-mentioned in the twelfth step, doth depend upon this following

THEOREM.

If there be two such numbers, that the greater exceeds the quadruple of the lesser by 3, and if unity be added to each number, the summs shall have such Reason between themselves as a Square to a Square, *viz.* the greater summ shall be to the lesser as 4 to 1.

Which Theorem may be easily demonstrated, thus,

$$\text{Suppose} \begin{cases} a \\ 4a + 3 \end{cases} \left\{ \text{Two numbers, whereof the greater exceeds the quadruple of the lesser by 3.} \right.$$
$$a + 1 = \text{The first number increased with unity.}$$
$$4a + 4 = \text{The second number increased with unity.}$$

I say $4a + 4$ hath such proportion to $a + 1$, as a Square to a Square, for,

$$4a + 4 . a + 1 :: 4 . 1.$$

In like manner, if there be two numbers whereof the greater exceeds nine times the lesser by 8, as 17 and 1, then if you add 1 to each number, the summs shall be to one another as a square number to a square number, *viz.* the greater summ shall be to the lesser as 9 to 1; the like is to be understood of other Squares.

2. After the two numbers prefixt before a in the Duplicate equality formed in the twentieth and twenty-first steps of this Question, are found such, that they have such Reason one to the other as a Square to a Square; then may any two square numbers in that Reason be used as is directed in the twenty-second step: So instead of 4 and 1 there taken, we may take 100 and 25, which have the same Reason between themselves as $\frac{42}{10}$ and $\frac{11}{10}$; For, $\frac{42}{10} . \frac{11}{10} :: 100 . 25$,

Therefore, $\frac{42}{10} \times 25 = \frac{11}{10} \times 100$.

Then by multiplying the Algebraick quantity in the twentieth step of the Resolution by 25, and that in the twenty-first step by 100, the following Duplicate equality (being that which *Diophantus* useth in solving this Question) will arise,

$$viz. \begin{cases} 130a + 105 = \square \\ 130a + 30 = \square \end{cases}$$

Hence, by the Canon in the seventh step of *Resolut.* 1. *Quest.* 8. (among innumerable values of a that might be found out,) you may find $a = \frac{7}{10}$ (as before) for the third number sought.

QUEST. 36. (Quæst. 20. Lib. 3. *Diophant.*)

To find two numbers, that the Product of their multiplication increased severally with each of them, and also with their summ, may make three Squares.

RESOLUTION.

1. For one of the numbers put . . . . . a
2. And for the other, ... $4a - 1$
3. Then their Product is . . . . . $4aa - a$

Whence it is evident, that if the first number a be added to the said Product, the summ is a Square, to wit, $4aa$.

4. It remains, that the second number and the summ of both being severally added to the said Product may make a Square; but the second number added to the Product makes $4aa + 3a - 1 = \square$; and the summ of both numbers, together with their Product, makes $4aa + 4a - 1$; therefore . . . . . $4aa + 3a - 1 = \square$, $4aa + 4a - 1 = \square$

5. Which Duplicate equality may be resolved by the method before explained in the preceding twentieth and twenty-first Questions. For the difference of those two Algebraick quantities which are to be equated to Squares is a, which is to be divided into two such quantities that the Product of their multiplication may make a, and that both in the half-summ and in the half-difference of those two quantities there may be found $2a$; but such are the quantities $4a$ and $\frac{1}{4}$, whose Product is a; also the half of their summ is $2a + \frac{1}{8}$, and the half-difference is $2a - \frac{1}{8}$: then by equating the Square of $2a + \frac{1}{8}$ to $4aa + 4a - 1$, or the Square of $2a - \frac{1}{8}$ to $4aa + 3a - 1$, from either of those Equations the value of a will be found $\frac{65}{224}$. Therefore the first number shall be $\frac{65}{224}$, and the second $\frac{16}{224}$; which numbers will solve the Question, as may easily be proved.

QUEST. 37. (Quæst. 22. Lib. 3. *Diophant.*)

To find four such numbers, that every one of them being added to, and subtracted from the Square of the summ of them all; as well the four summs as the four remainders shall be Squares.

RESOLUTION.

1. In every right-angled Triangle, if the Square of the Hypothenusal be increased or lessened by the quadruple of the Area, that is, by the double Product of the multiplication of the two sides about the right-angle, it makes a Square, (which Theorem is made manifest at the end of the Resolution.) Therefore the chief scope is to find four right-angled Triangles in numbers having equal Hypothenusals: But those may be found out thus;

First, (by the Canon in *Observat.* 8. *Resolut.* 2. of *Quest.* 1. of this *Book*,) find out two unlike right-angled Triangles in numbers, such are these,

5 , 4 , 3
13 , 12 , 5

2. Then multiply the three sides of the first Triangle by the Hypothenusal of the second, also multiply the three sides of the latter Triangle by the Hypothenusal of the first; so the Products will give these two right-angled Triangles having equal Hypothenusals,

viz. 65 , 52 , 39
65 , 60 , 25

3. By the help of the two unlike right-angled Triangles first found, to wit, 5, 4, 3 and 13, 12, 5, the Canon in *Observat.* 4. upon *Resolut.* 2, and 3. of *Quest.* 2. of this *Book* will give two other right-angled Triangles unlike to those in the second step, but having the same Hypothenusal 65, to wit, these;

65 , 56 , 33
65 , 16 , 63

4. Then assume a to represent a number unknown, and let it be multiplied by every one of the sides of those four Triangles having 65 for a common Hypothenusal, so the Products will be these,

$65a$, $52a$, $39a$
$65a$, $60a$, $25a$
$65a$, $56a$, $33a$
$65a$, $16a$, $63a$

5. Now for the summ of the four numbers sought by the Question put . . . . $65a$
6. Therefore the Square of the said summ is . . . . . $4225aa$
7. Then for the first number sought, take the quadruple of the Area of the first of the four Triangles in the fourth step, *viz.* multiplying $52a$ by $39a$, take the double of that Product for the first number, to wit, $4056aa$

8. In

8. In like manner for the second number take the double Product of $60a$ by $25a$, that is, ... $3000aa$
9. And for the third number take the double Product of $56a$ by $33a$, that is, ... $3696aa$
10. And for the fourth number take the double Product of $63a$ by $16a$, that is, ... $2016aa$
11. The summ of the four numbers expresst in the seventh, eighth, ninth and tenth steps is . . . . . $12768aa$
12. Which summ must be equal to $65a$, which in the fifth step was assumed for the summ of the four numbers sought, hence this Equation, ... $12768aa = 65a$
13. Which Equation reduced, gives . . . . . $a = \frac{65}{12768}$
14. Therefore from the thirteenth, seventh, eighth, ninth and tenth steps the four numbers required will be found these, *viz.* $\frac{4225}{163021824}$, $\frac{3380}{163021824}$, $\frac{1521}{163021824}$, $\frac{1296}{163021824}$, $\frac{208}{163021824}$; which four numbers will solve the Question, as will be evident to him that will take the pains of forming the Proof.

But because the Resolution of this Question is chiefly grounded upon a Theorem taken for granted in the first step, I shall here demonstrate the same

THEOREM.

15. If the Square of the Hypothenusal of a right-angled Triangle be increased or lessened by the quadruple of the Area, (that is, the double Product or the multiplication of the sides about the right-angle,) the summ, as also the remainder shall be a Square. For,

If a and e = the sides about the right-angle of a right-angled Triangle,
Then $2ae$ = the double Product of those sides,
Also aa and ee = the Squares of those sides,
And $aa + ee$ = the Square of the Hypothenusal, (*per* 47. *Prop.* 1. *Elem.* Euclid.)

Hence that which the Theorem asserts is manifest,

viz. $aa + ee + 2ae = \square$, whose side is $a + e$,
$aa + ee - 2ae = \square$, whose side is $a - e$.

It is also evident from the premisses, that this 37th Question may be extended to five, six, or as many numbers as shall be desired; but first of all, so many numbers as are required, so many right-angled Triangles in numbers must be found out having equal Hypothenusals; which Triangles in whole numbers may be readily discovered by the method delivered in *Observat.* 13. upon *Resolut.* 2. of *Quest.* 1. of this *Book*.

QUEST. 38.

[*This is* Quest. 20. *of the fourth Book of* Vieta's *Zeteticks, and the same with* Quest. 3. *in* Bachet's *Comment upon the fourth Book of* Diophantus.] Quest. 2. of

Two cube-numbers being given, such, that the double of the lesser exceeds the greater, to find two other cube-numbers whose difference shall be equal to the difference of the given Cubes. (But how to perform this when the double of the lesser Cube is less than the greater, I shall hereafter shew in *Quest.* 42.)

RESOLUTION.

1. Let the sides of the given Cubes be . . . d the greater, b the lesser.
2. Then the Cubes of those sides are . . . d^3 and b^3
3. And the difference of the said Cubes is . $d^3 - b^3$
4. For the side of the lesser Cube sought put . $a - d$
5. And for the side of the greater Cube sought put . . . . . $\frac{dd}{bb}a - b$
6. Therefore the greater Cube sought is . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$
7. And the lesser Cube sought is . . . . $aaa - 3daa + 3dda - d^3$
8. Therefore the difference of the two Cubes sought is . . . . . $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

9. Which difference must be equal to the difference of the given Cubes, therefore;
$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

10. From that Equation, after due Reduction, this ariseth, *viz.* . . . . . $a = \frac{3d^4b^3 - 3db^6}{d^6 - b^6}$

11. And by reducing the Fraction in the latter part of the last preceding Equation into its least Terms, by the common Divisor $d^3 - b^3$, it gives . . . . . $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the first, fourth, fifth and eleventh steps, the sides of the Cubes sought will be found equal to these known quantities,

viz.
$\frac{2db^3 - d^4}{d^3 + b^3}$ = the lesser side;
$\frac{2bd^3 - b^4}{d^3 + b^3}$ = the greater side.

13. The same sides will be produced, if you put $a - b$ for the side of the greater of the Cubes sought, and $\frac{bb}{dd}a - d$ for the lesser side, (instead of the Positions in the fifth and fourth steps;) and it's evident that each of the sides found out in the twelfth step will be greater than nothing, if $2b^3$ exceeds d^3, (that is, if the double of the lesser of the two Cubes given exceeds the greater, as the Question presupposeth.

The twelfth step affords this following

CANON.

14. Multiply the excess of the double of the lesser of the two Cubes given above the greater, by the side of the greater; multiply also the excess of the double of the greater Cube above the lesser, by the side of the lesser: then divide each of those Products by the summ of the given Cubes, and the Quotients shall be the sides of the Cubes sought.

Examples in Numbers.

15. Let two Cubes be given, such, that the double of the lesser exceeds the greater, as, ... $125 = d^3$, and $64 = b^3$

16. The sides of which Cubes are . . . . . : $5 = d$, and $4 = b$

17. Then by the Canon, $\frac{2db^3 - d^4}{d^3 + b^3} = \frac{1}{63}$, $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}$ the sides of the Cubes sought.

18. The Cubes of those sides $\frac{1}{63}$ and $\frac{248}{63}$ are $\frac{1}{250047}$ and $\frac{15252992}{250047}$, whose difference is 61, which is equal to the difference of the two given Cubes 125 and 64.

19. In like manner, if these two Cubes be given, to wit, 1728 and 1000, whose difference is 728, the foregoing Canon will give $\frac{408}{341}$ and $\frac{1270}{341}$, the sides of two Cubes whose difference is 728.

Observations upon Quest. 38.

First, the chief scope in the Resolution of this Question is, to raise an Equation between some number of *aaa* and some number of *aa*, that *a* may be found equal to a Rational number; to which purpose, the side of one of the Cubes sought may be feigned to be $a -$ one of the sides of the given Cubes, and the other side sought some number of $a -$ the other side given; but this latter number of *a* must be such as will cause equal numbers of *a* to arise in the Cubes of those feigned sides, that when the lesser of the feigned Cubes is subtracted from the greater, the numbers of *a* may vanish, and then the Remainder being equated to the difference of the given Cubes, this difference will likewise vanish, (because 'tis also found in the difference of the feigned Cubes,) and an Equation remain between some number of *aaa* and some number of *aa*. Now to cause that effect, supposing (as before in the Resolution) *d* to represent the side of the greater Cube given, and *b* the side of the lesser, we may put $a - d$ for the side of the lesser of the Cubes sought, and then the greater side must necessarily be $\frac{dd}{bb}a - b$; or $a - b$ may be put for the greater side

sought, and then the lesser must be $\frac{bb}{dd}a - d$; from either of which ways of framing the Positions there will arise, after due Reduction, an Equation between *aaa* and *aa*, whence *a* will be found equal to a Rational number. All which will be manifest to him that diligently examines the preceding Resolution.

Secondly, if two pairs of Cubes which shall have equal differences be desired in whole numbers, they may easily be found out by the help of the foregoing Canon, in this manner, *viz.* let *d* and *b* represent the sides of two such Cubes, that the double of the Cube of the lesser side *b* exceeds the Cube of the greater side *d*, then the said Canon gives this Equation,

$$\left\{\begin{array}{l} + \text{ Cube of } \frac{2ba^3 - b^4}{a^3 + b^3} \\ - \text{ Cube of } \frac{2da^3 - d^4}{a^3 + b^3} \end{array}\right\} = \text{Cube of } d - \text{Cube of } b.$$

Now to contract that Equation, suppose *f*, *h* and *g* to be equal to the Numerators and common Denominator, so that Equation will be converted into this, *viz.*
$$\frac{f^3}{g^3} - \frac{h^3}{g^3} = d^3 - b^3.$$

Whence, by multiplying every Term by the Denominator g^3, this Equation is produced,
viz. $f^3 - h^3 = g^3d^3 - g^3b^3$.

That is, $\left\{\begin{array}{l} + \text{ Cube of } f \\ - \text{ Cube of } h \end{array}\right\} = \text{Cube of } gd - \text{Cube of } gb.$

In which last Equation, if instead of *f*, *h* and *g*, you take $2bd^3 - b^4$, $2db^3 - d^4$, and $d^3 + b^3$, which were before supposed equal to *f*, *h* and *g* respectively, this following Equation will arise, *viz.*
$$\left\{\begin{array}{l} + \text{ Cube of } \overline{2bd^3 - b^4} \\ - \text{ Cube of } \overline{2db^3 - d^4} \end{array}\right\} = \left\{\begin{array}{l} + \text{ Cube of } \overline{d^4 + db^3}. \\ - \text{ Cube of } \overline{ba^3 + b^4}. \end{array}\right.$$

Which last Equation gives this following

CANON.

First, take two such Cubes in whole numbers that the double of the lesser may exceed the greater, and multiply the excess of the double of the greater above the less by the side of the lesser Cube; secondly, multiply the excess of the double of the lesser Cube above the greater by the side of the greater Cube; thirdly, multiply the summ of the same Cubes by the side of the greater; fourthly, multiply the summ of the said Cubes by the side of the lesser: then the difference of the Cubes of the first and second Products shall be equal to the difference of the Cubes of the third and fourth Products.

An Example in Numbers.

Let two such Cubes be taken, that the double of the lesser exceeds the greater, as, ... $125 = ddd$, $64 = bbb$

The sides of which Cubes are . . . . . $5 = d$, $4 = b$

Then by working according to the directions of the last preceding Canon, the four Products, that is, the sides of the four Cubes sought, in their least terms will be found these, to wit, 248, 5, 315, 252

Which four numbers will satisfie the Proposition; for the difference between the Cubes of 248 and 5 is equal to the difference between the Cubes of 315 and 252, as may easily be proved.

Hence it is easie to find four Cubes in whole numbers, such, that the summ of two of them shall be equal to the summ of the other two; for if two pairs of Cubes be found out by the last preceding Canon, such, that the first pair hath the same difference as the latter, then the summ of the greater Cube of the first pair and the lesser of the latter, shall be equal to the summ of the lesser Cube of the first pair and the greater of the latter.

Thirdly, *Albert Girard* (in his Comment in *Simon Stevin's* Arithmetick, upon the 19th of the fifth Book of *Diophantus,*) observes, but doth not demonstrate, that the Cubes found out by the preceding *Quest.* 38. are always less than the Cubes given; which property, since it will be useful in the following *Quest.* 39. I shall here demonstrate.

Suppose

Suppose
- $d = 5$, $b = 4$ } the ſides of two Cubes, ſuch, that the double of the leſſer exceeds the greater.
- $d^3 = 125$, $b^3 = 64$ } the Cubes of thoſe ſides.
- $d^3 - b^3 = 61$ } the difference of the ſame Cubes.
- $2b^3 \sqsubset d^3$

Then by the Canon in *Sect.* 14. *Queſt.* 38. the ſides of two Cubes whoſe difference is equal to the difference of the given Cubes, whoſe ſides are d ($=5$) and b ($=4$) will be found theſe that follow, to wit,

$$\frac{2db^3 - d^4}{d^3 + b^3} = \frac{5}{63}; \text{ and } \frac{2bd^3 - b^4}{d^3 + b^3} = \frac{248}{63}.$$

Now becauſe the Cubes given and found out have equal differences, if it be proved that the greater Cube found out ſhall be leſs than the greater Cube given, then conſequently the leſſer Cube found out ſhall be leſs than the leſſer Cube given: But that the ſide of the greater Cube found out, is leſs than the ſide of the greater Cube given, (and by conſequence the greater Cube found out leſs than the greater Cube given,) I prove thus;

The greater ſide found out (as before) is . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3}$

Therefore we muſt demonſtrate that . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3} \sqsupset d$

Demonſtration.

By ſuppoſition, ... $d \sqsubset b$

Therefore by multiplying d and b ſeverally by bb, it follows, that . . . . . $dbb \sqsubset b^3$

And by adding b^3 to each part, ... $b^3 + dbb \sqsubset 2b^3$

By ſuppoſition . . . . . $d^3 \sqsupset 2b^3$

Therefore from the two laſt preceding ſteps, ... $d^3 \sqsupset b^3 + dbb$

And by multiplying each part in the laſt ſtep by b, ... $bd^3 \sqsupset b^4 + db^3$

But by multiplying each number in the firſt ſtep of this Demonſtration by d^3, ... $bd^3 \sqsupset d^4$

Therefore by comparing the ſumm of the numbers in the firſt parts of the two laſt preceding ſteps, to the ſumm of thoſe in the latter parts, ... $2bd^3 \sqsupset d^4 + b^4 + db^3$

And by ſubtracting b^4 from each part of the laſt preceding ſtep, ... $2bd^3 - b^4 \sqsupset d^4 + db^3$

Wherefore by dividing each part of the laſt ſtep by $d^3 + b^3$, it's manifeſt that . . . . . $\frac{2bd^3 - b^4}{d^3 + b^3} \sqsupset d$

Which was to be demonſtrated.

Having proved that the greater of the two ſides found out by the Canon before diſcovered for reſolving *Queſt.* 38. is leſs than the greater of the two ſides given, it follows, that the Cube of that ſide found out is leſs than the Cube of the greater ſide given, and that the leſſer Cube found out is leſs than the leſſer Cube given, (becauſe by Conſtruction the two Cubes found out have the ſame difference as the Cubes given.) Therefore the truth of the property before affirmed is manifeſt.

Fourthly, if two pairs of numbers have equal differences, the leſſer number of the leſſer pair ſhall have leſſer Reaſon (or Proportion) to the greater number of the ſame pair, than the leſſer number of the greater pair hath to the greater number of this pair. To make this manifeſt,

Suppoſe
- $b = 8$, $c = 6$ } two unequal numbers taken at pleaſure.
- $b - c = 2$ } the difference of thoſe numbers.
- $d = 5$ } a number leſs than $c (=6)$ the leſſer of the two numbers firſt taken.
- $b - d = 3$, $c - d = 1$ } two numbers whoſe difference is equal to the difference of the numbers firſt taken.

Now I ſay that the Reaſon (or Proportion) of $c - d$ to $b - d$ is leſs than that of c to b; therefore,

The Propoſition to be demonſtrated, is, that . . . . . $\frac{c - d}{b - d} \sqsupset \frac{c}{b}$

Demon-

Demonſtration.

By ſuppoſition, ... $c \sqsupset b$

Therefore by multiplying c and b ſeverally by d, it follows that . . . . . $dc \sqsupset bd$

And by adding bc to each part in the laſt ſtep, ... $bc + dc \sqsupset bc + bd$

And by ſubtracting bd from each part, ... $bc + dc - bd \sqsupset bc$

And by ſubtracting dc from each part in the laſt preceding ſtep, ... $bc - bd \sqsupset bc - dc$

And by dividing each part of the laſt ſtep by $b - d$, ... $\frac{bc - bd}{b - d} \sqsupset c$

Wherefore by dividing each part of the laſt preceding ſtep by b, ... $\frac{c - d}{b - d} \sqsupset \frac{c}{b}$

Which was to be demonſtrated.

Fifthly and laſtly, from the preceding 3d and 4th Obſervations we may deduce this

COROLLARY.

If two cube-numbers be given, ſuch, that the double of the leſſer exceeds the greater, then (by the help of the preceding Canon in *Sect.* 14. *Queſt.* 38.) two cube-numbers may be found out, whoſe difference ſhall be equal to the difference of the given Cubes, and the double of the leſſer of the Cubes found out ſhall be leſs than the greater of them.

For if two given Cubes, (which I ſhall call the firſt pair) be ſuch that the double of the leſſer exceeds the greater, we may by the ſaid Canon find out a ſecond pair of Cubes, whoſe difference ſhall be equal to the difference of the firſt pair, and (by *Obſervat.* 3.) the Cubes of the ſecond pair ſhall be leſs than thoſe of the firſt pair, (*viz.* the greater Cube of the ſecond pair ſhall be leſs than the greater Cube of the *firſt* pair, and the leſſer Cube leſs than the leſſer;) and the leſſer Cube of the ſecond pair ſhall have leſs Reaſon or proportion to the greater Cube of the ſame pair, than the leſſer Cube of the firſt pair hath to the greater of the ſame pair, (by *Obſervat.* 4.) But if the double of the leſſer Cube of the ſecond pair doth yet happen to exceed the greater of the ſame pair, then by the help of the ſecond pair of Cubes and the ſaid Canon, we may find a third pair of Cubes, whoſe difference ſhall be equal to the common difference of the firſt and ſecond pairs; and by proceeding in like manner, the double of the leſſer of the two Cubes found out will at length neceſſarily be leſs than the greater, becauſe (as before hath been proved,) the leſſer Cube of each pair found out hath leſs proportion to the greater of the ſame pair, than the leſſer of the next precedent pair (by which the latter were found out) hath to the greater.

QUEST. 39.

Two cube-numbers being given, ſuch, that the double of the leſſer is either greater or leſs than the greater, to divide the difference of the given Cubes into two Rational cube-numbers.

Preparation.

1. When the double of the leſſer of the given Cubes exceeds the greater, two others muſt be found out, (according to the directions following the Corollary in *Obſervat.* 5. *Queſt.* 38.) ſuch, that the difference of theſe Cubes may be equal to the difference of thoſe given, and that the double of the leſſer of the Cubes found out may be leſs than the greater. Then two cube-numbers being given or found out, ſuch, that the double of the leſſer is leſs than the greater, their difference may be divided into two Rational cube-numbers by the following Reſolution, (which is the ſame in ſubſtance with that of the 18th of the 4th Book of *Vieta's Zeteticks*, and of the firſt Queſtion of *Bachet* in his Comment upon the fourth Book of *Diophantus.*) *Queſt. 2. of*

RESOLUTION.

2. Let the ſides of the given Cubes (qualified as above is ſuppoſed) be . . . . . d the greater, and b the leſſer.
3. Therefore the Cubes of thoſe ſides are . . . . d^3 and b^3
4. And the difference of the ſaid Cubes is . . . $d^3 - b^3$
5. For the ſide of one of the Cubes ſought put . . $d - a$

6. And

6. And for the ſide of the other Cube ſought put . . . . . $\frac{dd}{bb}a - b$

7. Therefore the firſt Cube is . . . . . $-a^3 + 3daa - 3dda + d^3$

8. And the latter Cube is . . . . . $\frac{d^6}{b^6}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

9. Therefore the ſumm of thoſe Cubes is . . $\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3$

10. Which ſumm muſt be equal to the difference of the given Cubes; therefore,

$$\frac{d^6a^3 - b^6a^3}{b^6} - \frac{3d^4}{b^3}aa + 3daa + d^3 - b^3 = d^3 - b^3.$$

11. Which Equation, after due Reduction, gives $a = \frac{3db^3}{d^3 + b^3}$

12. Therefore from the firſt, fifth, ſixth and eleventh ſteps, the ſides of the two Cubes ſought will be found equal to theſe known Quantities,

Viz. $\frac{d^4 - 2db^3}{d^3 + b^3}$ = the firſt ſide,

$\frac{2bd^3 - b^4}{d^3 + b^3}$ = the other ſide.

13. The ſame ſides will be produced, if inſtead of the Poſitions in the fifth and ſixth ſteps, there be put $a - b$ and $d - \frac{bb}{dd}a$ for the ſides of the Cubes ſought. And 'tis evident that each of the ſides found out in the twelfth ſtep will be greater than nothing if $2b^3$ be leſs than d^3, that is, if the double of the leſſer of the two Cubes given be leſs than the greater, as is ſuppoſed in the Preparation to the Reſolution of this Queſtion. The ſides in the twelfth ſtep, being expreſt by words, will give this

CANON.

14. Multiply the exceſs of the greater of the two given Cubes above the double of the leſſer by the ſide of the greater; multiply alſo the exceſs of the double of the greater Cube above the leſſer by the ſide of the leſſer; then divide each of thoſe Products by the ſumm of the ſaid Cubes, and the Quotients ſhall be the ſides of the Cubes ſought.

Example 1. *in Numbers.*

15. Let two ſuch Cubes be given, that the double of the leſſer is leſs than the greater, as, ... $8 = d^3$ and $1 = b^3$

16. The ſides of thoſe Cubes are . . . . . $2 = d$ and $1 = b$

17. Then by the Canon, $\frac{d^4 - 2db^3}{d^3 + b^3} = \frac{4}{3}$, $\frac{2bd^3 - b^4}{d^3 + b^3} = \frac{5}{3}$ the ſides of the Cubes ſought.

The Cubes of thoſe ſides $\frac{4}{3}$ and $\frac{5}{3}$, are $\frac{64}{27}$ and $\frac{125}{27}$, whoſe ſumm is 7, which is equal to the difference of the two given Cubes 8 and 1, as was required by the Queſtion.

Example 2.

18. Let it be required to divide 61, which is the difference of theſe Cubes 125 and 64, into two Rational cube-numbers.

Here, becauſe the double of the leſſer Cube exceeds the greater, the Canon above expreſt in *Sect.* 14. is of no force; therefore by the help of the given Cubes, (according to the directions following the Corollary in *Obſervat.* 5. *Queſt.* 38.) two other Cubes muſt be found out, ſuch, that their difference may be equal to 61, to wit, the difference of the given Cubes 125 and 64, and that the double of the leſſer of the two Cubes found out may be leſs than the greater of them: But two ſuch Cubes are theſe, *viz.* $\frac{1122122922}{2500047}$ and $\frac{-1221}{2500047}$, whoſe ſides are $\frac{2428}{63}$ and $\frac{-1}{63}$; then uſing theſe Cubes as the Canon in the preceding *Sect.* 14. of this *Queſt.* 39. doth direct, you will find $\frac{1121212221}{960946371}$ and $\frac{1782682216}{960946371}$ for the ſides of the Cubes required; for the ſumm of the Cubes of the ſaid ſides is 61, which is equal to the difference of the given Cubes, 125 and 64.

19. Hence

19. Hence it is eaſie to find four cube numbers, the greateſt of which ſhall be equal to the ſumm of the other three; for when the difference of two given Cubes is divided into two rational Cubes, theſe two, together with the leſſer of the two given Cubes ſhall make the greater Cube given. But if four ſuch Cubes be deſired in whole numbers, they may be readily found out by the following Canon, which is raiſed by the like manner of arguing as was before uſed in the ſecond Obſervation upon the preceding *Queſt.* 38.

CANON.

20. Firſt, take two Cubes in whole numbers, with this caution, That the double of the leſſer may be leſs than the greater, and multiply the exceſs of the greater Cube above the double of the leſſer by the ſide of the greater; ſecondly, multiply the exceſs of the double of the greater Cube above the leſs by the ſide of the leſſer; thirdly, multiply the ſumm of the ſaid Cubes by the ſide of the greater; fourthly, multiply the ſumm of the ſame Cubes by the ſide of the leſſer: Then the ſumm of the Cubes of the firſt, ſecond and fourth Products ſhall be equal to the Cube of the third Product.

An example in Numbers.

21. Let two ſuch Cubes be taken, that the double of the leſſer is leſs than the greater, as, ... $8 = ddd$, $1 = bbb$

22. The ſides of which Cubes are . . . . . $2 = d$, $1 = b$

23. Then by the laſt preceding Canon, the ſides of the four Cubes ſought, in their leaſt terms will be found theſe, *viz.* 4, 5, 6, 3

24. I ſay the ſumm of the Cubes of 4, 5 and 3 is equal to the Cube of 6, *viz.* . . . . . $64 + 125 + 27 = 216$

QUEST. 40.

To divide any cube-number, ſuppoſe 8, into three Rational cube-numbers.

RESOLUTION.

Take any cube-number leſs than the given Cube 8, as 1; then (by the preceding *Queſt.* 39.) divide 7 the difference of thoſe Cubes into two Cubes, ſuppoſe into theſe, $\frac{64}{27}$ and $\frac{125}{27}$, whoſe ſumm 7 is equal to the difference of the given Cubes 8 and 1.

Therefore $\frac{64}{27} + \frac{125}{27} = 8 - 1$,

$\frac{64}{27} + \frac{125}{27} + 1 = 8$.

Whereby 'tis manifeſt that three Cubes, to wit, $\frac{64}{27}$, $\frac{125}{27}$ and 1 are found out, whoſe ſumm is equal to the given Cube 8, as was required.

Hence you may eaſily perceive a way to divide a given Cube into any odd number of Cubes; as to divide a Cube into five Cubes, firſt divide the given Cube into three Cubes, and then divide one of thoſe three into three Cubes, ſo the other two, with the three Cubes laſt found out are five Cubes, whoſe ſumm is equal to the Cube firſt given; in like manner you may divide a Cube into 7, 9, 11, 13, *&c.* Cubes. But there is not any Rational cube-number that can be divided into two Rational Cube numbers; which negative Propoſition the Learned Dr *Wallis* hath demonſtrated.

QUEST. 41.

[*This is the* 19th *of the* 4th *Book of* Vieta's *Zeteticks, and the ſame with* Queſt. 2. *of* Bachet *in his Comment upon* Queſt. 2. *of the* 4th *Book of* Diophantus.]

Two cube-numbers being given, to find two other cube-numbers whoſe difference ſhall be equal to the ſumm of the given Cubes.

RESOLUTION.

1. Let the ſides of the given Cubes be . . . . . d the greater, b the leſſer.

2. Therefore the Cubes of thoſe ſides are . . . . . d^3 and b^3

3. For the ſide of the greater Cube ſought put . . . . $a + d$

4. And for the ſide of the leſſer Cube ſought put . . . $\frac{dd}{bb}a - b$

5. Therefore the greater Cube ſought is . . . $\}$ $a^3 + 3daa + 3dda + d^3$

6. And the leſſer Cube ſought is . . . . $\}$ $\frac{d^3}{b^3}aaa - \frac{3d^4}{b^3}aa + 3dda - b^3$

7. Therefore the difference of the Cubes ſought is . . . . . $\}$ $\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3$

8. Which difference muſt be equal to the ſumm of the given Cubes, therefore

$$\frac{b^6a^3 - d^6a^3}{b^6} + \frac{3d^4}{b^3}aa + 3daa + d^3 + b^3 = d^3 + b^3.$$

9. Which Equation, after due Reduction, gives $\}$ $a = \frac{3db^3}{d^3 - b^3}$

10. Therefore from the firſt, third, fourth and tenth ſteps, the ſides of the two Cubes ſought will be found equal to theſe known quantities,

Viz. $\}$ $\frac{2db^3 + d^4}{d^3 - b^3}$ = the greater ſide,
$\frac{2bd^3 + b^4}{d^3 - b^3}$ = the leſſer ſide.

The ſame ſides will be produced, if inſtead of the Poſitions in the third and fourth ſteps there be put $d + \frac{bb}{dd}a$ and $a - b$; and thoſe ſides above-expreſt by Letters give this

CANON.

11. Add the greater of the two Cubes given to the double of the leſſer, and multiply the ſumm by the ſide of the greater Cube; add alſo the leſſer Cube to the double of the greater, and multiply this ſumm by the ſide of the leſſer Cube; laſtly, divide each of thoſe Products by the difference of the given Cubes, and the Quotients ſhall be the ſides of the Cubes ſought.

An Example in Numbers.

Let two Cubes be given, as, ... $\}$ $8 = ddd$, and $1 = bbb$
The ſides whereof are . . . . . $\}$ $2 = d$, and $1 = b$

Then by the Canon, $\}$ $\frac{2db^3 + d^4}{d^3 - b^3} = \frac{32}{7}$, $\frac{2bd^3 + b^4}{d^3 - b^3} = \frac{17}{7}$ $\}$ the ſides of the Cubes ſought.

The Cubes of thoſe ſides $\frac{32}{7}$ and $\frac{17}{7}$ are $\frac{32768}{343}$ and $\frac{4913}{343}$, whoſe difference 9 is equal to the ſumm of the given Cubes 8 and 1.

12. Hence 'tis eaſie to find out four cube-numbers, the greateſt of which ſhall be equal to the ſumm of the other three: For when by this Queſtion two Cubes are found out, having their difference equal to the ſumm of two given Cubes, the leſſer of the two Cubes found out, together with the two given Cubes ſhall be equal to the greater of the Cubes found out. But if four ſuch Cubes be deſired in whole numbers, they may be readily found out by the following *Canon*, which is raiſed by the like manner of arguing as was before uſed in *Obſervat.* 2. *Queſt.* 38.

CANON.

13. Firſt take any two Cubes in whole numbers, add the greater to the double of the leſſer and multiply the ſumm by the ſide of the greater; ſecondly, add the leſſer Cube to the double of the greater, and multiply this ſumm by the ſide of the leſſer; thirdly, multiply the difference of thoſe Cubes by the ſide of the greater; fourthly, multiply the ſaid difference by the ſide of the leſſer Cube: Then the ſumm of the Cubes of the three latter Products ſhall be equal to the Cube of the firſt Product.

An Example in Numbers.

Let two Cubes in whole numbers be taken at pleaſure, as $\}$ $8 = ddd$, and $1 = bbb$
The ſides of thoſe Cubes are . . . . . $\}$ $2 = d$, and $1 = b$
Then by the *Canon* laſt afore-going, the ſides of the four Cubes ſought will be found theſe, to wit, . . . $\}$ 20, 17, 14, 7
I ſay the Cube of 20 is equal to the ſumm of the Cubes of 17, 14 and 7, *viz.* . . . . . $\}$ $8000 = 4913 + 2744 + 343$

QUEST. 42.

QUEST. 42.

Two cube-numbers being given, ſuch, that the double of the leſſer is leſs than the greater, to find out two other Cubes whoſe difference ſhall be equal to the difference of the given Cubes. (But how this is to be done when the double of the leſſer Cube exceeds the greater, hath already been ſhewn in *Queſt.* 38.)

RESOLUTION.

1. Let there be two Cubes given, to wit, ... $\}$ $ddd = 8$, $bbb = 1$

2. By the Canon in *Sect.* 14. of the foregoing *Queſt.* 39. find out two Cubes whoſe ſumm ſhall be equal to the difference of the given Cubes, ſuch are theſe, ... $\}$ $ccc = \frac{64}{27}$, $ggg = \frac{125}{27}$

Therefore by that Conſtruction, . . . $\}$ $ddd - bbb = ccc + ggg = 8 - 1 = 7$.

3. By the Canon in *Sect.* 11. of the foregoing *Queſt.* 41. find two Cubes whoſe difference ſhall be equal to the ſumm of the Cubes *ccc* and *ggg*, (found out in the preceding ſecond ſtep,) ſuch are theſe,

$\}$ $kkk = \frac{2024284625}{6128487}$, whoſe ſide is $\frac{1265}{183}$.
$lll = \frac{1981385216}{6128487}$, whoſe ſide is $\frac{1256}{183}$.

Therefore by this Conſtruction, . . . $\}$ $kkk - lll = ccc + ggg = 7$.

4. Therefore from the ſecond and third ſteps, (*per* 1. *Axiom.* 1. *Elem.* Euclid.)

$$kkk - lll = ddd - bbb = 7.$$

I ſay *kkk* and *lll*, that is, $\frac{2024284625}{6128487}$ and $\frac{1981385216}{6128487}$, (whoſe ſides are $\frac{1265}{183}$ and $\frac{1256}{183}$,) will ſolve the Queſtion propoſed; for their difference 7 is equal to the difference of the given Cubes 8 and 1.

Note. Although by the preceding Reſolutions of this and *Queſt.* 38. innumerable pairs of cube-numbers may be found out, ſuch, that the difference of each pair ſhall be equal to the difference of two Cubes given, yet neither of thoſe Reſolutions will find out all the pairs of Cubes that have the ſame difference with two given Cubes; for example, if the Cubes 1728 and 1000 be given, whoſe difference is 728, the Canon in the 14th ſtep of the foregoing *Queſt.* 38. will not find out the Cubes 729 and 1, whoſe difference is 728; although that Canon, with the help of the Reſolution of this *Queſt.* 42. will find out innumerable pairs of Cubes, ſuch, that the difference of each pair ſhall be 728.

QUEST. 43.

To divide a given number 28 compos'd of two cube numbers 27 and 1, into two other Rational cube-numbers.

[*This Queſtion was propounded in* 1657. *by Monſ.* Fermat, (*as appears by an Epiſtolical Commerce printed at* Oxford *in* 1658.) *but his way of ſolving it came not to light, till it was publiſh'd (after his death) among other his Analytical Inventions, by way of Supplement to Monſ.* Bachet's *Comment upon* Diophantus, *printed at* Tholoſe *in* 1670. *yet the very ſame way of ſolving this Queſtion was found out long before by our Learned* Dr John Wallis, (*though, it ſeems, not timely enough to have been inſerted in the little Book above-mentioned,*) *and likewiſe by my ſelf, before I had ſeen or heard of any Solution to the ſaid Queſtion, in ſuch manner as here it follows.*]

RESOLUTION.

1. Let the Cubes 27 and 1, whoſe ſumm makes the given number 28, be repreſented by *ddd* and *bbb*, viz. . . . . $\}$ $ddd = 27$, $bbb = 1$

2. By the Canon in *Sect.* 11. of the foregoing *Queſt.* 41. find two cube-numbers whoſe difference may be equal to 28 the ſumm of the given Cubes 27 and 1, that is, *ddd* and *bbb*,) ſuch are theſe Cubes,

Viz. $\}$ $ggg = \frac{658503}{17576}$, whoſe ſide is $\frac{87}{26}$.
$ccc = \frac{166375}{17576}$, whoſe ſide is $\frac{55}{26}$.

Therefore, ... $\}$ $ggg - ccc = ddd + bbb = 27 + 1 = 28$.

3. By

3. By the foregoing *Queſt.* 39. find out two Cubes whoſe ſumm ſhall be equal to 28 the difference of the two Cubes *ggg* and *ccc*, ſuch are theſe,

$$kkk = \frac{25345232527341298070 2625}{9864820937041015055552}.$$
$$lll = \frac{22762660963735440852831}{9864820937041015055552}.$$

The ſides of which Cubes are theſe, to wit, ... $\begin{cases} k = \frac{61284705}{21446828}. \\ l = \frac{28340511}{21446828}. \end{cases}$

Therefore, $ggg - ccc = kkk + lll$.

4. But by Conſtruction in the 2d ſtep, $ggg - ccc = ddd + bbb = 27 + 1 = 28$.
5. Therefore from the two laſt Equations, (*per* 1. *Axiom.* 1. *l. lem.* Euclid.) $kkk + lll = ddd + bbb = 27 + 1 = 28$.

Whence it is manifeſt that the two Cubes found out, to wit, *kkk* and *lll*, (which with their ſides are before ſeverally expreſt by numbers in the third ſtep,) will ſolve the Queſtion, for their ſumm makes 28, which is the ſumm of the given Cubes 27 and 1. And becauſe by the help of the known Cubes *ggg* and *ccc* in the ſecond ſtep, divers pairs of Cubes having the ſame difference with the laid *ggg* and *ccc* may be found out, (by the 38th or 42d Queſtion aforegoing:) Therefore by the help of any of the pairs of Cubes ſo found out, their difference may be divided into two Cubes whoſe ſumm ſhall be equal to the ſumm of the given Cubes 27 and 1.

Another Example.

Let it be required to divide 9, which is compos'd of the Cubes 8 and 1, into two other Cubes.

RESOLUTION.

1. Let the Cubes 8 and 1, whoſe ſumm makes the given number 9, be represented by *ddd* and *bbb*, viz. . . . . $\begin{cases} ddd = 8 \\ bbb = 1 \end{cases}$
2. By the Canon in *Sect.* 11. *Queſt.* 41. of this *Book*, find out two Cubes whoſe difference may be equal to 9, the ſumm of the given Cubes 8 and 1; ſuch are theſe Cubes,

$$Viz. \begin{cases} ggg = \frac{8000}{343}, & \text{whoſe ſide is } \frac{20}{7}. \\ ccc = \frac{4913}{343}, & \text{whoſe ſide is } \frac{17}{7}. \end{cases}$$

Therefore, $ggg - ccc = ddd + bbb = 8 + 1 = 9$.

3. Then by the preceding 39th Queſtion divide the difference of the Cubes *ggg* and *ccc* into two rational Cubes, *viz.* divide 9 the difference of the Cubes $\frac{8000}{343}$ and $\frac{4913}{343}$ into two Cubes: But here becauſe the double of the leſſer Cube $\frac{4913}{343}$ exceeds the greater $\frac{8000}{343}$, two Cubes muſt firſt be found out, (by the help of the foregoing *Queſt.* 38.) that the difference of theſe may be equal to the difference of thoſe, and that the double of the leſſer of the Cubes found out may be leſs than the greater; ſuch are theſe Cubes,

$$Viz. \begin{cases} mmm = \frac{669559084262639}{7385426376464 71}, & \text{whoſe ſide is } \frac{188479}{90391}. \\ nnn = \frac{48707103808000}{7385426376464 71}, & \text{whoſe ſide is } \frac{36520}{90391}. \end{cases}$$

4. Now foraſmuch as the double of the leſſer of the two Cubes laſt found out is leſs than the greater, we may by the help of the preceding *Queſt.* 39. divide 9 the difference of thoſe Cubes *mmm* and *nnn*, (and likewiſe of *ggg* and *ccc*) into two rational Cubes, whoſe ſides will be found theſe,

$$Viz. \begin{cases} k = \frac{1243617773399009 4836481}{6096238356761 37297449}, \\ l = \frac{4872671717143523 36560}{6096238356761 37297449}. \end{cases}$$

5. Therefore by Conſtruction in the two laſt preceding ſteps, ... $ggg - ccc = kkk + lll$
6. But by Conſtruction in the ſecond ſtep, $ggg - ccc = ddd + bbb = 8 + 1 = 9$
7. Therefore from the two laſt Equations, . . $kkk + lll = ddd + bbb = 8 + 1 = 9$

Thus two Cubes (whoſe ſides *k* and *l* are above expreſt in numbers) are found out, which added together make 9, the ſumm of the given Cubes 8 and 1, as was required.

QUEST. 44.

QUEST. 44.

To divide the double of any given cube-number into four cube-numbers.

For example, let it be required to divide 54 the double of the Cube 27, into four cube-numbers.

RESOLUTION.

1. For the given Cube 27 put *ddd*, *viz.* ſuppoſe . . . . . $ddd = 27$
2. Therefore the double of that Cube is . . . . . $2ddd = 54$
3. Take any cube-number leſs than the given Cube 27, as 1, for which put *bbb*, *viz.* ſuppoſe . . . . . $bbb = 1$
4. By the foregoing *Queſt.* 43. find two Cubes whoſe ſumm may be equal to *ddd* + *bbb*, (to wit, 27 + 1,) ſuppoſe thoſe which ſolved the ſaid *Queſt.* 43. in *Example* 1.

$$Viz. \begin{cases} kkk = \frac{25345232527341298070 2625}{9864820937041015055552}, \\ lll = \frac{22762660963735440852831}{9864820937041015055552}. \end{cases}$$

The ſides of which Cubes are theſe, to wit, ... $\begin{cases} k = \frac{61284705}{21446828}, \\ l = \frac{28340511}{21446828}. \end{cases}$

Therefore by that Conſtruction, ... $ddd + bbb = kkk + lll = 28$

5. By *Queſt.* 39. of this *Book* divide 26 the difference of the Cubes 27 and 1, to wit, *ddd* — *bbb* into two Cubes, ſuppoſe into theſe,

$$Viz. \begin{cases} rrr = \frac{421875}{21952}, & \text{whoſe ſide is } \frac{75}{28}. \\ sss = \frac{148877}{21952}, & \text{whoſe ſide is } \frac{53}{28}. \end{cases}$$

Therefore by this Conſtruction, ... $ddd - bbb = rrr + sss = 26$

6. Therefore by adding together the Equations in the fourth and fifth ſteps, this will ariſe, *viz.* $2ddd = kkk + lll + rrr + sss = 54$

Therefore four Cubes are found out, to wit, *kkk*, *lll*, *rrr* and *sss*, which with their ſides are before expreſt in numbers in the fourth and fifth ſteps, and the ſumm of thoſe Cubes makes 54, which is equal to the double of the Cube 27 firſt given, as was required by the Queſtion.

QUEST. 45. (Quæſt. 17. Lib. 4. *Diophant.*)

To find out three numbers whoſe ſumm may make a Square; and that the ſecond number added to the Square of the firſt may make a Square; alſo, that the third number added to the Square of the ſecond may make a Square; and laſtly, that the firſt number added to the Square of the third may make a Square.

RESOLUTION.

1. For the firſt number ſought put *a* — any known number, as, . . . $a - 1$
2. The Square thereof is . . . . . $aa - 2a + 1$
3. To which Square if + 4*a* be added, (to wit, the double of — 2*a*, but with the contrary ſign +,) it makes a Square, to wit, . . . $aa + 2a + 1$
4. Therefore for the ſecond number put . . . . . $4a$

Whereby one of the conditions in the Queſtion is ſatisfied; for the ſecond number 4*a* added to the Square of the firſt number *a* — 1 makes the Square *aa* + 2*a* + 1, whoſe Root is *a* + 1.

5. Then form a Square from 4*a* + 1, (which is the ſumm of the ſecond number 4*a* and the known number 1 in the firſt aſſumed number *a* — 1, but with the contrary ſign +,) ſo the Square of 4*a* + 1 will be 16*aa* + 8*a* + 1; from which ſubtract the Square of the ſecond number 4*a*, to wit 16*aa*, and put the Remainder 8*a* + 1 for the third number: Whence it is evident, that if this third number be added to the Square of the ſecond, the ſumm is a Square, whereby another of the conditions in the Queſtion is ſatisfied.
6. From the firſt, fourth and fifth ſteps the ſumm of the three numbers ſought is 13*a*, which according to the Queſtion muſt be a Square, let it therefore be equated to ſome Square, *viz.* ſuppoſe $13a = 169aa$, whence $a = \frac{13}{aa}$; now according to this value

value of a, the first number which was put $a - 1$ will be $13aa - 1$, the second number which was put $4a$ will be $52aa$, and lastly, the third number which was assumed $8a + 1$ will be $104aa + 1$. It remains that the Square of the third number $104aa + 1$, to wit, $10816aaaa + 208aa + 1$, added to the first number $13aa - 1$ may make a Square; but it makes $10816aaaa + 221aa$, this therefore must be equated to a Square, or the same divided by aa gives $10816aa + 221$ to be equated to a Square, whose side, to the end that a may be greater than $\sqrt{\frac{1}{13}}$, and consequently $13aa$ greater than 1, may be feigned to be $104a +$ any known number less than $4\frac{3}{10}$, or $104a -$ any known number greater than $60\frac{6}{13}$; let therefore the side of the said Square be feigned $104a + 1$, whence the Square it self is $10816aa + 208a + 1$, which being equated to the aforesaid $10816aa + 221$, will give $a = \frac{21}{13}$. Therefore the positions being resolved, the first number will be $\frac{1666}{2704}$, the second $\frac{1911}{2704}$, the third $\frac{1911}{2704}$; which three numbers will solve the Question, for their summ is $\frac{4488}{2704}$, the Square of the side $\frac{21}{52}$; also the Square of the first number, to wit, $\frac{2775636841}{7311616}$ added to the second makes the Square $\frac{3442580641}{7311616}$ from the side $\frac{58729}{2704}$; moreover the Square of the second, to wit, $\frac{3652264000000}{7311616}$ added to the third makes the Square $\frac{2160112200016}{7311616}$, from the side $\frac{1460004}{2704}$; lastly, the Square of the third, to wit, $\frac{1006818828816}{7311616}$ added to the first makes the Square $\frac{2002828281600}{7311616}$, from the side $\frac{1417460}{2704}$.

QUEST. 46.

To find three numbers, that as well the summ of every two, as of all three, may make a Square.

RESOLUTION.

1. Let b represent any known number, and a some number unknown, then from $a + $ some even number of b, (for avoiding Fractions) as from $a + 2b$ form a Square, which will be . . . $aa + 4ba + 4bb$
2. Then for the first number sought put the two first terms of the said Square, as . . . . . $aa + 4ba$
3. Then take the half of the said $4ba$, to wit, $2ba$, and prefixing the sign $-$ to it, it makes $-2ba$, to which add bb the Square of half the Coefficient $2b$, and take the summ for the second number sought, to wit, ... $-2ba + bb$
4. Subtract bb in the said second number from $4bb$ part of the Square first formed, and add the Remainder $3bb$ to $+ 2ba$, to wit, the same multitude of ba as is in the second number, but with a contrary sign, and put this summ for the third number sought, to wit, ... $+ 2ba + 3bb$
5. Then from the premisses it necessarily follows, that the summ of the first and second numbers (in the second and third steps) makes a Square, to wit, ... $aa + 2ba + bb$
6. And the summ of the second and third numbers (in the third and fourth steps) is a Square, to wit, ... $+ 4bb$
7. Also the summ of all the three numbers is a Square, to wit, ... $aa + 4ba + 4bb$
8. It remains that the summ of the first and third numbers make a Square, but it makes $aa + 6ba + 3bb$, which must be equated to a Square, yet so as the value of a may be less than $\frac{1}{2}b$, to the end that the second number $-2ba + bb$ may be greater than nothing. Now to cause that effect, the side of the said Square may be feigned $-a +$ any number between $\sqrt{3bb}$ and $3b$, (as may be collected from the Canon in *Sect.* 15. *Quest.* 12. of this *Book*:) Let therefore the said side be $-a + 2b$, and then its Square $aa - 4ba + 4bb$ being equated to $aa + 6ba + 3bb$ (the summ of the first and third numbers) this Equation ariseth, to wit,
$$aa + 6ba + 3bb = aa - 4ba + 4bb.$$
9. Whence after due Reduction, the value of a is made known, *viz.* $a = \frac{1}{10}b$
10. Therefore, the positions in the second, third and fourth steps being resolved according to that value of a, the three numbers sought are discovered, to wit,
$$\frac{41}{100}bb \quad , \quad \frac{4}{5}bb \quad \text{and} \quad \frac{16}{5}bb.$$

Hence this

CANON.

CANON.

Take any square number, then $\frac{41}{100}$ of that Square, also $\frac{4}{5}$ of the same Square, and $\frac{16}{5}$ thereof, will give three numbers to solve the Question.

As, for example, if 10 be taken for the side of a Square, then these three numbers will be found out by the Canon, to wit, 41, 80 and 320, which will solve the Question: For the summ of 41 and 80 makes the Square 121, whose side is 11; also the summ of 80 and 320 makes the Square 400, whose side is 20; and the summ of 320 and 41 makes the Square 361, whose side is 19; lastly, the summ of all the three numbers 41, 80 and 320 makes the Square 441, whose side is 21. In like manner you may find out as many Answers in whole numbers as you please, by taking 20, 30, 40 or 50, *&c.* for the side of a Square, and then taking such parts thereof as the Canon directs.

QUEST. 47. (Quæst. 23. Lib. 4. *Diophant.*)

To find three numbers, that if they be severally added to the Solid produced by their continual multiplication, every one of the three summs may be a Square.

(I shall wave Diophantus's *Resolution, and use that of* Fermat *in his Observation upon this Question, which is much easier.)*

RESOLUTION.

1. Let a Square be formed from $a -$ any known number, as from $a - 1$, whose Square is . . . . . $aa - 2a + 1$
2. Then for the Solid of the three numbers sought put the two first terms of that Square, to wit, ... $aa - 2a$
3. And for the first number sought put the last term of the said Square, to wit, ... 1

Whence one of the conditions is satisfied; for if the said first number 1 be added to $aa - 2a$, (that is, the Solid of all the three numbers,) the summ is a Square, to wit, that first formed.

5. For the second number put . . . . . $2a$

This added to the said Solid $aa - 2a$ makes the Square aa, whereby another of the conditions in the Question is satisfied.

5. Then divide $aa - 2a$, (the Solid of all the three numbers) by $2a$ the Product of the first and second, so the Quotient is
$$\frac{1}{2}a - 1, \text{ (the third number.)}$$
6. Which third number added to the Solid of all the three must also make a Square, but it makes
$$aa - \frac{1}{2}a - 1.$$
7. Therefore $aa - \frac{1}{2}a - 1$ must be equated to a Square, yet so, as the value of a may be greater than 2, to the end that the third number $\frac{1}{2}a - 1$ may be greater than nothing: But to cause that effect, the side of the said Square may be feigned $a -$ any number less than 2, but greater than $\frac{1}{4}$; or $a -$ any number greater than 2; let then the said side be feigned $a - 3$, whose Square equated to $aa - \frac{1}{2}a - 1$, will give $a = \frac{23}{9}$. According to which value, the Positions being resolved, the first number sought is 1, the second $\frac{40}{9}$, and the third $\frac{1}{9}$, which will solve the Question: For the solid Product of their multiplication one into another, to wit, $\frac{920}{81}$, taking to it severally the said three numbers, makes the Squares $\frac{1681}{81}$, $\frac{3600}{81}$ and $\frac{289}{81}$.

QUEST. 48. (Quæst. 31. Lib. 4. *Diophant.*)

To find four square numbers, whose summ added to the summ of their sides may make a number given, suppose 12.

RESOLUTION.

Forasmuch as (by the first *Proposition* in the following *Observation* upon this *Quest.*) every Square increased with his side and $\frac{1}{4}$ of unity makes a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square; therefore the summ of the four Squares sought together with four times $\frac{1}{4}$ will make four Squares; but the given number 12 increased with four times $\frac{1}{4}$, to wit, 1, makes 13. Therefore we must divide 13 into four Squares; then if from every one of their sides we subtract $\frac{1}{2}$, there will remain the sides of the four Squares sought. But 13 is compos'd of two Squares 4 and 9; therefore

therefore (by the first Question of this Book) each of these may be divided into two Squares, *viz.* 4 into $\frac{64}{25}$ and $\frac{36}{25}$, and 9 into $\frac{144}{25}$ and $\frac{81}{25}$: now the four Roots of those Squares are $\frac{8}{5}$, $\frac{6}{5}$, $\frac{12}{5}$ and $\frac{9}{5}$, from each of which Roots if $\frac{1}{2}$ be subtracted there will remain the sides of the four Squares sought, to wit, the sides $\frac{11}{10}$, $\frac{7}{10}$, $\frac{19}{10}$ and $\frac{14}{10}$, whose Squares $\frac{121}{100}$, $\frac{49}{100}$, $\frac{361}{100}$ and $\frac{196}{100}$ will solve the Question: For if their summ 7 be added to 5 the summ of their sides, it makes the given number 12; which was required.

Observations upon Quest. 48.

The preceding Resolution depends upon two Propositions, *viz.*

First, if any square number be increased with its side and $\frac{1}{4}$ of unity the summ will be a Square, whose side lessened by $\frac{1}{2}$ of unity gives the side of the former Square. This may be demonstrated thus;

Let there be a Square	aa
Then to that Square add its side and $\frac{1}{4}$ of unitty, to wit,	$a \dotplus \frac{1}{4}$
So the summ makes this Square,	$aa \dotplus a \dotplus \frac{1}{4}$
Whose Root is	$a \dotplus \frac{1}{2}$
From which Root if you subtract	$\dotplus \frac{1}{2}$
The Remainder is the side of the first Square, to wit,	a

Therefore the Proposition is manifest.

Secondly, the said Resolution takes this Proposition for granted, *viz.* That any given whole number increased with 1 may be divided into four Squares; how this may be generally done *Diophantus* doth not shew: But 'tis evident, that if a given square number increased with 1 makes a Square, or a number composed of two Squares, then the summ may easily be divided into four Squares, or into as many as you please, (by the first or second Question of this Book.) But if 13 be given, how shall 13 increased with 1, that is 14, which is neither a Square nor composed of two Squares, be divided into four Squares? This at first sight seems to be a very hard Task, but if the matter be narrowly considered, the difficulty will soon vanish; for 14 is compos'd of three Squares, to wit, 1, 4, 9; wherefore if one of these be divided into two Squares, then consequently 14 is divided into four Squares. But *Fermat* in his Observation upon this *Quest.* 31. of the fourth Book of *Diophantus*, affirms that every whole number is either a Square, or else compos'd of two, three or four Squares, and he there promiseth to give the Demonstration of this and other abstruse Mysteries in Numbers in a particular Treatise. *Bachet* confesseth he could not demonstrate the same, but gives Examples of the certainty thereof in all whole numbers from 1 to 120, and saith he had made experiment of all whole numbers to 325. If this *Prop.* be granted, then the Question may be easily extended to five, six, or as many Squares as you please, without any Determination. But if two Squares only be desired, whose summ with the summ of their sides may make a Square, then after 2 is added to the quadruple of the given number, the summ must be compos'd of two Squares: And if three Squares be sought, then after 3 is added to the quadruple of the given number the summ must be compos'd of three Squares, which conditions are manifest from the first Proposition above expresst.

QUEST. 49.

[*This is the* 34th *of the fourth Book of* Diophantus, *and the* 13th *of the fifth Book of* Vieta's *Zeteticks.*]

To divide a given number x into two such parts, that if the first part be increased with a given number b, and the other part with a given number d; the Product made by the multiplication of the two summs one into the other may be a square number.

RESOLUTION.

1. For the first part sought put	$a - b$
2. Therefore the other part, to the end the summ of the parts may make x, shall be	$x - a \dotplus b$
3. Then (according to the Question) adding b to the first part, the summ is	$\therefore a$
4. And adding d to the second part, the summ is	$x - a \dotplus b \dotplus d$

5. There-

5. Therefore the Product made by the multiplication of those two summs one into the other will be	$xa - aa \dotplus ba \dotplus da$
6. That is	$\overline{x \dotplus b \dotplus d}$ into $a, -aa$

7. Which Product must be equated to a Square, whose side may be feigned sa, and then the Square of sa, to wit, $ssaa$, being equated to the Product in the sixth step, this Equation ariseth, *viz.*

$\overline{x \dotplus b \dotplus d}$ into $a, -aa = ssaa$.

8. Which Equation, after due Reduction, gives	$a = \frac{x \dotplus b \dotplus d}{ss \dotplus 1}$
9. Therefore from the eighth and first steps, the first part sought will be made known, to wit,	$\frac{x \dotplus d - ssb}{ss \dotplus 1}$
10. And from the eighth and second steps the second part sought will be discovered, to wit,	$\frac{ssx \dotplus ssb - d}{ss \dotplus 1}$

But to the end that each of the two parts in the ninth and tenth steps may be greater than nothing, the number s cannot be taken at pleasure, but within the limits hereafter discovered, *viz.*

11. Forasmuch as the numerator in the ninth step requires that	$x \dotplus d \sqsubset ssb$
12. Therefore by dividing each part in the last step by b,	$\frac{x \dotplus d}{b} \sqsubset ss$
13. That is,	$ss \sqsupset \frac{x \dotplus d}{b}$
14. Therefore by extracting the square Root out of each part in the last step,	$s \sqsupset \sqrt{\frac{x \dotplus d}{b}}$
15. Again, the Numerator in the tenth step shews, that	$ssx \dotplus ssb \sqsubset d$
16. Therefore by dividing each part by $x \dotplus b$, it follows that	$ss \sqsubset \frac{d}{x \dotplus b}$
17. Therefore by extracting the square Root out of each part in the last step,	$s \sqsubset \sqrt{\frac{d}{x \dotplus b}}$

18. Thus in the fourteenth and seventeenth steps it is discovered, that for the number s we may take any number between $\sqrt{\frac{x \dotplus d}{b}}$ and $\sqrt{\frac{d}{x \dotplus b}}$, and then the two desired parts whose summ is equal to x the number to be divided will be such as are before expresst in the ninth and tenth steps.

An Example in Numbers.

Suppose $\left\{\begin{array}{l} 4 = x \\ 12 = b \\ 20 = d \end{array}\right\}$ numbers given in the Question;

Whence $\left\{\frac{4}{3} = s\right\}$ a number chosen within the limits in the eighteenth step.

Then by the help of those known numbers, the ninth and tenth steps will give $\frac{256}{25}$ and $\frac{144}{25}$ the two parts sought, whose summ is 4, (or x;) the former of which parts increased with 12 (or b,) and the latter with 20 (or d,) make the two summs $\frac{144}{25}$ and $\frac{196}{25}$; these multiplied one by the other produce a Square whose side is $\frac{168}{25}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 50. (Quæst. 35. Lib. 4. *Diophant.*)

To divide a given number into three numbers, such, that if the Product of the multiplication of the first into the second be increased and lessened by the third, as well the summ as the remainder may be a square number.

RESOLUTION.

1. Let the given number be	6
2. For the third number sought put	a
3. For the second number sought put some known number less than the given number 6, as	2
4. Therefore, because all the three numbers sought must make 6, the first number shall be	$4 - a$

5. Then (according to the Queſtion) the Product of the firſt and ſecond numbers ſought, together with the third, muſt make a Square, *viz.* $8 - a = \square$
6. Alſo the ſame Product leſſened by the third number muſt make a Square, *viz.* . . . . . $8 - 3a = \square$
7. So in the two laſt ſteps we are faln upon a Duplicate equality, but 'tis inexplicable; *Diophantus* therefore ſeeks out another Duplicate equality wherein the numbers prefixt to a may have ſuch proportion to one another as a ſquare number hath to a ſquare number, for then it will be reſolvable like that Duplicate equality which hath been already explain'd in *Queſt.* 35. of this *Book*. Firſt, then inſtead of 2 which was aſſumed for the ſecond number ſought, ſome other number leſs than 6 muſt be taken, ſuch, that if it be increaſed and leſſened by unity, the ſumm may be to the remainder as a ſquare number to a ſquare number; (for if the riſe of 3 and 1, which are prefixt to a in the Duplicate equality above expreſt, be examined, it will appear that 3 ariſeth by adding 1 to 2, and 1 ariſeth by ſubtracting 1 from the ſame number 2.) Therefore let e repreſent ſome number to be taken inſtead of 2 for the ſecond number ſought; then $e+1$ muſt be to $e-1$, as a Square to a Square, ſuppoſe as bb to dd.
   Therefore, as . . . . . $e+1 \,.\, e-1 :: bb \,.\, dd$
8. Therefore by comparing the Product of the extremes to the Product of the means, ... $dde+dd = bbe-bb$
9. Whence after due Reduction, ... $e = \frac{bb+dd}{bb-dd}$
10. But the value of e muſt be leſs than 6, which I ſhall call f, therefore . . . . . $\frac{bb+dd}{bb-dd} \sqsupset f$
11. Therefore by multiplying each part by f, ... $bb+dd \sqsupset f \times bb-dd$:
   Hence this Canon to find out the number e, *viz.*
12. Take any two ſquare numbers whoſe ſumm may be leſs than the Product of their difference multiplied into the number given in the Queſtion; then divide the ſumm of the ſaid Squares by their difference, ſo the Quotient ſhall be the number to be put for the ſecond number ſought by the Queſtion propoſed, for it ſhall be leſs than the number given in the Queſtion; and if it be increaſed with 1 and leſſened by 1, the ſumm ſhall be to the remainder as the greater of the Squares taken is to the leſſer. Therefore I take the Squares 4 and 1 and divide their ſumm 5 by their difference 3, ſo there ariſeth $\frac{5}{3}$ for the ſecond number. Now let the Poſitions be renewed thus, *viz.*
13. The number given to be divided into three numbers is . . . 6
14. For the ſecond number put . . . . . $\frac{5}{3}$
15. And for the third number . . . . . a
16. Then the ſumm of the ſecond and third numbers ſubtracted from 6 (the ſumm of all three) leaves the firſt number . . . $4\frac{1}{3} - a$
17. Now (according to the Queſtion) the Product of the firſt and ſecond numbers together with the third, muſt make a Square, *viz.* . . . . . $4\frac{5}{9} - \frac{5}{3}a = \square$
18. And the ſame Product leſſened by the third number muſt leave a Square, *viz.* . . . . . $4\frac{5}{9} - \frac{8}{3}a = \square$
19. So in the two laſt ſteps there is a Duplicate equality wherein the numbers prefixt to a have ſuch proportion one to another as a Square to a Square, for $\frac{5}{3}$ is to $\frac{8}{3}$ as 1 to 4; and therefore this Duplicate equality may be reſolved like that in the preceding *Queſt.* 35. in this manner, *viz.* Foraſmuch as a ſquare number multiplied by a Square produceth a Square, therefore to take away the Fractions, multiply all the Quantities in the 17th and 18th ſteps by the Denominator 9, ſo this Duplicate equality ariſeth,

$$\textit{viz.} \begin{cases} 65 - 6a = \square \\ 65 - 24a = \square. \end{cases}$$

20. Then the former of thoſe two Equations multiplied by 4 produceth $260 - 24a = \square$, ſo at length this Duplicate equality remains to be reſolved,

$$\textit{viz.} \begin{cases} 260 - 24a = \square \\ 65 - 24a = \square. \end{cases}$$

21. Now the difference of theſe two Equations being 195, I ſeek by *Canon* 2. *Queſt.* 7. of this *Book* two ſuch ſquare numbers that their difference may be 195, and that the greater Square may be leſs than 260. But the only pair of Squares in whole numbers ſo

ſo qualified are 196 and 1, the greater of which being equated to $260 - 24a$, or the leſſer to $65 - 24a$, will give $a = \frac{5}{3}$ for the third number ſought; and conſequently, by the Poſitions in the fourteenth and ſixteenth ſteps, the firſt and ſecond numbers are $\frac{1}{3}$ and $\frac{1}{3}$, which three numbers will ſolve the Queſtion, as is evident by the Proof, for their ſumm is 6; alſo if $2\frac{1}{9}$ the Product of the firſt and ſecond be increaſed with the third number $\frac{8}{3}$ it makes the Square $4\frac{5}{9}$; but if the ſame Product be leſſened by the ſaid $\frac{8}{3}$ it leaves the Square $\frac{1}{9}$.

QUEST. 51. (Quæſt. 36. Lib. 4. *Diophant.*)

To find three numbers, whereof the third may be ſuch a Fraction of unity, that if the firſt number takes from the ſecond ſuch part or parts as the Fraction expreſſeth, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as b to d. Alſo, that the ſecond number taking the ſame part or parts from the firſt, the ſumm may be to the remainder in a given Reaſon, ſuppoſe as f to g. But the Product made by the mutual multiplication of the firſt term of each Reaſon muſt exceed the Product of the latter terms one into the other, *viz.* bf muſt be greater than dg.

Preparation.

Let a and e repreſent the firſt and ſecond numbers, and u the third, or Fraction ſought; then becauſe to take any part or parts of a number, the number muſt be multiplied by the Fraction expreſſing the parts, the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $a + ue \,.\, e - ue :: b \,.\, d$,
2. And . . . . . $e + ua \,.\, a - ua :: f \,.\, g$.
   What are the numbers, a, e, u?

RESOLUTION.

3. By comparing the Product of the extremes to the Product of the means in the firſt Analogy, this Equation is produced, to wit,
   $$da + due = be - bue.$$
4. Likewiſe from the latter Analogy this Equation is produced, *viz.*
   $$ge + gua = fa - fua.$$
5. From the Equation in the third ſtep by tranſpoſition of due, this ariſeth,
   $$da = be - bue - due.$$
6. And by dividing each part of the laſt Equation by d, this ariſeth,
   $$a = \frac{be - bue - due}{d}.$$
7. Then by exchanging a in the fourth ſtep, for that which is equal to a in the laſt Equation, and multiplying all into d, the Equation in the fourth ſtep will be converted into this, *viz.*
   $$\left.\begin{matrix} dge + bgue - bguue \\ - dguue \end{matrix}\right\} = \left\{\begin{matrix} bfe - 2bfue - dfue \\ + bfuue + dfuue \end{matrix}\right.$$
8. From which Equation, after due Reduction, this following Equation ariſeth, wherein u only is unknown, *viz.*
   $$\frac{bg + 2bf + df}{bg + dg + bf + df} u - uu = \frac{bf - dg}{bg + dg + bf + df}.$$
9. Which laſt Equation may be reſolved (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) in this manner, *viz.* half the Coefficient drawn into u in the ſaid Equation is
   $$\frac{\frac{1}{2}bg + bf + \frac{1}{2}df}{bg + dg + bf + df}.$$
10. The Square of the ſaid half Coefficient is
    $$\frac{\frac{1}{4}bbgg + bbff + \frac{1}{4}ddff + bbfg + bdff + \frac{1}{2}bdfg}{bg + dg + bf + df} \text{ into } \frac{bg + dg + bf + df}{bg + dg + bf + df}.$$
11. Then to reduce the known Abſolute quantity which ſolely poſſeſſeth the latter part of the Equation in the eighth ſtep to the ſame Denominator with the Square in the tenth ſtep, I multiply as well the Numerator as the Denominator of the ſaid Abſolute quantity, by its Denominator $bg + dg + bf + df$, and it makes
    $$\frac{bbfg + bbff + bdff - bdgg - ddgg - ddfg}{bg + dg + bf + df} \text{ into } \frac{bg + dg + bf + df}{bg + dg + bf + df}.$$

12. Which

12. Which Fraction laſt above expreſt being ſubtracted from the Fraction in the tenth ſtep will leave this that follows, to wit,

$$\frac{\frac{1}{4}bbgg \dashv \frac{1}{4}ddff \dashv ddgg \dashv \frac{1}{2}bdfg \dashv bdgg \dashv ddfg}{bg \dashv dg \dashv bf \dashv af} \text{ into } \frac{bg \dashv ag \dashv bf \dashv df}{bg \dashv dg \dashv bf \dashv af}.$$

13. The ſquare Root of the Fraction in the twelfth ſtep is

$$\frac{\frac{1}{2}bg \dashv \frac{1}{2}df \dashv dg}{bg \dashv dg \dashv bf \dashv df}.$$

14. Which ſquare Root being added to and ſubtracted from the half-Coefficient in the ninth ſtep, the ſumm and remainder ſhall be the two values of u in the Equation in the eighth ſtep, *viz.*

$$u = 1; \text{ alſo, } u = \frac{bf - dg}{bg \dashv dg \dashv bf \dashv df}.$$

The latter of which values of u, to wit, $\frac{bf - dg}{bg \dashv dg \dashv bf \dashv af}$ is the Fraction ſought by the Queſtion.

15. Then according to the ſaid leſſer value of u, the compound quantity $\frac{b - bu - du}{d}$ into which e is multiplied in the latter part of the Equation in the ſixth ſtep will be reduced into this fractional quantity, to wit, $\frac{bbg \dashv bdg \dashv ddg}{bdg \dashv ddg \dashv bdf \dashv ddf}$, which multiplied into any number taken at pleaſure for the value of e, will give the number a; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a, and the Denominator for e, or any two whole numbers in the ſame proportion with the ſaid Numerator and Denominator, and the leſſer of the two values of u before found for the Fraction ſought.

An Example in Numbers.

16. Let there be given . . . . . $\begin{cases} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{cases}$

17. Then the three numbers ſought will be theſe, *viz.* $\begin{cases} 16 = bbg \dashv 2bdg \dashv ddg \\ 24 = bdg \dashv ddg \dashv bdf \dashv ddf \\ \frac{7}{12} = \frac{bf - dg}{bg \dashv dg \dashv bf \dashv af} \end{cases}$

18. Which three numbers, to wit, 16, 24 and $\frac{7}{12}$ will ſolve the Queſtion, as will be manifeſt by

The Proof.

19. If the firſt number 16 receive $\frac{7}{12}$ of the ſecond number 24, that is, 14, the ſumm 30 will be to the remainder 10 as 3 to 1, (*viz.* as b to d;) Again, if the ſecond number 24 take $\frac{7}{12}$ parts of the firſt number 16, to wit, $9\frac{1}{3}$, the ſumm $\frac{40}{3}$ will be to the remainder $\frac{20}{3}$ as 5 to 1, (to wit, as f to g:) Or inſtead of 16 and 24 you may take 2 and 3, or any two numbers in the ſame Proportion.

20. Again, if $b = 3$. $d = 2$. $f = 4$. and $g = 5$. then the literal quantities in the preceding 17th ſtep will give theſe three numbers, 125, 90 and $\frac{2}{45}$ to ſolve the Queſtion; or inſtead of 125 and 90 you may take 25 and 18, or any two numbers in the ſame Proportion.

Note. If in reducing the Equation in the ſeventh ſtep, bf were ſuppoſed either equal to dg or leſs than dg, there would come forth an Equation wherein the value of u would be either unity or greater than unity; but according to the import of the Queſtion it ought to be leſs than unity, and ſuch is the leſſer value of u in the Equation in the eighth ſtep, where bf is ſuppoſed greater than dg, as the Determination annexed to the Queſtion requires.

QUEST. 52. (Quæſt. 41. Lib. 4. *Diophant.*)

To find two numbers, that the Product of their multiplication may be to their ſumm in a given Reaſon (or Proportion,) ſuppoſe as r to s.

RESO-

RESOLUTION.

1. For the firſt number put . . . . . a
2. And for the ſecond . . . . . e
3. Then their ſumm is . . . . . $a \dashv e$
4. And the Product of their multiplication is . . . . . ae
5. But according to the Queſtion the Product muſt be to the ſumm as r to s, therefore . . . . . $ae . a+e :: r . s$
6. Therefore by comparing the Product of the extremes to the Product of the means this Equation ariſeth, ... $sae = ra \dashv re$
7. And by ſubtracting re from each part of that Equation, this ariſeth, *viz.* . . . . . $sae - re = ra$
8. And by dividing each part of the laſt Equation by $sa - r$, this ariſeth, *viz.* . . . . . $\therefore e = \frac{ra}{sa - r}$

Which Equation gives this

CANON.

9. For the firſt number ſought take any number greater than the Quotient that ariſeth by dividing the former term of the given Reaſon by the latter; then multiply the firſt number ſo taken by the latter term, and from the Product ſubtract the former term; laſtly, by the Remainder divide the Product made by the multiplication of the firſt number into the former term, and the Quotient ſhall be the ſecond number ſought. For example, if two numbers be deſired that their Product may be to their ſumm as 3 to 2, (that is, as r to s in the Reſolution,) you may take any number greater than $\frac{2}{1}$, as 2, for the firſt number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are ſuch, that their Product 12 is to their Summ 8, as 3 to 2, as was deſired. After the ſame manner you may find out innumerable Anſwers to the Queſtion.

10. But if it were deſired to find out two numbers that the Product of their multiplication might be equal to their ſumm, and that the ſumm or Product might be a ſquare number; the numbers may be found out thus, *viz.*

For the firſt number ſought put . . . . . a
And for the ſecond number . . . . . e
Then according to the Queſtion, ... $ae = a \dashv e$
Therefore by tranſpoſition of e, ... $\therefore ae - e = a$
Therefore each part of the laſt Equation being divided by $a - 1$, there will ariſe . . . . . $e = \frac{a}{a - 1}$
Which laſt Equation multiplied by a gives . . . . . $ae = \frac{aa}{a - 1} (= a \dashv e)$

Which $\frac{aa}{a - 1}$ muſt (according to the Queſtion) be a ſquare number. But the Numerator aa is a Square; it remains then to equate the Denominator $a - 1$ to ſome ſquare number, let it be dd, *viz.* ſuppoſe $a - 1 = dd$, whence $a = dd \dashv 1$; according to which value of a, the number e which was before found equal to $\frac{a}{a - 1}$, will be $\frac{dd \dashv 1}{dd}$. Therefore for the two numbers ſought take $dd \dashv 1$ and $\frac{dd \dashv 1}{dd}$, which in words give this

CANON.

For one of the numbers ſought take any ſquare number increaſed with unity; then divide that ſumm by the ſquare number taken, and the Quotient ſhall be the other number ſought.

As, for example, you may take $4 \dashv 1$, that is, 5 for one of the numbers ſought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ ſhall be the other number ſought. So 5 and $\frac{5}{4}$ will ſolve the Queſtion laſt propoſed; for their Product is $\frac{25}{4}$, their Summ alſo is $\frac{25}{4}$, which is a ſquare number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares, and that the ſumm of the Squares of the two numbers may be a Square.

RESO-

12. Which Fraction laſt above expreſt being ſubtracted from the Fraction in the tenth ſtep will leave this that follows, to wit,

$$\frac{\frac{1}{2}bbgg \dashv\vdash \frac{1}{4}ddff \dashv\vdash ddgg \dashv\vdash \frac{1}{2}bdfg \dashv\vdash bbgg \dashv\vdash ddfg}{bg \dashv\vdash dg \dashv\vdash bf \dashv\vdash af \text{ into } bg \dashv\vdash ag \dashv\vdash bf \dashv\vdash df}.$$

13. The ſquare Root of the Fraction in the twelfth ſtep is

$$\frac{\frac{1}{2}bg \dashv\vdash \frac{1}{2}df \dashv\vdash dg}{bg \dashv\vdash dg \dashv\vdash bf \dashv\vdash df}.$$

14. Which ſquare Root being added to and ſubtracted from the half-Coefficient in the ninth ſtep, the ſumm and remainder ſhall be the two values of u in the Equation in the eighth ſtep, *viz.*

$$u = 1; \text{ alſo, } u = \frac{bf - dg}{bg \dashv\vdash dg \dashv\vdash bf \dashv\vdash df}.$$

The latter of which values of u, to wit, $\frac{bf-dg}{bg \dashv\vdash dg \dashv\vdash bf \dashv\vdash df}$ is the Fraction ſought by the Queſtion.

15. Then according to the ſaid leſſer value of u, the compound quantity $\frac{b-bu-du}{d}$ into which e is multiplied in the latter part of the Equation in the ſixth ſtep will be reduced into this fractional quantity, to wit, $\frac{bbg \dashv\vdash bdg \dashv\vdash ddg}{bdg \dashv\vdash ddg \dashv\vdash bdf \dashv\vdash ddf}$, which multiplied into any number taken at pleaſure for the value of e, will give the number a; and therefore to find out a and e in whole numbers we may take the Numerator of that fractional quantity for a, and the Denominator for e, or any two whole numbers in the ſame proportion with the ſaid Numerator and Denominator, and the leſſer of the two values of u before found for the Fraction ſought.

An Example in Numbers.

16. Let there be given $\begin{cases} 3 = b \\ 1 = d \\ 5 = f \\ 1 = g \end{cases}$

17. Then the three numbers ſought will be theſe, *viz.* $\begin{cases} 16 = bbg \dashv\vdash 2bdg \dashv\vdash ddg \\ 24 = bdg \dashv\vdash ddg \dashv\vdash bdf \dashv\vdash ddf \\ \frac{7}{12} = \frac{bf-dg}{bg \dashv\vdash dg \dashv\vdash bf \dashv\vdash df} \end{cases}$

18. Which three numbers, to wit, 16, 24 and $\frac{7}{12}$ will ſolve the Queſtion, as will be manifeſt by

The Proof.

19. If the firſt number 16 receive $\frac{7}{12}$ of the ſecond number 24, that is, 14, the ſumm 30 will be to the remainder 10 as 3 to 1, (*viz.* as b to d;) Again, if the ſecond number 24 take $\frac{7}{12}$ parts of the firſt number 16, to wit, $9\frac{1}{3}$, the ſumm $\frac{100}{3}$ will be to the remainder $\frac{20}{3}$ as 5 to 1, (to wit, as f to g:) Or inſtead of 16 and 24 you may take 2 and 3, or any two numbers in the ſame Proportion.

20. Again, if $b = 3$. $d = 2$. $f = 4$. and $g = 5$. then the literal quantities in the preceding 17th ſtep will give theſe three numbers, 125, 90 and $\frac{2}{45}$ to ſolve the Queſtion; or inſtead of 125 and 90 you may take 25 and 18, or any two numbers in the ſame Proportion.

Note. If in reducing the Equation in the ſeventh ſtep, bf were ſuppoſed either equal to dg or leſs than dg, there would come forth an Equation wherein the value of u would be either unity or greater than unity; but according to the import of the Queſtion it ought to be leſs than unity, and ſuch is the leſſer value of u in the Equation in the eighth ſtep, where bf is ſuppoſed greater than dg, as the Determination annexed to the Queſtion requires.

QUEST. 52. (Quæſt. 41. Lib. 4. *Diophant.*)

To find two numbers, that the Product of their multiplication may be to their ſumm in a given Reaſon (or Proportion,) ſuppoſe as r to s.

RESO-

RESOLUTION.

1. For the firſt number put . . . . . a
2. And for the ſecond . . . . . e
3. Then their ſumm is . . . . . $a \dashv\vdash e$
4. And the Product of their multiplication is . . . . . ae
5. But according to the Queſtion the Product muſt be to the ſumm as r to s, therefore . . . . . $ae \,.\, a+e :: r \,.\, s$
6. Therefore by comparing the Product of the extremes to the Product of the means this Equation ariſeth, ... $sae = ra \dashv\vdash re$
7. And by ſubtracting re from each part of that Equation, this ariſeth, *viz.* . . . . . $sae - re = ra$
8. And by dividing each part of the laſt Equation by $sa - r$, this ariſeth, *viz.* . . . . . $\therefore e = \frac{ra}{sa-r}$

Which Equation gives this

CANON.

9. For the firſt number ſought take any number greater than the Quotient that ariſeth by dividing the former term of the given Reaſon by the latter; then multiply the firſt number ſo taken by the latter term, and from the Product ſubtract the former term; laſtly, by the Remainder divide the Product made by the multiplication of the firſt number into the former term, and the Quotient ſhall be the ſecond number ſought. For example, if two numbers be deſired that their Product may be to their ſumm as 3 to 2, (that is, as r to s in the Reſolution,) you may take any number greater than $\frac{2}{1}$, as 2, for the firſt number; then (by the Canon) the other number will be found 6; which numbers 2 and 6 are ſuch, that their Product 12 is to their Summ 8, as 3 to 2, as was deſired. After the ſame manner you may find out innumerable Anſwers to the Queſtion.

10. But if it were deſired to find out two numbers that the Product of their multiplication might be equal to their ſumm, and that the ſumm or Product might be a ſquare number; the numbers may be found out thus, *viz.*

For the firſt number ſought put . . . . . a
And for the ſecond number . . . . . e
Then according to the Queſtion, ... $ae = a \dashv\vdash e$
Therefore by tranſpoſition of e, ... $\therefore ae - e = a$
Therefore each part of the laſt Equation being divided by $a-1$, there will ariſe . . . . . $e = \frac{a}{a-1}$
Which laſt Equation multiplied by a gives . . . . . $ae = \frac{aa}{a-1} \; (= a \dashv\vdash e)$

Which $\frac{aa}{a-1}$ muſt (according to the Queſtion) be a ſquare number. But the Numerator aa is a Square; it remains then to equate the Denominator $a-1$ to ſome ſquare number, let it be dd, *viz.* ſuppoſe $a - 1 = dd$, whence $a = dd \dashv\vdash 1$; according to which value of a, the number e which was before found equal to $\frac{a}{a-1}$, will be $\frac{dd \dashv\vdash 1}{dd}$. Therefore for the two numbers ſought take $dd \dashv\vdash 1$ and $\frac{dd \dashv\vdash 1}{dd}$, which in words give this

CANON.

For one of the numbers ſought take any ſquare number increaſed with unity; then divide that ſumm by the ſquare number taken, and the Quotient ſhall be the other number ſought.

As, for example, you may take $4 \dashv\vdash 1$, that is, 5 for one of the numbers ſought; then dividing 5 by 4 (the Square taken,) the Quotient $\frac{5}{4}$ ſhall be the other number ſought. So 5 and $\frac{5}{4}$ will ſolve the Queſtion laſt propoſed; for their Product is $\frac{25}{4}$, their Summ alſo is $\frac{25}{4}$, which is a ſquare number as was required.

QUEST. 53.

To find two numbers, that their difference may be equal to the difference of their Squares, and that the ſumm of the Squares of the two numbers may be a Square.

RESO-

RESOLUTION.

1. For the greater number put . . . . . a
2. And for the leſſer . . . . . e
3. Then their difference is . . . . . $a-e$
4. And the difference of their Squares is . . . . . $aa-ee$
5. Therefore according to the Queſtion, ... $aa-ee=a-e$
6. Therefore by dividing each part of that Equation by $a-e$, the Quotient gives . . . . . $a+e=1$
7. Therefore by tranſpoſition of a, ... $e=1-a$
8. Now taking $1-a$ inſtead of e, the two numbers ſought are . . a and $1-a$
9. The difference of which numbers is either $2a-1$ or $1-2a$, and the ſame is the difference of their Squares: But the ſumm of their Squares muſt make a Square; therefore $2aa-2a+1$ muſt be equated to a Square, yet ſo, as the value of a may be leſs than 1. Now to cauſe that effect the ſaid ſide may be feigned $-1+$ any multitude of a greater than $2a$; let therefore the ſaid ſide be feigned $3a-1$, then the Square of $3a-1$ being equated to the ſaid $2aa-2a+1$ the value of a will be found $\frac{4}{7}$, which ſubtracted from 1 leaves $\frac{3}{7}$; therefore $\frac{4}{7}$ and $\frac{3}{7}$ are the numbers ſought. For as well their difference as the difference of their Squares is $\frac{1}{7}$; and the ſumm of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$ makes a Square, to wit, $\frac{25}{49}$.

From the premiſſes ariſeth this

CANON.

10. Take any number greater than 2; then divide the exceſs of that number above unity, by the exceſs of half the Square of the ſame number above unity, and the Quotient ſhall be one of the numbers ſought, which ſubtracted from unity leaves the other number ſought. As, for example, take the number 3; then dividing the exceſs of 3 above 1, that is, 2, by the exceſs of half the Square of 3 above 1, that is, by $\frac{2}{2}$; the Quotient $\frac{4}{7}$ is one of the numbers ſought, which ſubtracted from 1 leaves $\frac{3}{7}$ for the other number.

11. The ſame Queſtion may be propounded thus, *viz.* To find a right-angled Triangle in Rational numbers, that the difference of the ſides about the right-angle may be equal to the difference of the Squares of the ſame ſides. For ſolving this Queſtion, take any two numbers found out by the ſaid Canon, as $\frac{4}{7}$ and $\frac{3}{7}$ for the ſides about the right-angle, whence the Hypothenuſal (to wit, the ſquare Root of the ſumm of the Squares of $\frac{4}{7}$ and $\frac{3}{7}$) is $\frac{5}{7}$.

Moreover, from the foregoing Reſolution of *Queſt.* 53. we may deduce this

THEOREM.

12. If unity be divided into any two parts, the difference of the parts is equal to the difference of the Squares of the ſame parts: And if the Product made by the mutual multiplication of the parts be ſubtracted from each of them, each Remainder will be a Square: Alſo the exceſs of the greater part above its Square is equal to the exceſs of the leſſer part above its Square, and each exceſs is equal to the Product of the parts.

This will eaſily be manifeſted by theſe two numbers a and $1-a$, whoſe ſumm is unity.

QUEST. 54. (Quæſt. 45. Lib. 4. *Diophant.*)

To find three numbers, that the exceſs of the greateſt above the mean may be to the exceſs of the mean above the leaſt in a given Reaſon, ſuppoſe as 3 to 1; and that the ſumm of every two of the three numbers may be a Square.

RESOLUTION.

1. Foraſmuch as the ſumm of the mean and leaſt of the three numbers muſt be a Square, let it be 4
2. Therefore the mean is greater than 2; for if we ſhould put it 2, the leaſt would be alſo 2, which is abſurd; therefore for the mean let there be put . . $a+2$
3. Therefore from the ſaid poſitions the leaſt number is . . . . . $2-a$
4. And the exceſs of the mean above the leaſt is . . . . . $2a$
5. But the exceſs of the greateſt above the mean muſt be the triple of the exceſs of the mean above the leaſt; therefore from the laſt ſtep the exceſs of the greateſt above the mean is . . . . . $6a$

6. Which

6. Which laſt mentioned exceſs, to wit, $6a$, being added to the mean $a+2$, gives the greateſt of the three numbers, to wit, ... $7a+2$
7. But the Queſtion requires two things more, to wit, that the greateſt with the mean may make a Square; and that the greateſt with the leaſt may make a Square; hence ariſeth this Duplicate equality to be reſolved, to wit, ... $8a+4=\square$ / $6a+4=\square$
8. Which Duplicate equality hath already been reſolved in the preceding *Queſt.* 33. of this *Book*, where the number $\frac{441}{242}$, (among innumerable other numbers that might be diſcovered, and each to be leſs than 2, as the third ſtep of the preceding Reſolution requires,) was found out for the value of a. Therefore $\frac{441}{242}$ being taken for the value of a, and the poſitions in the ſixth, ſecond and third ſteps reſolved accordingly, the three numbers ſought will be found theſe, to wit, $\frac{213528}{48841}$, $\frac{182788}{48841}$ and $\frac{151848}{48841}$, which will ſolve the Queſtion. For firſt, the exceſs of the greateſt above the mean, to wit, $\frac{151840}{48841}$, is the triple of $\frac{151820}{48841}$ the exceſs of the mean above the leaſt; ſecondly, the ſumm of the greateſt and mean is the Square $\frac{213249}{48841}$, whoſe ſide is $\frac{462}{221}$; thirdly, the ſumm of the greateſt and leaſt is the Square $\frac{213276}{48841}$, whoſe ſide is $\frac{462}{221}$; laſtly, the ſumm of the mean and leaſt is the Square $\frac{121104}{48841}$, whoſe ſide is $\frac{442}{221}$, that is, 2.

QUEST. 55. (Quæſt. 2. Lib. 5. *Diophant.*)

To find three numbers in Geometrical proportion, that every one of them increaſed with a given number d may make a Square.

RESOLUTION.

1. Firſt, ſeek a Square which added to the given number d may make a Square, and whoſe $\frac{1}{4}$ part may exceed d; ſuppoſe it be found bb, let this be put for one of the extreme Proportionals ſought, to wit, ... bb
2. For the other extreme put . . . . . aa
3. Then becauſe the Product of the extremes is equal to the Square of the mean, therefore the Square of the mean is $bbaa$, whoſe Root is the mean it ſelf, to wit, ... ba
4. By Conſtruction in the firſt ſtep the firſt extreme bb increaſed with the given number d makes a Square; but according to the Queſtion the other extreme and the mean being ſeverally increaſed with the ſame given number d muſt alſo make a Square, whence this Duplicate equality ariſeth,

$$viz.\ \begin{cases} aa+d=\square \\ ba+d=\square \end{cases}$$

5. Now to reſolve this Duplicate equality I proceed as in former Queſtions, *viz.* Firſt, the difference of thoſe two Equations is $aa-ba$, which is equal to the Product of a into $a-b$; then if the Square of half the ſumm of a and $a-b$ be equated to $aa+d$, or the Square of half the difference of a and $a-b$ to $ba+d$, from either of thoſe Equations the value of a will be made known: But half the difference of the ſaid a and $a-b$ is $\frac{1}{2}b$, whoſe Square is $\frac{1}{4}bb$; let this be equated to $ba+d$, and it will be

$$ba+d=\tfrac{1}{4}bb.$$

6. Whence, after due Reduction, the value of a will be diſcovered, *viz.* . . . . . $a=\frac{\frac{1}{4}bb-d}{b}$
7. And by multiplying each part of the laſt Equation by b, it gives . . $ba=\frac{1}{4}bb-d$

From the premiſſes ariſeth the following Canon to find out the three Proportionals ſought.

CANON.

8. Take for one of the extreme Proportionals ſought ſuch a ſquare number that if it be increaſed with the given number it may make a Square, and that a quarter of the firſt Square may exceed the given number; then from a quarter of the firſt Square ſubtract the given number and there will remain the mean Proportional; laſtly, divide the mean by the ſide of the Square firſt taken, and the Square of the Quotient ſhall be the other extreme.

An Example in Numbers.

Suppoſe $19=d$ the number given in the Queſtion; then find a Square that if it be increaſed with 19 may make a Square, and that a quarter of the Square found out may exceed

exceed the said 19, (or that the side of the said Square may exceed $\sqrt{76}$.) But such is the Square 81, (found out after the manner of resolving the ninth Question of this Book,) for 81 increased with 19 makes the Square 100, also $\frac{1}{4}$ of 81 is greater than 19; therefore 81 shall be the first of the three Proportionals sought. Then by the Canon above expreſt, the other two will be found $\frac{1}{4}$ and $\frac{361}{1296}$. I say, 81, $\frac{1}{4}$ and $\frac{361}{1296}$ will solve the Question: For first they are continual Proportionals, in regard the Product of the extremes is equal to the Square of the mean; secondly, the first Proportional 81 increased with the given number 19 makes the Square 100; thirdly, the mean Proportional $\frac{1}{4}$ increased with the said 19 makes the Square $\frac{81}{4}$; lastly, the third Proportional $\frac{361}{1296}$ increased with the said 19 makes the Square $\frac{24649}{1296}$, whose side is $\frac{157}{36}$. Therefore the Question is solved, and manifestly capable of innumerable Answers.

QUEST. 56. (Quæst. 7. Lib. 5. *Diophant.*)

To find two numbers, that the Product of their multiplication added to the summ of their Squares may make a Square.

RESOLUTION.

1. For one of the numbers sought take any known number, as . . . . . b
2. For the other number put . . . . . a
3. The Square of the first is . . . . . bb
4. The Square of the second is . . . . . aa
5. The Product of the multiplication of the two numbers is . . . . . ba
6. Therefore the summ of the said Squares and Product is . . . $aa + ba + bb$
7. Which summ must be equal to a Square, the side whereof may be feigned to be a — any known number greater than b, let it be $a - d$, and then the Square of $a - d$ being equated to the said summ, this Equation ariseth, *viz.* . . . . . $aa + ba + bb = aa - 2da + dd$
8. Which Equation after due Reduction gives . . . $a = \frac{dd - bb}{2d + b}$
9. Therefore from the first, second and eighth steps the two numbers sought are equal to these known numbers, *viz.* . . . . . b and $\frac{dd - bb}{2d + b}$
10. But to avoid Fractions multiply those two numbers severally by the Denominator $2d + b$, and take the Products for two numbers to solve the Question, *viz.* . . . $2db + bb$ and $dd - bb$

The Proof.

11. The Square of $dd - bb$ is . . . . . $dddd - 2ddbb + bbbb$
12. The Square of $2db + bb$ is . . . . . $4ddbb + 4dbbb + bbbb$
13. The Product of $dd - bb$ into $2db + bb$ is . . . $2bddd - 2dbbb + bbdd - bbbb$
14. The summ of the said Squares and Product is . . . $d^4 + b^4 + 3ddbb + 2db^3 + 2bd^3$
15. Which summ is a Square whose Root is . . . . $dd + db + bb$

From the tenth step ariseth

CANON 1.

16. Take any two unequal square numbers, then their difference shall be one of the numbers sought; and the lesser Square increased with the double Product of the multiplication of the sides of those Squares shall be the other number sought.

Moreover, because the Product of the multiplication of the summ of any two numbers into their difference, is equal to the difference of their Squares, therefore from the preceding Canon ariseth

CANON 2.

17. Take any two unequal numbers, then the Product of their summ multiplied into their difference shall be one of the two numbers sought, and the double Product made by the mutual multiplication of the two numbers first taken, together with the Square of the lesser number shall be the other number sought.

From

From the premisses 'tis evident that the Question is capable of innumerable Answers in whole numbers, of which (for the Learners exercise) I shall exhibit six, with their Proofs, in the following Table.

	d, b	s, r	ss	rr	sr	qq	q
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	3401	49

18. The numbers under d and b in this Table are six pairs of numbers taken at pleasure, by which the latter of the two preceding Canons gives six pairs of numbers under s and r to solve *Quest.* 56. As, for example, if 2 and 1 be taken, then *Canon* 2. gives 5 and 3, (that is, s and r) to solve the Question: For 25 and 9, the Squares of 5 and 3, together with 15 the Product of 5 and 3, (that is, $ss + rr + sr$) make the Square 49, (that is qq,) whose side is 7, (to wit, q.)

But for a further Proof, you may observe from the fourteenth and fifteenth steps of the Resolution, that every number standing under q is equal to its respective $dd + db + bb$; so 7 in the Columel of q is equal to the Squares of 2 and 1 standing under d, b, together with the Product of 2 into 1. The like is to be understood of the other five Answers in the Table.

Observat. 1. *upon the foregoing* Quest. 56.

Whereas it is taken for granted in the tenth step of the preceding Resolution of *Quest.* 56. that two numbers in the same Reason (or Proportion) with those found out to solve the said Question will likewise satisfie the same, I shall here demonstrate the truth thereof.

Suppositions.

1. Let two numbers capable of solving *Quest.* 56. suppose 5 and 3, be represented by . . . . . a and b
2. And let their Squares be signified by . . . . . aa and bb
3. Then according to the import of *Quest.* 56. . . . . $aa + ba + bb = \square$
4. Let two other numbers having the same Proportion to one another as a to b be represented by d and c, viz. suppose . . . . . $a . b :: d . c$
5. Then add cd the Product of c and d to cc and dd the Squares of c and d, so the summ is . . . . . $dd + cd + cc$
6. Now we must demonstrate that . . . . . $dd + cd + cc = \square$ (a Square.)

Demonstration.

7. By *Prop.* 17. *Elem.* 7. Euclid. . . . . $a . b :: aa . ba$
8. And by *Prop.* 11. *Elem.* 8. . . . . $aa . ba :: ba . bb$
9. Therefore out of the two last preceding Analogies, ... $a . b :: aa . ba :: ba . bb$
10. And by the like argumentation, ... $d . c :: dd . cd :: cd . cc$
11. And because by supposition in the fourth step, . . . $a . b :: d . c$
12. Therefore out of the ninth, tenth and eleventh steps, (*per Prop.* 11. *Elem.* 5.) . . $aa . ba :: ba . bb :: dd . cd :: cd . cc$
13. Therefore alternately (*per Prop.* 13. *Elem.* 7.) $aa . dd :: ba . cd :: bb . cc$
14. Therefore (*per Prop.* 12. *Elem.* 7.) . . . $aa + ba + bb . dd + cd + cc :: aa . dd$
15. And because by supposition . . . . . aa and dd are Squares.
16. And by supposition in the third step, . . . $aa + ba + bb = \square$
17. Therefore from the fourteenth, fifteenth and sixteenth steps, (*per Prop.* 24. *Elem.* 8.) . . $dd + cd + cc = \square$

Which was to be proved.

Obſervat. 2. *upon* Queſt. 56.

Albert Girard in *pag.* 618. of *Simon Stevin*'s Arithmetick printed in the *French* Tongue at *Leyden*, in 1625. doth from the ſaid ſeventh Queſtion of the fifth Book of *Diophantus* deduce this following

THEOREM.

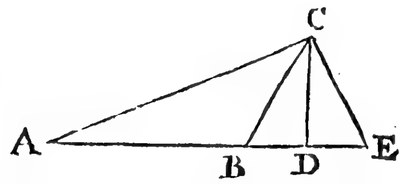
If a plain Triangle be made of three ſuch ſides, that the ſumm of the Squares of two of thoſe ſides, together with the Rectangle (or Product of the multiplication) of the ſame two ſides, is equal to the Square of the third ſide; then the angle oppoſite to ſuch third ſide hath for its meaſure exactly 120 degrees. But if the ſaid Rectangle (or Product) be ſubtracted from the ſumm of the ſaid Squares, and the Remainder be equal to the Square of the third ſide, then the angle oppoſite to ſuch third ſide ſhall have for its meaſure infallibly 60 degrees.

This may eaſily be demonſtrated by *Prop.* 12, & 13. *Elem.* 2. *Euclid.* but waving the Demonſtration, I ſhall explain the Theorem by Numbers.

Firſt then, if three numbers be deſired to expreſs the meaſures of the ſides of a plain Triangle that ſhall have one angle whoſe meaſure is 120 degrees, the preceding Table will furniſh you with ſix ſuch Triangles; for in every rank of numbers in that Table, the three numbers which anſwer to s, r and q will conſtitute the Triangle deſired. As, for example, 5, 3 and 7; likewiſe, 8, 7, 13 | 16, 5, 19, *&c.* will expreſs the Quantities of the ſides of Triangles, in every one of which, the meaſure of the angle oppoſite to the greateſt ſide is exactly 120 degrees.

Otherwiſe, without the help of the ſaid Table, if two unequal numbers be taken at pleaſure, as $2 = d$, and $1 = b$, then theſe three following numbers ſhall expreſs the meaſures of the ſides of a Triangle having an angle of 120 degrees, *viz.*

$$\begin{cases} 2db + bb = 5 = \text{AB}, \\ dd - bb = 3 = \text{BC} = \text{BE} = \text{EC}, \\ dd + db + bb = 7 = \text{AC}. \end{cases}$$



Which three numbers, (as is manifeſt by the preceding Reſolution of *Queſt.* 56.) have this property, *viz.* the ſumm of the Squares of the two leſſer numbers together with the Product of their multiplication is equal to the Square of the third or greateſt number.

Moreover, if three unequal numbers be deſired to expreſs the Quantities of the ſides of a plain Triangle that ſhall have for the meaſure of one of its angles exactly 60 degrees, you may readily find them out by the help of the preceding Table: For the numbers anſwering to $s + r$, r, and q are the three numbers deſired; ſo from the firſt rank of numbers in the Table, you may take

$$\begin{cases} 8 = s + r = \text{AE}, \\ 3 = r = \text{EC} = \text{EB} = \text{BC}, \\ 7 = q = \text{AC}. \end{cases}$$

Which three numbers 8, 3, 7 are the meaſures of the ſides of a Triangle having one of its angles, (to wit, that oppoſite to 7 or q,) exactly 60 degrees.

Otherwiſe, without the help of the ſaid Table, if two unequal numbers be taken at pleaſure, as $2 = d$, and $1 = b$, then theſe three following numbers ſhall be the meaſures of the ſides of a Triangle having one angle of 60 degrees, to wit, that oppoſite to the ſide $dd + db + bb$.

$$\begin{cases} 2db + dd = 8 = \text{AE}, \\ dd - bb = 3 = \text{EC}, \\ dd + db + bb = 7 = \text{AC}. \end{cases}$$

In which Triangle the Square of the ſide $dd + db + bb$ is equal to the ſumm of the Squares of the other two ſides $2db + dd$ and $dd - bb$, leſs by the Rectangle (or Product) of the ſame two ſides, as is evident by the following

Proof.

The Square of $2db + dd$ is . . . $\rangle$ $4ddbb + 4bddd + dddd$
The Square of $dd - bb$ is . . . . $\rangle$ $dddd - 2ddbb + bbbb$

The

The ſumm of thoſe Squares is . . . . $\rangle$ $2ddbb + 4bddd + 2dddd + bbbb$
The Product of $2db + dd$ into $dd - bb$ is $\rangle$ $2bddd + dddd - 2dbbb - ddbb$
Which Product being ſubtracted from the ſaid ſumm of the Squares, leaves . . . $\rangle$ $dddd + bbbb + 3ddbb + 2dbbb + 2bddd$

Which Remainder is the Square of $dd + db + bb$, as was affirmed.

QUEST. 57. (Quæſt. 8. Lib. 5. *Diophant.*)

To find three right-angled Triangles in Rational numbers that ſhall have equal Area's.

RESOLUTION.

1. Firſt, by either of the Canons in *Sect.* 16, 17. of the foregoing *Queſt.* 56. find out two numbers capable of ſolving that Queſtion, ſuppoſe . . $\rangle$ s the greater, r the leſſer.
2. Therefore, (according to the import of the ſaid *Queſt.* 56.) the Squares of s and r, with the Product of s into r is equal to ſome Rational ſquare number, let it be qq, whence . . . . . $\rangle$ $ss + sr + rr = qq$
3. Therefore by Conſtruction in the firſt and ſecond ſteps, ... $\rangle$ $\sqrt{:ss + sr + rr:} = q$, (Rational.)
4. By the Canon in *Obſervat.* 8. *Reſolut.* 2. *Queſt.* 1. form a right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and r, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$\text{viz.} \begin{cases} ss + sr + 2rr = \text{Hypothenuſal}, \\ ss + sr = \text{Baſe}, \\ 2qr = \text{Perpendicular}. \end{cases}$$

5. In like manner form a ſecond right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and s, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$\text{viz.} \begin{cases} 2ss + sr + rr = \text{Hypothenuſal}, \\ sr + rr = \text{Baſe}, \\ 2qs = \text{Perpendicular}. \end{cases}$$

6. Likewiſe, form a third right-angled Triangle from $\sqrt{:ss + sr + rr:}$ (that is, q) and $s + r$, ſo the three ſides will be expreſſible by theſe Rational numbers,

$$\text{viz.} \begin{cases} 2ss + 3sr + 2rr = \text{Hypothenuſal}, \\ sr = \text{Baſe}, \\ 2qs + 2qr = \text{Perpendicular}. \end{cases}$$

7. I ſay thoſe three Triangles will ſolve this 57th Queſtion: For firſt, by Conſtruction they are right-angled Triangles; ſecondly, all the ſides are expreſſible by Rational numbers, for s, r and q are Rational numbers by Conſtruction; thirdly, if in every one of thoſe three Triangles the Baſe be multiplied by the Perpendicular, every one of the three Products will manifeſtly be equal to $2qrss + 2qsrr$; therefore the halves of thoſe Products, that is, the Area's of thoſe three right-angled Triangles are equal to one another, as was required.
8. And ſince the foregoing *Queſt.* 56. gives innumerable whole numbers anſwering to s, r and q, you may find out as many Ternions of right-angled Triangles in whole numbers as ſhall be deſired to ſolve this 57th Queſtion. But to have the nine ſides of every Ternion expreſſible by whole numbers in the leaſt Terms, every pair of the ſaid numbers s, r muſt be the leaſt Terms of a Reaſon, that is, two ſuch numbers as have no common Diviſor beſides unity.
9. The premiſſes give the following Canon to find out innumerable Ternions of right-angled Triangles in whole numbers to ſolve the Queſtion propoſed, after the Rational whole numbers repreſented by s, r and q are firſt found out by either of the Canons of *Queſt.* 56.

CANON.

$$\left.\begin{array}{l} ss + sr + 2rr = \text{h} = \text{Hypoth.} \\ ss + sr = \text{b} = \text{Baſe}, \\ 2qr = \text{p} = \text{Perp.} \end{array}\right\} \text{of Triangle I,}$$

$$\left.\begin{array}{rcl} 2ss + sr + rr & = & h = \text{Hypoth.} \\ sr + rr & = & b = \text{Base,} \\ 2qs & = & p = \text{Perp.} \end{array}\right\} \text{of Triangle II.}$$

$$\left.\begin{array}{rcl} 2ss + 3sr + 2rr & = & H = \text{Hypoth.} \\ sr & = & B = \text{Base,} \\ 2qs + 2qr & = & P = \text{Perp.} \end{array}\right\} \text{of Triangle III.}$$

Observations upon the Canon *last aforegoing.*

Divers properties, besides the equality of Area's, in the three right-angled Triangles found out by the said Canon, do present themselves to your view, and are worthy of Observation. The principal Properties are these four, *viz.*

1. $h + b = h + b.$
2. $H + B = 2b + 2b.$
3. $H - B = h + b = h + b.$
4. $P = p + p.$

That is to say, in words,

1. The summ of the Hypothenusal and Base of the first Triangle, is equal to the summ of the Hypothenusal and Base of the second.

2. The summ of the Hypothenusal and Base of the third Triangle, is equal to the double-summ of the Bases of the first and second.

3. The excess of the Hypothenusal above the Base of the third Triangle is equal to the summ of the Hypothenusal and Base of the second, and likewise to the summ of the Hypothenusal and Base of the first.

4. The Perpendicular of the third Triangle is equal to the summ of the Perpendiculars of the first and second.

By the first of those three Triangles is meant that which hath the shortest Hypothenusal; by the second, that whose Hypothenusal is next greater than the shortest; and by the third, that which hath the longest Hypothenusal; in which order they are set in the Table. But the better to explain the Canon and Properties, I shall resume the Table belonging to *Quest.* 56. and call it *Table* I. whence six Answers in whole numbers to this *Quest.* 57. are deduced, and inserted in the following *Table* II.

Table I. *brought from* Quest. 56.

	d, b	s, r	ss	rr	sr	qq	q
1	2, 1	5, 3	25	9	15	49	7
2	3, 1	8, 7	64	49	56	169	13
3	3, 2	16, 5	256	25	80	361	19
4	4, 3	33, 7	1089	49	231	1369	37
5	5, 1	24, 11	576	121	264	961	31
6	5, 3	39, 16	1521	256	624	2401	49

Table II. *deduced from* Tab. I. *by the Canon in* Sect. 9. Quest. 57.

	h.	b.	p.	*h.*	*b*	*p.*	H.	B.	P.
1	58	40	42	74	24	70	113	15	112
2	218	120	182	233	105	208	394	56	390
3	386	336	190	617	105	608	802	80	798
4	1418	1320	518	2458	280	2442	2969	231	2960
5	1082	840	682	1537	385	1488	2186	264	2170
6	2657	2145	1568	3922	880	3822	5426	624	5390

The Construction of *Table* I. hath already been expresst in *Quest.* 56. whence the latter *Table* is deduced according to the Canon in *Sect.* 9. of this 57th Question, and contains six Answers to it; the first of which is to be understood thus,

In

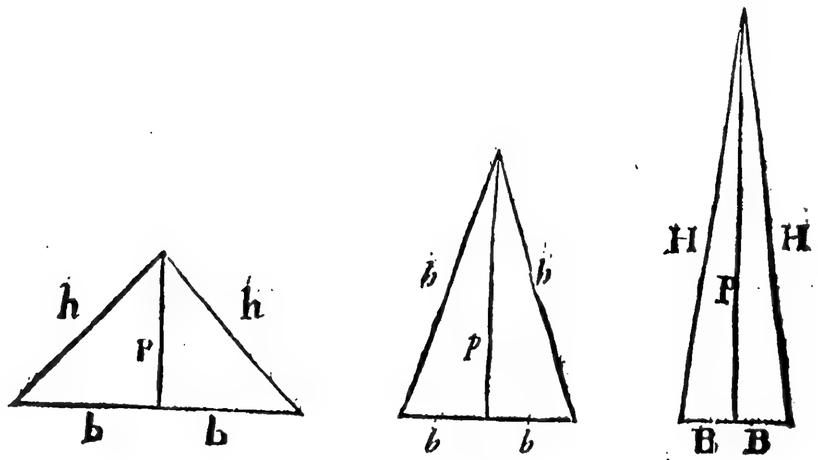
In the first rank, under	h, b, p.	*h, b, p.*	H, B, P.
you will find	58, 40, 42.	74, 24, 70.	113, 15, 112.

Which three Triangles have equal Area's, and such other properties as before have been declared, *viz.* $h + b = h + b$, *&c.* as will easily appear by comparing the numbers answering to those Equations. The like is to be understood of the other five Answers.

QUEST. 58.

[*This is* Probl. 29. *in* pag. 131. *of the Introduction to* Algebra, *translated out of High* Dutch *into* English *in* 1668. *by* Tho. Brancker, *M. A.*]

To find three equicrural Triangles equal to one another in Area; and that the Perimeters of two of those Triangles may be equal to one another; and that the sides and Perpendiculars of every one of those three Triangles may be expressible by rational numbers.



RESOLUTION.

1. By the Canon in the ninth step of the Resolution of the foregoing *Quest.* 57. find out three right-angled Triangles in rational numbers, and equal to one another in Area; such are the three Triangles in any one of the six ranks of numbers in *Table* II. belonging to the said *Quest.* 57. for example, take those in the first rank,

$$viz. \left\{\begin{array}{c|c|c} h, b, p. & h, b, p. & H, B, P. \\ 58, 40, 42. & 74, 24, 70. & 113, 15, 112. \end{array}\right.$$

2. Then (as is evident by the Diagram belonging to this Question) the sides of the three equicrural Triangles desired shall be these following,

$$viz. \left\{\begin{array}{c|c|c} h, h, 2b. & h, h, 2b. & H, H, 2B. \\ 58, 58, 80. & 74, 74, 48. & 113, 113, 30. \end{array}\right.$$

3. And the Perpendiculars falling upon the Bases, *viz.* upon $\left\{\begin{array}{c} p, p, P. \\ 42, 70, 112. \end{array}\right.$ 2b, 2b, 2B of those three equicrural Triangles, are these,

Which three equicrural Triangles in rational numbers above expresst in the second step will solve the Question, as will be evident by

The Proof.

By Construction in the first step the three right-angled Triangles h, b, p. | *h, b, p.* | H, B, P are equal to one another in Area, therefore their double Area's are equal to one another; but the said double Area's are the Area's of the three equicrural Triangles h, h, 2b. | *h, h, 2b.* | H, H, 2B, and therefore the Area's of those three equicrural Triangles are equal between themselves,

$$viz. \left\{\begin{array}{rcl} bp & = & bp = BP, \\ 40 \times 42 & = & 24 \times 70 = 15 \times 112 = 1680. \end{array}\right.$$

5. Moreover, by the first property in the Observations upon the Canon for resolving *Quest.* 57. this Equation is manifest,

$$viz. \left\{\begin{array}{rcl} h + b & = & h + b, \\ 58 + 40 & = & 74 + 24 = 98. \end{array}\right.$$

6. There-

6. Therefore the double of the first part of that Equation shall be equal to the double of the latter part,

$$\textit{viz.}\ \begin{cases} 2h + 2b = 2h + 2b, \\ 116 + 80 = 148 + 48 = 196. \end{cases}$$

7. But the said double summs (if you view the Diagram belonging to this Question) are manifestly equal to the Perimeters of the two equicrural Triangles h, h, 2b and h, h, 2b, therefore those two equicrural Triangles are equal to one another in their Perimeters as well as in their Area's, and each Area is equal to the Area of the third equicrural Triangle H, H, 2B; also all their sides and Perpendiculars are expresst by rational numbers, as the Question required. In like manner five Answers more to this 58th Question may be collected from *Table* II. in *Quest.* 57. and 'tis evident from the premisses, that innumerable Ternions of equicrural Triangles in rational whole numbers may be found out to solve the said *Quest.* 58.

QUEST. 59.

The three sides of any plain right-angled Triangle being given in rational numbers; to find out another right-angled Triangle in rational numbers, which shall have the same Area with the former.

[*Monsr.* de Fermat, *in his Observation upon* Quest. 8. *of the fifth Book of* Diophantus, *gives a Canon to solve the Question above proposed, but shews not the rise thereof; I shall therefore resolve the Question at large by Literal Algebra, upon the same gronnds by which it is resolved by Numeral Algebra in* pag. 11. *of his Analytical Inventions, prefixed to the late Edition of* Diophantus *printed at* Tholose *in* 1670.]

RESOLUTION.

Let there be given a right-angled Triangle in rational numbers, as 3, 4, 5, which may be represented by b, p, h, whose Area is $\frac{1}{2}bp$; then let b and $a+p$ be assumed for the sides about the right-angle of a second right-angled Triangle, whence the Hypothenusal will be $\sqrt{:aa+2pa+pp+bb}$: that is, (because $hh = pp+bb$) $\sqrt{:aa+2pa+hh}$: and the Area of this latter Triangle is $\frac{1}{2}ba+\frac{1}{2}bp$: Now if this latter Area be divided by the former Area $\frac{1}{2}bp$; and if by the square Root of the Quotient $\frac{a}{p}+1$, *viz.* by $\sqrt{:\frac{a}{p}+1}$: the three sides of the second Triangle be severally divided, the Quotients shall be the three sides of a thirdright-angled Triangle, whose Area is equal to the Area of the right-angled Triangle first given: For if the sides of the second right-angled Triangle, to wit, b, $a+p$ and $\sqrt{:aa+2pa+hh}$: be severally divided by $\sqrt{:\frac{a}{p}+1}$: the Quotients $\frac{b}{\sqrt{:\frac{a}{p}+1}}$; $\frac{a+p}{\sqrt{:\frac{a}{p}+1}}$ and $\frac{\sqrt{:aa+2pa+hh}}{\sqrt{:\frac{a}{p}+1}}$: are the sides of a third right-angled Trianlge whose Area $\frac{ba+bp}{\frac{2a}{p}+2}$ is equal to $\frac{1}{2}bp$ the Area of the first right-angled Triangle; (for $\frac{2a}{p}+2$ into $\frac{1}{2}bp$ makes $ba+bp$:) So that if $aa+2pa+hh$ and $\frac{a}{p}+1$ were square numbers, then the Question were solved: But how to make those two Algebraick quantities to be square numbers, the following Resolution shews.

1. Let there be given, as before, the three sides of a right-angled Triangle, as	3 : 4 : 5 / b . p : h
2. Then for one of the sides about the right-angle of a second Triangle put	b
3. And for the other side about the right-angle,	$a+p$
4. Then because the summ of the Squares of those two sides must be equal to the Square of the Hypothenusal, the Square of the Hypothenusal of the second right-angled Triangle shall be	$aa+2pa+pp+bb$
5. That is, (because $hh = pp+bb$,)	$aa+2pa+hh$

6. From

6. From the second and third steps the Area of the second right-angled Triangle is	$\frac{1}{2}ba+\frac{1}{2}bp$
7. Which Area divided by $\frac{1}{2}bp$, the Area of the first right-angled Triangle, gives the Quotient	$\frac{a}{p}+1$
8. Now according to the scope before-mentioned, each of the quantities in the fifth and seventh steps must be equated to a Square, so we are fallen upon this Duplicate equality, *viz.*	$aa+2pa+hh = \square$ / $\frac{a}{p}+1 = \square$
9. In order to resolve that Duplicate equality, I multiply the said $\frac{a}{p}+1$ by the Square hh, to the end there may be the same known Square hh in each Equation, so it produceth	$\frac{hh}{p}a+hh = \square$
10. Then the difference of the said $aa+2pa+hh$ and $\frac{hh}{p}a+hh$, *viz.* the difference of the two Squares sought is	$aa+\frac{2pp-hh}{p}a$
11. Which difference is equal to the Product of the multiplication of these two quantities, to wit,	$a+\frac{2pp-hh}{p}$ and a
12. The half-summ of the two quantities last mentioned is	$a+\frac{2pp-hh}{2p}$

13. Then the Square of the said half-summ being equated to $aa+2pa+hh$, (assumed to be the greater of the two quantities in the eighth step) gives this Equation, *viz.*

$$aa+2pa+hh = aa+\frac{2pp-hh}{p}a+\frac{4pppp+hhhh-4hhpp}{4pp}.$$

14. Which Equation, after due Reduction, gives $a = \frac{pppp+\frac{1}{4}hhhh-2hhpp}{hhp}$

15. Therefore from the second, third and fourteenth steps the two sides about the right-angle of the second right-angled Triangle sought are	b and $\frac{pppp+\frac{1}{4}hhhh-hhpp}{hhp}$

16. The summ of the Squares of those two sides, by taking $bb+pp$ instead of the Factor hh, and $bbbb+2bbpp+pppp$ instead of the Factor $hhhh$, will be found

$$\frac{\frac{1}{16}b^8+\frac{1}{16}p^8+\frac{3}{8}b^4p^4+\frac{3}{4}b^6pp+\frac{3}{4}bbp^6}{b^4pp+2bbp^4+p^6}.$$

17. The square Root of the summ of the Squares in the last step is the Hypothenusal of the second right-angled Triangle sought, to wit,	$\frac{\frac{1}{4}bbbb+\frac{1}{4}pppp+\frac{1}{2}bbpp}{bbp+ppp}$
18. Which Hypothenusal, by taking hh instead of $bb+pp$; and $hhhh$ instead of $bbbb+2bbpp+pppp$, may be expresst thus,	$\frac{\frac{1}{4}hhhh+bbpp}{hhp}$

19. Therefore from the fifteenth and eighteenth steps, the three sides of the second right-angled Triangle sought are these, to wit,

$$b\ ,\ \frac{pppp+\frac{1}{4}hhhh-hhpp}{hhp}\ \text{and}\ \frac{\frac{1}{4}hhhh+bbpp}{hhp}.$$

20. Therefore the Area of the said second right-angled Triangle is	$\frac{ppppb+\frac{1}{4}hhhhb-hhppb}{2hhp}$
21. Which Area divided by $\frac{bp}{2}$ the Area of the right-angled Triangle first given, gives the Quotient	$\frac{pppp+\frac{1}{4}hhhh-hhpp}{hhpp}$
22. The square Root of that Quotient is	$\frac{pp-\frac{1}{2}hh}{hp}$

23. By which square Root, if the three sides of the second right-angled Triangle before expresst in the nineteenth step be severally divided, the Quotients will be the three sides of the third right-angled Triangle sought, to wit,

1. $\frac{hhpb}{pph-\frac{1}{2}hhh}$, (or $\frac{hpb}{pp-\frac{1}{2}hh}$.)
2. $\frac{pppp+\frac{1}{4}hhhh-hhpp}{pph-\frac{1}{2}hhh}$.
3. $\frac{\frac{1}{4}hhhh+bbpp}{pph-\frac{1}{2}hhh}$.

24. But

24. But $bbpp + pppp = hhpp$, therefore inſtead of ſubtracting $hhpp$ in the Numerator of the ſecond of the three ſides laſt before expreſt, we may ſubtract $bbpp + pppp$, whence that ſide may be expreſt thus $\frac{\frac{1}{4}hhhh - bbpp}{pph - \frac{1}{2}hbh}$, and conſequently the three ſides of the third right-angled Triangle ſought ſhall be theſe, to wit,

$$\frac{hhpb}{pph - \frac{1}{2}hbh}, \quad \frac{\frac{1}{4}hhhh - bbpp}{pph - \frac{1}{2}hbh}, \quad \frac{\frac{1}{4}hhhh + bbpp}{pph - \frac{1}{2}hbh}.$$

25. Or, (to avoid Fractions) we may multiply the Numerator and Denominator of every one of the three ſides laſt above expreſt, by 4, ſo theſe three following ſides (which are of the ſame value with thoſe) will be produced for the third right-angled Triangle ſought, to wit,

$$\frac{4hhpb}{4pph - 2hbh}, \quad \frac{hhhh - 4bbpp}{4pph - 2hbh}, \quad \frac{hhhh + 4bbpp}{4pph - 2hbh}.$$

Hence ariſeth,

CANON 1.

26. Let b, p, h repreſent the three ſides of any right-angled Triangle given in rational numbers, whereof the Hypothenuſal is h, and p the greater of the two ſides about the right-angle; then from hh and $2bp$ form a right-angled Triangle, (by the Canon in *Obſervat.* 8. upon the Reſolution by literal Algebra of the firſt Queſtion of this Book;) that done, divide ſeverally the three ſides found out by $4pph - 2hbh$, ſo there will ariſe the three ſides of a right-angled Triangle whoſe Area ſhall be equal to the Area of the given right-angled Triangle b, p, h.

27. Again, by ſuppoſition, $hh + pp = bb + 2pp$, and $p \sqsubset b$, therefore $pp - bb = 2pp - hh$; whence by multiplying each part into $2h$ it follows that $2pph - 2bbh = 4pph - 2hhh$: Therefore inſtead of the Diviſor $4pph - 2hhh$ in *Canon* 1. we may take $2pph - 2bbh$, and ſo there will ariſe the following Canon, (which is the ſame with Monſ. *Fermat*'s in the place before cited.)

CANON 2.

$$\frac{4hhpb}{2pph - 2bbh}, \quad \frac{hhhh - 4bbpp}{2pph - 2bbh}, \quad \frac{hhhh + 4bbpp}{2pph - 2bbh}.$$

In words thus,

28. Let b, p, h repreſent the three ſides of aay right-angled Triangle given in rational numbers, *viz.* h the Hypothenuſal, p the greater ſide about the right-angle, and b the leſſer; then from hh and $2bp$ form a right-angled Triangle, and divide ſeverally the three ſides found out by $2pph - 2bbh$, ſo the Quotients ſhall be the three ſides of a right-angled Triangle whoſe Area is equal to the Area of the given right-angled Triangle b, p, h.

An Example in Numbers.

	b	p	h
Let there be given the three ſides of a right-angled Triangle in rational numbers, to wit,	3	4	5
Then by either of the Canons theſe three ſides of a right-angled Triangle will be found out, to wit,	$\frac{42}{70}$	$\frac{1200}{70}$	$\frac{1201}{70}$

Which latter Triangle hath the ſame Area with the former, to wit, 6.

Again,

	b	p	h
Let there be given this right-angled Triangle in numbers, to wit,	5	12	13
Then by either of the preceding Canons this right-angled Triangle will be found out, to wit,	$\frac{14161}{3094}$	$\frac{40160}{3094}$	$\frac{42961}{3094}$

Which latter Triangle hath the ſame Area with the former, to wit, 30.

From the premiſſes it is evident, that from any right-angled Triangle given in rational numbers another of the ſame Area may be found out; and from the ſecond a third; and from the third a fourth, *&c.* all which right-angled Triangles ſhall have one common Area, to wit, that of the given right-angled Triangle, and all their ſides expreſſible by rational numbers.

29. Moreover, from the nineteenth ſtep of the preceding Reſolution, ſeveral Canons may be deduced, to ſhew how by the help of any right-angled Triangle given in rational numbers,

numbers, to find out another, whoſe Area divided by the Area of the given Triangle will give the Quotient a ſquare number. But I ſhall exhibit only this following

CANON.

Let h repreſent the Hypothenuſal, p the greater ſide about the right-angle, and b the leſſer in any right-angled Triangle given in rational numbers; then from hh and $2bp$ form a right-angled Triangle, and divide ſeverally the three ſides found out by $4hhp$; ſo there will come forth the three ſides of a right-angled Triangle, whoſe Area divided by the Area of the given Triangle, will give the Quotient a ſquare number whoſe Root is $\frac{pp - \frac{1}{2}bh}{hp}$ or $\frac{2pp - hh}{2hp}$.

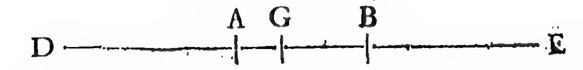
An Example in Numbers.

	b	p	h
Let there be given the three ſides of a right-angled Triangle in numbers, to wit,	3	4	5
Then by the laſt Canon, this right-angled Triangle will be found out, to wit,	3	$\frac{42}{400}$	$\frac{1205}{400}$
The Area of the latter Triangle is			$\frac{147}{800}$
Which Area divided by 6 the Area of the given right-angled Triangle 3, 4, 5 gives a ſquare number, to wit,			$\frac{49}{1600}$
The ſquare Root whereof is			$\frac{7}{40}$

QUEST. 60. (Quæſt. 12. Lib. 5. *Diophant.*)

To divide unity into two ſuch parts, that if to each part a given number, ſuppoſe 6, be added, the two ſumms may be ſquare numbers. But the ſumm of the double of the given number and unity muſt either be a ſquare number, or elſe compoſed of two Squares.

RESOLUTION.



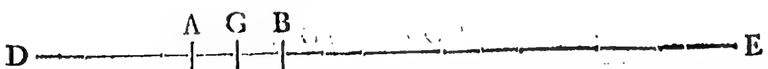
Let AB be 1, and AD = BE be the given number 6; therefore DE = 13. Now we muſt divide AB, to wit, unity, into two parts, ſuppoſe in the point G, that GD and GE may be ſquare numbers; ſo that in effect we muſt divide DE, that is, 13 into two ſuch Squares that one of them may be greater than 6, but leſs than 7. But that may be done by the fourth Queſtion of this Book, where 13 is divided into the Squares $\frac{114442}{1681}$ and $\frac{102424}{1681}$, whoſe ſides are $\frac{107}{41}$ and $\frac{128}{41}$, and each of thoſe Squares is greater than 6, but leſs than 7; therefore taking each Square from 7, the remainders $\frac{3118}{1681}$ and $\frac{11561}{1681}$, (that is GA and GB,) are the deſired parts of unity: For if to each of thoſe parts the given number 6 be added, the two ſumms will be the Squares $\frac{114442}{1681}$ and $\frac{102424}{1681}$.

It is alſo evident, that inſtead of unity, any number may be given to be divided; provided that the ſumm of this number and of the double of the other number given make a Square, or elſe a number compoſed of two Squares.

QUEST. 61. (Quæſt. 13. Lib. 5. *Diophant.*)

To divide unity into two ſuch parts, that to the one adding 2, (a number given,) and to the other 6, (another number given,) the ſumms may be ſquare numbers. But the ſumm of the given numbers with unity muſt either make a ſquare number, or elſe a number compos'd of two Squares.

RESOLUTION.



Let AB be 1, AD = 2. BE = 6; therefore DE = 9. Now we muſt divide AB, to wit, 1 into two parts, ſuppoſe in the point G, that GD and GE may be ſquare numbers: So that in effect we muſt divide DE, that is, 9 into two ſuch Squares that one may be greater than 2, but leſs than 3; or that one Square may be greater than 6, but leſs than 7. But the third Queſtion of this Book ſhews how to find out two Squares ſo qualified, as the Square $\frac{10609}{2809}$ for GD, and the Square $\frac{11881}{2809}$ for GE; the ſides of

which Squares are $\frac{84}{13}$ and $\frac{121}{13}$: then ſubtracting 2 from the Square $\frac{1156}{2809}$, and 6 from the Square $\frac{18225}{2809}$, the remainders $\frac{1428}{2809}$ and $\frac{1378}{2809}$ are the deſired parts GA, GB of unity. For if the former part be added to 2, and the latter to 6, the ſumms are Squares, to wit, $\frac{1156}{2809}$ and $\frac{18225}{2809}$, as was required.

QUEST. 62. (Quæſt. 14. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if to every one of them a given number 3 be added, the three ſumms may be Squares. But the ſumm of the triple of the given number and unity muſt either be a ſquare number, or elſe a number compoſed of two or three Squares.

RESOLUTION.

It is eaſie to apprehend that the ſumm of the three Squares ſought makes 10, and that the ſcope of the ſearch muſt be to find out three Squares, every one of which may fall between 3 and 4, and that their ſumm may be 10; for then the three exceſſes of thoſe Squares above 3 will be the three deſired parts of unity. First then, foraſmuch as 10 is compos'd of two Squares 9 and 1; I divide 10 into two other Squares whereof one may be greater than 3, but leſs than 4; ſuch are the Squares $3\frac{224}{1369}$ and $6\frac{441}{1369}$, whoſe ſides are $\frac{73}{37}$ and $\frac{24}{37}$; for the ſumm of thoſe Squares is 10, and the firſt Square $3\frac{224}{1369}$ is between 3 and 4: Then I take the Fraction $\frac{224}{1369}$, (to wit, the exceſs of the firſt Square above 3) for the firſt of the three deſired parts of unity; for if to that Fraction the given number 3 be added, it makes the Square $3\frac{224}{1369}$.

Then I divide the latter of the Squares firſt found, to wit, $6\frac{441}{1369}$, whoſe ſide is $\frac{93}{37}$, into two ſuch Squares that one may be greater than 3, but leſs than $3\frac{441}{1369}$, or leſs than $3\frac{1}{4}$; for if ſuch Square be leſs than $3\frac{1}{4}$, it will neceſſarily be leſs than $3\frac{224}{1369}$, becauſe $\frac{1}{4}$ is leſs than $\frac{441}{1369}$. But the ſides of two ſuch Squares are $\frac{1233}{1073}$ and $\frac{1860}{1073}$, therefore the Squares themſelves are $\frac{3462229}{1151329}$ and $3\frac{1611329}{1151329}$; from each of which if 3 be ſubtracted, the remainders $\frac{1462229}{1151329}$ and $\frac{161329}{1151329}$, with the Fraction $\frac{224}{1369}$ firſt found, that is, (in the ſame Denominator with the former) $\frac{281424}{1151329}$ ſhall be the three deſired parts of unity to ſolve the Queſtion.

After the ſame manner any number given inſtead of unity may be divided into three ſuch parts, that a number given being added to every one of them may make three Squares. But the ſumm of the number given to be divided, and the triple of the number to be added, muſt be either a Square, or a number compos'd of two or three Squares.

As, if it were deſired to divide 2 into three parts, that each part increaſed with 4 may make a Square: First, foraſmuch as 14 the ſumm of 2 and the triple of 4 is compos'd of three Squares 1, 4 and 9, let 10 the ſumm of 9 and 1 (two of thoſe three Squares) be divided into two other Squares that the firſt may exceed 4 the number given to be added, but beleſs than 6; to the end the exceſs may be leſs than 2 the number given to be divided; then add the other of the two Squares found out to 4, (the other of the three Squares before mentioned, whoſe ſumm made 14,) and divide the ſumm into two ſuch Squares that each may be greater than 4; laſtly, from each of theſe two Squares laſt found out, as alſo from the firſt Square before found, ſubtract 4, ſo the Remainders ſhall be the deſired parts of 2. But the Operation I leave to the Learner's exerciſe.

QUEST. 63. (Quæſt. 15. Lib. 5. *Diophant.*)

To divide unity into three ſuch parts, that if the firſt be increaſed with 2, the ſecond with 3, and the third with 4, the three ſumms may be ſquare numbers. But the ſumm of the three numbers given and unity muſt either be a Square, or compos'd of two or three Squares.

RESOLUTION.

First we muſt divide 10 (the ſumm of the three numbers given with unity) that the firſt may exceed 2, the ſecond 3, and the third 4. To which end, firſt, (by *Queſt.* 4. of this *Book*) divide 10 into two Squares that one may fall between 2 and 3; ſuch are the Squares $\frac{1}{841}\frac{842}{841}$ and $\frac{6}{841}\frac{161}{841}$, whoſe Roots are $\frac{41}{29}$ and $\frac{11}{29}$; then from the firſt Square $\frac{1}{841}\frac{842}{841}$ ſubtract 2, and take the Remainder $\frac{1}{841}\frac{67}{841}$ for one of the deſired parts of unity.

It remains to divide the other Square $\frac{6}{841}\frac{161}{841}$ into two other Squares, that one may fall between

between 3 and 4: But two ſuch Squares will be found $\frac{2}{707}\frac{2}{281}\frac{3}{281}$ and $\frac{2}{707}\frac{6}{281}\frac{4}{281}$, whoſe ſides are $\frac{1}{841}\frac{701}{841}$ and $\frac{1}{841}\frac{620}{841}$; then from the latter of the ſaid Squares ſubtract 3, and from the former 4; ſo the two Remainders $\frac{2}{707}\frac{2}{281}\frac{5}{281}$ and $\frac{-}{707}\frac{6}{281}\frac{4}{281}$ with $\frac{1}{841}\frac{67}{841}$ before found, that is, (in the ſame Denominator with the two former) $\frac{1}{707}\frac{40}{281}\frac{447}{281}$ ſhall be the three deſired numbers, which will ſolve the Queſtion. For firſt, their ſumm makes unity; moreover if 2 be added to $\frac{1}{707}\frac{40}{281}\frac{447}{281}$, that is, $\frac{1}{841}\frac{67}{841}$, (the firſt number) the ſumm is the Square $\frac{1}{841}\frac{842}{841}$; and if 3 be added to the ſecond number $\frac{2}{707}\frac{2}{281}\frac{5}{281}$, it makes the Square $\frac{2}{707}\frac{6}{281}\frac{4}{281}$; laſtly, if 4 be added to the third number $\frac{-}{707}\frac{6}{281}\frac{4}{281}$, it makes the Square $\frac{2}{707}\frac{8}{281}\frac{2}{281}\frac{1}{281}$.

QUEST. 64. (Quæſt. 16. Lib. 5. *Diophant.*)

To divide a given number 10 into three numbers, that the ſumm of every two may be a Square; but the double of the given number muſt be either a Square, or elſe compos'd of two or three Squares.

RESOLUTION.

Foraſmuch as the three numbers ſought are to be ſuch that the ſumm of the firſt and ſecond muſt make a Square, alſo the ſumm of the firſt and third muſt make a Square, likewiſe the ſumm of the ſecond and third muſt make a Square; therefore theſe three Squares are equal to the ſaid three numbers twice taken. And becauſe the ſumm of the three numbers is 10, therefore twice their ſumm, to wit, 20 ſhall be the ſumm of the three Squares. We muſt therefore divide 20 into three Squares, each of which may be leſs than 10; (for every one of theſe Squares muſt be equal to two of the three numbers, and conſequently leſs than 10 the ſumm of all the three numbers.) But 20 is compos'd of two Squares 16 and 4; therefore we may take 4 (which is leſs than 10) for one of the three Squares ſought, and then divide 16 into two Squares, one of which may fall between 10 and 6, for then the other will alſo be leſs than 10, (becauſe both muſt make 16.) But (by *Queſt.* 4. of this *Book*) the ſides of two ſuch Squares will be found $\frac{8}{13}$ and $\frac{4}{13}$; wherefore the three Squares ſought are 4, $\frac{2}{28}\frac{10}{561}$ and $\frac{2}{28}\frac{26}{561}$, which ſubtracted ſeverally from 10 leave three Remainders, 6, $\frac{5}{28}\frac{52}{561}$ and $\frac{1}{28}\frac{92}{561}$ for the three deſired numbers, whoſe ſumm is 10, and every two of them added together makes a Square, as was deſired.

But to make it more clearly evident that three numbers ſo found out will ſolve the Queſtion, let *bb*, *cc*, *dd* repreſent three ſquare numbers, then

Suppoſe . . . . . $bb + cc + dd = 20$

And conſequently . . . . . $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd = 10$

Then from 10, that is, from $\frac{1}{2}bb + \frac{1}{2}cc + \frac{1}{2}dd$ ſubtract
bb, *cc* and *dd* ſeverally, ſo the three Remainders are . . . .
$\frac{1}{2}cc + \frac{1}{2}dd - \frac{1}{2}bb$
$\frac{1}{2}bb + \frac{1}{2}dd - \frac{1}{2}cc$
$\frac{1}{2}bb + \frac{1}{2}cc - \frac{1}{2}dd$

I ſay thoſe three Remainders ſhall be three numbers to ſolve the Queſtion; for the ſumm of the firſt and ſecond makes the Square *dd*, the ſumm of the ſecond and third makes the Square *bb*, and the ſumm of the firſt and third makes the Square *cc*.

QUEST. 65. (Quæſt. 17. Lib. 5. *Diophant.*)

To divide a given number, ſuppoſe 10, into four numbers, that the ſumm of every three may make a Square.

RESOLUTION.

Foraſmuch as the ſumm of every three of the four numbers ſought muſt make a Square, therefore the four Squares ſought are equal to the four deſired numbers thrice taken. But the four numbers thrice taken make 30, therefore 30 muſt be divided into four ſuch Squares that every one of them may be leſs than 10, (for every one of the four Squares muſt be equal to three of the numbers ſought, and conſequently be leſs than 10 the ſumm of all four.) But 30 is compos'd of four Squares, 16, 9, 4 and 1, two of which, to wit, 9 and 4 may be taken for two of the Squares ſought, and then 17 (the ſumm of the other two Squares 16 and 1) muſt be divided into two Squares that one may be leſs than 10, but greater than 7, and then the other will be alſo leſs than 10; but the ſides of two ſuch Squares will (by *Queſt.* 4. of this *Book*) be found $\frac{15}{17}$ and $\frac{8}{17}$, and the Squares themſelves are $\frac{2}{2}\frac{2}{89}\frac{5}{89}$ and $\frac{2}{2}\frac{8}{89}\frac{9}{89}$, each of which is leſs than 10. Therefore the four Squares

ſought are 9, 4, $\frac{2704}{289}$ and $\frac{2209}{289}$, which ſubtracted ſeverally from 10, leave the Remainders 1, 6, $\frac{1456}{289}$ and $\frac{4811}{289}$ for the four numbers ſought, whoſe ſumm makes 10, and every three of them added together make the Squares $\frac{2209}{289}$, 9, 4 and $\frac{2704}{289}$, whoſe ſides are $\frac{47}{17}$, 3, 2 and $\frac{43}{17}$.

Note. If the quadruple of the number given be a whole number, this Queſtion may be extended to five numbers, or as many as you pleaſe: for every whole number is compos'd of four Squares, which may be divided into any multitude of Squares within any poſſible limits, by the help of the third Queſtion of this Book.

QUEST. 66. (*A* Lemma, *uſed in the following* Queſt. 67.)

To find three ſuch Cube-numbers, that if from every one of them a number given, ſuppoſe 1, be ſubtracted, the ſumm of the Remainders may be a Square.

RESOLUTION.

1. For the ſide of the firſt Cube put $a+1$ any abſolute number, as, . . . } $a+1$
2. Then take ſome ſquare number, as 9, and from its $\frac{1}{3}$, to wit, from 3 ſubtract 1 the abſolute number in the ſide of the firſt Cube, and let the Remainder 2 be connected by + with $-a$ for the ſide of the ſecond Cube; the reaſon whereof will appear in *Obſervation* 1. upon this Queſtion, ... } $-a+2$
3. Let the ſide of the third Cube be any known number whoſe Cube exceeds the number given in the Queſtion, to wit, the ſide . . . } 2
4. Then the Cubes of thoſe three ſides are theſe, to wit,

1.	$aaa+3aa+3a+1$,
2.	$-aaa+6aa-12a+8$,
3.	. . . . . $+8$.

5. From every one of which Cubes ſubtract 1 the number given in the Queſtion, and add the Remainders together, ſo the ſumm will be $9aa-9a+14$, which muſt be equated to a Square; but the ſide thereof muſt be ſo feigned that the value of a may be leſs than 2, to the end that $-a+2$, or $2-a$ the ſide of the ſecond Cube may be greater than nothing. Now to cauſe that effect, the ſaid ſide may be feigned $3a-$ any abſolute number between $\sqrt{14}$ and $\sqrt{32}+6$, (which limits are found out after the method before delivered in divers Queſtions of this Book;) let therefore the ſide of the ſaid Square be feigned $3a-4$, and then the Square of $3a-4$ being equated to $9aa-9a+14$ above mentioned, this Equation ariſeth, to wit,
$$9aa-9a+14=9aa-24a+16.$$
6. From which Equation, the value of a will be made known, *viz.* . . } $a=\frac{2}{15}$
7. Therefore from the ſixth, firſt, ſecond and third ſteps the ſides of the three Cubes ſought are $\frac{17}{15}$, $\frac{28}{15}$ and 2; wherefore the Cubes themſelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, which will ſolve the Queſtion. For if from every one of thoſe Cubes the given number 1 be ſubtracted, the ſumm of the three Remainders in its leaſt terms is $\frac{324}{25}$, which is a Square, as was required.

Obſervations upon Queſt. 66.

1. Two things are remarkable in the preceding poſitions for the ſides of the Cubes ſought; Firſt, in the firſt ſide $a+1$ there is put $+a$, and in the ſecond ſide $-a+2$ there is put $-a$, to the end that in adding together the Remainders mentioned in the fifth ſtep, $+aaa$ and $-aaa$ may deſtroy one another: Secondly, the unities in the ſecond ſide muſt be choſen with ſuch Caution that the number prefixt to aa in the ſumm of the ſaid three Remainders may be a Square, for if $9aa$ had not been a Square, then $9aa-9a+14$ could not have been equated to a Square. Therefore that the number prefixt to aa in the ſaid ſumm of the Remainders may infallibly come forth a ſquare number, the number of unities to be connected with $-a$ by + in the ſide of the ſecond Cube, muſt be the exceſs of one third part of ſome ſquare number above the unit or unities in the ſide of the firſt Cube: The reaſon whereof may be thus manifeſted, Suppoſe $a+b$, and $-a+d$, and 2, to be the ſides of the three Cubes ſought; then from the Cubes of thoſe ſides ſubtracting the given number 1 ſeverally, the ſumm of the Remainders is $3baa+3daa+3bba-3dda+bbb+ddd+5$; which ſumm cannot be equated to a Square, unleſs the number ſignified by $3b+3d$, which is multiplied into aa be a ſquare number;

number; ſuppoſe therefore $3b+3d=ff$, whence by due Reduction, $d=\frac{1}{3}ff-b$; which ſhews that d the number of unities in $-a+d$ the ſide of the ſecond Cube, muſt be the exceſs of one third part of ſome ſquare number, above b the unit or unities in $a+b$ the feigned ſide of the firſt Cube, as was directed in the ſecond ſtep of the Reſolution.

2. It is eaſie to apprehend that this Queſtion may be extended to as many numbers as you pleaſe: For having put (as before) $a+1$ for the ſide of the firſt Cube, and (for the reaſons before given) $-a+2$ for the ſide of the ſecond Cube, you may put 2 for the ſide of a third Cube, 3 for the ſide of a fourth, *&c.* provided that the Cubes of theſe abſolute numbers 2, 3, *&c.* do every one of them exceed the number given to be ſubtracted; then from the ſumm of all thoſe Cubes ſubtracting the given number, the ſumm of the Remainders may be equated to a Square, becauſe by Conſtruction the number prefixt to aa is a Square.

QUEST. 67. (Queſt. 18. Lib. 5. *Diophant.*)

To find three numbers, that if they be ſeverally added to the Cube of their ſumm, the three ſumms made by thoſe additions may be Cubes.

RESOLUTION.

1. For the ſumm of the three numbers ſought put . . . } a
2. Then the Cube of their ſumm is . . . . . } aaa
3. For the three numbers ſought put . . . . . } $7aaa$, $26aaa$ and $63aaa$
4. Whence it is manifeſt, that if each of the three laſt mentioned quantities aſſumed for the three numbers ſought, be increaſed with aaa which was put for the Cube of their ſumm, there will come forth the Cubes $8aaa$, $27aaa$ and $64aaa$; but the ſumm of theſe three quantities $7aaa$, $26aaa$ and $63aaa$ muſt be equal to a; therefore $96aaa=a$, and conſequently $96aa=1$: where if 96 were a ſquare number, the value of a would be expreſſible by a rational number, and conſequently the Queſtion ſolved. Whence therefore comes 96? examine the Poſitions and you will find that by ſubtracting 1 (to wit, unity) from the three Cubes 8, 27 and 64, the remainders 7, 26 and 63 added together make 96. Therefore we muſt ſeek three ſuch Cubes that if 1 be ſubtracted from every one of them, the ſumm of the three remainders may be a Square: But the ſides of three ſuch Cubes are $\frac{17}{15}$, $\frac{28}{15}$ and 2, (found out by the preceding *Queſt.* 66.) and the Cubes themſelves are $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8, from every one of which if 1 be ſubtracted, the three remainders will be $\frac{1538}{3375}$, $\frac{18577}{3375}$ and 7, whoſe ſumm in its leaſt terms is the Square $\frac{324}{25}$, whoſe ſide is $\frac{18}{5}$. Now by the help of thoſe remainders, let the work be renewed thus, *viz.*

5. For the ſumm of the three numbers ſought put . . . } a
6. Whence the Cube of their ſumm is . . . . . } aaa
7. Then for the firſt number put . . . . . } $\frac{1538}{3375}aaa$
8. For the ſecond . . . . . } $\frac{18577}{3375}aaa$
9. And for the third . . . . . } $7aaa$
10. Then the ſumm of the three numbers laſt expreſt is . . } $\frac{324}{25}aaa$
11. Which ſumm muſt be equal to a, which in the fifth ſtep was put for the ſumm of the three numbers ſought, therefore } $\frac{324}{25}aaa=a$
12. Which Equation, after due Reduction, diſcovers the ſumm of the three numbers ſought, *viz.* . . . . . } $a=\frac{25}{324}$

Therefore from the twelfth, ſeventh, eighth and ninth ſteps, the three numbers ſought will be $\frac{1538}{157464}$, $\frac{18577}{157464}$, $\frac{21632}{157464}$, whoſe ſumm is $\frac{25}{324}$, the Cube whereof is $\frac{15625}{5971968}$; which being added to every one of the ſaid three numbers, the ſumms will be Cubes, to wit, $\frac{4913}{157464}$, $\frac{21952}{157464}$, $\frac{373248}{157464}$, whoſe ſides are $\frac{17}{54}$, $\frac{28}{54}$ and $\frac{120}{54}$; therefore the Queſtion is ſatisfied, and by the help of the preceding *Queſt.* 66. may be extended to four, five or as many numbers as ſhall be deſired.

QUEST. 68.

To find two Cube-numbers, that if their difference be increaſed with a given number, ſuppoſe 2, it may make a Square, and that the ſide of the greater Cube may be leſs than 1 a number given.

RESO-

RESOLUTION.

1. For the side of the lesser Cube sought put . . . . . } a
2. For the side of the greater Cube put $a \dashv$ one third part of some square number; but such third part must be less than 1 the prescribed limit, therefore let the side of the greater Cube be . . } $a \dashv \frac{1}{3}$
3. Therefore the greater Cube is . . . . . } $aaa \dashv aa \dashv \frac{1}{3}a \dashv \frac{1}{27}$
4. And the lesser Cube is . . . . . } aaa
5. Therefore the difference of the said Cubes is . . . . . } . . $aa \dashv \frac{1}{3}a \dashv \frac{1}{27}$
6. To which difference add 2 the number first given in the Question, and the summ is . . . . . } . . $aa \dashv \frac{1}{3}a \dashv 2\frac{1}{27}$
7. Which summ must be equal to a Square, the side whereof must be so feigned that the value of a may be less than $\frac{2}{3}$, for then $a \dashv \frac{1}{3}$ (the side of the greater Cube) will be less than 1, as the Question requires. Now to cause that effect, the side of the said Square may be feigned to be either $a \dashv$ any known number between $\frac{228}{1000}$ and $1\frac{427}{1000}$, or else $-a \dashv$ any known number between $1\frac{428}{1000}$ and $2\frac{31}{100}$, (which limits may be found out by the method directed in the preceding *Quest.* 12.) let therefore the said side be feigned $a \dashv \frac{4}{3}$, and then the Square of $a \dashv \frac{4}{3}$ being equated to the summ in the sixth step, this Equation ariseth, *viz.*

$$aa \dashv \tfrac{8}{3}a \dashv \tfrac{16}{9} = aa \dashv \tfrac{1}{3}a \dashv 2\tfrac{5}{27}.$$

8. Which Equation after due Reduction gives . . . . . } $a = \frac{1}{9}$
9. Therefore from the eighth, first and second steps the sides of the two Cubes sought are . . . . . } $\frac{1}{9}$ and $\frac{4}{9}$
10. The Cubes of which sides $\frac{1}{9}$ and $\frac{4}{9}$, *viz.* $\frac{1}{729}$ and $\frac{64}{729}$ will solve the Question; for if to their difference $\frac{63}{729}$ you add 2, the summ $\frac{1682}{81}$ is a Square, and $\frac{4}{9}$ the side of the greater Cube is less than 1, as was required.

Example. 2.

Let it be required to find two Cube-numbers whose difference added to 1458 (a given number) may make a Square, and that the side of the greater Cube may be less than 9 (a number given.)

Resolution.

1. For the side of the lesser Cube put . . . . . } a
2. For the side of the greater Cube put $a \dashv$ one third part of some square number, but such third part must be less than 9 the limit above prescribed; therefore let the said side be . . } $a \dashv 3$
3. Therefore the greater Cube is . . . . . } $aaa \dashv 9aa \dashv 27a \dashv 27$
4. And the lesser Cube is . . . . . } aaa
5. Therefore the difference of those Cubes is . . . . . } . $\dashv 9aa \dashv 27a \dashv 27$
6. To which difference adding the number first given in this *Example* 2. to wit, ... } 1458
7. The summ will be . . . . . } $\dashv 9aa \dashv 27a \dashv 1485$
8. Which summ must be equal to a Square, the side whereof must be so feigned that the value of a may be less than 6, for then $a \dashv 3$ (which in the second step was assumed for the side of the greater Cube sought) will be less than the prescribed limit 9; Now to cause that effect, the side of the said Square may be feigned to be $3a \dashv$ any number between 26 and 39, or $-3a \dashv$ any number between 38 and 63; suppose therefore the said side be feigned $3a \dashv 36$, then the Square of $3a \dashv 36$ being equated to the summ in the seventh step, this Equation ariseth, *viz.*

$$9aa \dashv 216a \dashv 1296 = 9aa \dashv 27a \dashv 1485.$$

9. Which Equation after due Reduction gives . . . . . } $a = 1$
10. Therefore from the ninth, first and second steps of this second Example, the sides of the Cubes sought are . . . } 1 and 4
11. The Cubes of which sides 1 and 4, *viz.* 1 and 64 will solve the Question; for if their difference 63 be added to 1458 the number given in *Example* 2. it makes the Square 1521, whose side is 39; and 4 the side of the greater Cube is less than 9, as was required.

Example 3.

Example 3.

Again, the same numbers 1458 and 9 being given as before in *Example* 2. the side of the Square mentioned in the eighth step may be feigned to be $-3a \dashv 48$, (which is within the limits there expresst,) and then the Square of $-3a \dashv 48$ being equated to $9aa \dashv 27a \dashv 1485$, (before expresst in the seventh step,) after due Reduction the sides of two Cubes to solve *Quest.* 68. as it is before propos'd in *Example* 2. will be found $\frac{228}{125}$ and $\frac{332}{125}$; therefore the Cubes themselves are $\frac{11852352}{1953125}$ and $\frac{36594368}{1953125}$, whose difference $\frac{24741016}{1953125}$ added to the given number 1458 makes a Square, to wit, $\frac{2848766216}{1953125}$, whose side is $\frac{53384}{1125}$ and $\frac{228}{125}$ the side of the greater Cube is less than the prescribed number 9.

QUEST. 69.

To find two such cube-numbers, that if each of them be subtracted from a given squared cube-number, the summ of the remainders may be a Square.

RESOLUTION.

1. Let the given squared cube-number be . . . . } $ddddd$ or d^5
2. The Root or side whereof is . . . . . } d
3. For the side of the first Cube sought put . . . } a
4. The Cube thereof is . . . . . } aaa
5. For the side of the other Cube sought put $-a \dashv$ the cube-root of the given squared Cube, *viz.* . . } $-a \dashv dd$
6. The Cube thereof is . . . . . } $-aaa \dashv 3ddaa - 3d^4a \dashv d^6$
7. Then by subtracting the Cube in the fourth step from the given squared Cube in the first, there will remain } $\dashv d^6 - aaa$
8. And by subtracting the Cube in the sixth step from the given squared Cube in the first, there will remain } $\dashv aaa - 3ddaa \dashv 3d^4a$
9. The summ of those remainders (in the seventh and eighth steps) is . . . . . } $-3ddaa \dashv 3d^4a \dashv d^6$
10. Which summ must be equal to a Square, the side whereof (in regard d^6 is a Square) we may feign to be either $ea \dashv ddd$, or $ea - ddd$, (where e represents a number yet unknown, and to be chosen according to the limit hereafter discovered:) First then let the said side be feigned $ea \dashv ddd$, and then the Square of $ea \dashv ddd$ being equated to the summ in the ninth step, will give

$$-3ddaa \dashv 3d^4a \dashv d^6 = eeaa \dashv 2ed^3a \dashv d^6.$$

11. Which Equation after due Reduction gives . . } $a = \frac{3d^4 - 2ed^3}{3dd \dashv ee}$
12. Therefore from the eleventh, first, second, third and fifth steps the sides of the two Cubes sought will be found equal to these quantities, *viz.* . . } $\frac{3d^4 - 2ed^3}{3dd \dashv ee}$ and $\frac{eedd \dashv 2ed^3}{3dd \dashv ee}$
13. Again, forasmuch as the side of the Square mentioned in the tenth step may be feigned to be $ea - ddd$ (as well as $ea \dashv ddd$,) let the said side be $ea - ddd$, and then its Square being equated to the summ in the ninth step, this Equation ariseth, *viz.*

$$-3ddaa \dashv 3d^4a \dashv d^6 = eeaa - 2ed^3a \dashv d^6.$$

14. Which Equation after due Reduction gives . . } $a = \frac{3d^4 \dashv 2ed^3}{3dd \dashv ee}$
15. Therefore from the fourteenth, first, second, third and fifth steps the sides of the two Cubes sought will be found equal to these quantities, *viz* . . . } $\frac{3d^4 \dashv 2ed^3}{3dd \dashv ee}$ and $\frac{ddee - 2ed^3}{3dd \dashv ee}$

The two quantities expresst by letters in the twelfth step will give

CANON 1.

16. Supposing d to be the side or Root of the squared cube-number given, take some known number, with this Caution, That its double may be less than the triple of d, and call the number taken e, then the sides of the two Cubes sought shall be these, *viz.*

$$\frac{3d^4 - 2ed^3}{3dd \dashv ee} \text{ and } \frac{eedd \dashv 2ed^3}{3dd \dashv ee}.$$

An

An Example in Numbers.

Let there be given any squared cube-number, as . . . . . $\therefore$ $64 = d^6$
The Root or side whereof is . . . . . $\therefore$ $2 = d$
Then take a number for e, according to the Caution in the *Canon*, as $\therefore$ $2 = e$

Then by the *Canon* you will find $\left\{\begin{array}{l}\frac{3d^4 - 2ed^3}{3dd + ee} = 1 \\ \frac{eedd + 2ed^3}{3dd + ee} = 3\end{array}\right\}$ the sides of the Cubes sought.

The Cubes of which sides 1 and 3, *viz.* 1 and 27 will solve the Question proposed; for if each of those Cubes be subtracted from the given squared Cube 64, the sum of the remainders 63 and 37 makes a Square, to wit, 100.

The two quantities expresst by letters in the fifteenth step will give

CANON 2.

17. Supposing d to be the side or Root of a given squared cube-number, take some known number with this Caution, That it be greater than the double of d, and call the number taken e, then the sides of the two Cubes sought shall be these, *viz.*

$$\frac{3d^4 + 2ed^3}{3dd + ee} \text{ and } \frac{ddee - 2ed^3}{3dd + ee}.$$

An Example of Canon 2. *in Numbers.*

Let there be given any squared Cube, as . . . . . $\therefore$ $1 = d^6$
The side or Root whereof is . . . . . $\therefore$ $1 = d$
Then take a number for e, according to the Caution in *Canon* 2. as $\therefore$ $4 = e$

Then by *Canon* 2. you will find $\left\{\begin{array}{l}\frac{3d^4 + 2ed^3}{3dd + ee} = \frac{11}{19} \\ \frac{ddee - 2ed^3}{3dd + ee} = \frac{8}{19}\end{array}\right\}$ the sides of the Cubes sought.

The Cubes of which sides $\frac{11}{19}$ and $\frac{8}{19}$, *viz.* $\frac{1331}{6859}$ and $\frac{512}{6859}$ will solve *Queſt.* 69. for if each of those Cubes be subtracted from 1 the given squared Cube, the sum of the remainders makes a Square, to wit, $\frac{4225}{361}$, whose side is $\frac{65}{19}$.

QUEST. 70.

To find three such cube-numbers, that if every one of them be subtracted from a given Cube, suppose 1, the sum of the three remainders may be a Square.

RESOLUTION.

1. First, by the foregoing *Queſt.* 68. find two such cube-numbers, that their difference being added to 2, (the double of the Cube given in this Question,) the sum may be a Square, and that the greater of those two Cubes may be less than the given Cube 1. But two such Cubes are $\frac{64}{729}$ and $\frac{1}{729}$, whose sides are $\frac{4}{9}$ and $\frac{1}{9}$, (found out in the first *Example* of *Queſt.* 68.) for if the difference of the said Cubes be added to 2, the sum is $\frac{1682}{81}$, which is a square number whose side is $\frac{41}{9}$.
2. Then for the side of the first of the three Cubes sought put $a - \frac{4}{9}$; ($\frac{4}{9}$ being the side of the greater of the two Cubes found out in the first step,) . . . . . $a - \frac{4}{9}$
3. For the side of the second Cube sought put $-a + 1$, (1 being the side of the Cube given in the Question,) . . . . $-a + 1$
4. And let the side of the third Cube be the side of the lesser of the two Cubes found out in the first step, to wit, ... $\frac{1}{9}$
5. Then from the second step the first Cube will be . . . . $\therefore$ $aaa - \frac{12}{9}aa + \frac{16}{27}a - \frac{64}{729}$
6. And from the third step, the second Cube will be . . . . $\therefore$ $-aaa + 3aa - 3a + 1$
7. And from the fourth step the third Cube will be . . . . $\therefore$ . . $\frac{1}{729}$
8. Then, according to the Question subtract those three Cubes severally from the given Cube 1, so the three remainders shall be these, to wit,

1.	$-aaa + \frac{12}{9}aa - \frac{16}{27}a + \frac{728}{729}$,
2.	$+aaa - 3aa + 3a$,
3.	. . . . . $+ \frac{728}{729}$.

9. The

9. The sum of the said three remainders is . . . . $\therefore$ $-\frac{1}{3}aa + \frac{65}{27}a + \frac{1682}{81}$
10. Which sum must be equal to a Square, whose side, to the end the value of a may be greater than $\frac{4}{9}$, but less than 1, (as the Positions in the second and third steps do require) may be feigned to be either $\frac{13}{9} +$ any number of a between $\frac{2356}{1000}a$ and $\frac{813}{1000}a$, or else $\frac{13}{9} -$ any number of a greater than $3\frac{125}{1000}a$, (which limits may be found out by the method before delivered in *Queſt.* 13. of this *Book*;) suppose therefore the said side to be $\frac{1}{3}a + \frac{13}{9}$, then the Square of $\frac{1}{3}a + \frac{13}{9}$ being equated to the sum of the three remainders in the ninth step, this Equation ariseth, *viz.*

$$-\tfrac{1}{3}aa + \tfrac{65}{27}a + \tfrac{1682}{81} = \tfrac{1}{9}aa + \tfrac{26}{27}a + \tfrac{169}{81}.$$

11. Which Equation after due Reduction will give . . $\therefore$ $a = \frac{13}{16}$
12. Therefore from the eleventh, second, third and fourth steps, the sides of the three Cubes sought will be these, $\left\{\begin{array}{l}-\frac{52}{144}, -\frac{27}{144}, \text{ and } -\frac{16}{144}\end{array}\right.$
13. The Cubes of which sides are these . . . . . $\therefore$ $-\frac{148877}{2985984}, -\frac{12683}{2985984}, -\frac{4096}{2985984}$

Which three Cubes will solve the Question, as will appear by

The Proof.

By subtracting severally the said three Cubes found out from 1, (the Cube given in the Question,) the three remainders will be these,

$$\frac{2837107}{2985984}, \quad \frac{2966301}{2985984}, \quad \frac{2981888}{2985984}.$$

The sum of those remainders is . . . . . $\therefore$ . . $\frac{8785296}{2985984}$
Which sum reduced to its least terms by the common Divisor 144, will be . . . . . $\left\{\begin{array}{l}\text{. . } \frac{61022}{20736}\end{array}\right.$

Which $\frac{61022}{20736}$ is a Square, whose side is $\frac{247}{144}$; therefore the Question is solved, and (as is evident by the tenth step) capable of innumerable Answers, the positions in the second, third and fourth steps standing unaltred.

Observations upon the preceding Resolution *of* Queſt. 70.

1. The chief scope in the said Resolution is, to form the positions for the sides of the three Cubes sought in such manner, that when the said Cubes are severally subtracted from the Cube given in the Question, there may be a possibility of equating the sum of the three remainders to a Square, which sum (as you see in the ninth step) is $-\frac{1}{3}aa + \frac{65}{27}a + \frac{1682}{81}$, which could not be equated to a Square if $\frac{1682}{81}$ were not an affirmative square number; I shall therefore shew how the said $\frac{1682}{81}$ doth necessarily become an affirmative square number by the preceding Operation.
2. If the subtraction of every one of the three feigned Cubes in the fifth, sixth and seventh steps from the given Cube 1, as also the adding of the remainders together be well examined, it will appear, that by adding the Cube $\frac{64}{729}$ to 1, and by subtracting the Cube $\frac{1}{729}$ from 1, and then by adding that sum and remainder together, their sum is $\frac{1682}{81}$, which (in regard by Construction the greater of the said Cubes, to wit, $\frac{64}{729}$ is added to 1, and the lesser $\frac{1}{729}$ is subtracted from 1) is the same with the sum that will arise by adding the difference of those Cubes unto 2, (the double of 1.) For if the greater of two numbers be added unto, and the lesser be subtracted from a third number, the sum and remainder added together will make the same sum that ariseth by adding the difference of those two numbers to the double of the said third number: But by Construction in the first step of the Resolution, the said Cubes $\frac{64}{729}$ and $\frac{1}{729}$ are found such that their difference added to 2 makes a Square, to wit, $\frac{1681}{81}$. Whence it is manifest that the Algebraick quantity $-\frac{1}{3}aa + \frac{65}{27}a + \frac{1682}{81}$ is capable of being equated to a Square, and that variously, as you see in the tenth step of the Resolution.

Example 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from a given Cube, suppose 729, the sum of the three remainders may be a square number.

Resolution.

1. First, by *Queſt.* 68. find two such Cubes that if their difference be added to 1458, to wit, the double of the given Cube 729, the sum may be a Square, and that the side of the greater of those two Cubes may be less than 9 the side of the given Cube 729: But two such Cubes are 64 and 1, (found out in the second *Example* of *Queſt.* 68.)

for if their difference 63 be added to the prescribed number 1458, the summ 1521 is a Square whose side is 39.

2. Then for the side of the first of the three Cubes sought let there be put $a-4$, (4 being the side of the Cube 64, the greater of the two Cubes found out in the first step,) . . $a-4$
3. For the side of the second Cube sought put $-a+9$, (9 being the side of the given Cube 729,) . . . $-a+9$
4. Let the side of the third Cube be 1, to wit, the side of the lesser of the two Cubes fouhd out in the first step, ... 1
5. Then (according to the Question) subtract severally the Cubes of those three sides (assumed in the three last steps) from the given Cube 729, and add the three remainders together, so the summ will be

$$-15aa+195a+1521.$$

6. Which summ must be equal to a Square, whose side, to the end the value of a may be greater than 4, but less than 9, as the second and third steps require, may be feigned to be either $39+$ any number of a between $\frac{210}{1000}a$ and $1\frac{522}{1000}a$, or else $39-$ any number of a between $9\frac{111}{1000}a$ and $21\frac{22}{1000}a$, (which limits may be found out by the method delivered in *Queſt.* 13. of this Book.) Suppose therefore the said side be feigned $39+a$, then the Square thereof being equated to the summ in the fifth step, this Equation will arise, to wit,

$$aa+78a+1521 = -15aa+195a+1521.$$

7. Which Equation after due Reduction gives . . . . $a = \frac{111}{16}$
8. Therefore from the seventh, second, third and fourth steps the sides of the three Cubes sought are these, . . $\frac{53}{16}$, $\frac{27}{16}$ and 1 (or $\frac{16}{16}$)
9. And consequently the Cubes themselves are . . . $\frac{148877}{4096}$, $\frac{12681}{4096}$, $\frac{4096}{4096}$ (or 1)

Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting every one of the said three Cubes in the ninth step from the given Cube 729, the three remainders will be these,

$$\frac{2837107}{4096}, \quad \frac{2966301}{4096}, \quad \frac{2981888}{4096}.$$

The summ of those remainders is . . . . . $\frac{8785296}{4096}$
Which summ being reduced to its least terms by the common Divisor 16, will be . . . . . $\frac{549081}{4096}$
Which is a Square, whose side is . . . . . $\frac{741}{16}$

Example 3.

1. Again, the same things remaining as before in the second Example from the first to the sixth step, we may feign the side of the Square mentioned in the said sixth step to be $39-10a$, and then the Square of $39-10a$ being equated to $-15aa+195a+1521$ will give $a = \frac{192}{23}$.
2. Therefore from the second, third and fourth steps of *Example* 2. the sides of three other Cubes to solve the Question propounded in the said second Example will be found these, to wit, $\frac{1021}{23}$, $\frac{12}{23}$, and 1 (or $\frac{23}{23}$.)
3. And consequently the Cubes themselves are

$$\frac{1092727}{12167}, \quad \frac{1728}{12167}, \quad 1 \text{ (or } \frac{12167}{12167}.)$$

4. Which three Cubes being severally subtracted from the given Cube 729, the summ of the three remainders in its least terms will be $\frac{1102809}{529}$, which is a Square, whose side is $\frac{1051}{23}$, as was required in *Example* 2.

QUEST. 71.

[*Another way of solving the preceding* Queſt. 70. *when the given Cube is a squared Cube, or the Cube of a Square.*]

Let it be required to find three Cube-numbers, such, that if every one of them be subtracted from a given squared Cube-number, suppose 64, the summ of the three remainders may be a Square.

RESO-

RESOLUTION.

1. First, by the foregoing *Queſt.* 69. find two such cube-numbers, that if each of them be subtracted from the given squared Cube 64, the summ of the remainders may be a Square; such are the Cubes 1 and 27, (found out in the Example of *Canon* 1. of the said *Queſt.* 69.) for if each of them be subtracted from 64, the summ of the remainders makes the Square 100.
2. Then for the side of the first of the three Cubes sought put $a+$ either of the sides of the two Cubes found out in the first step, *viz.* . . . . . $a+1$
3. For the side of the second Cube put $-a+4$, (4 being the side of the given squared Cube 64,) . . . . . $-a+4$
4. Let the side of the third Cube, be the side of the other of the two Cubes found out in the first step, to wit, ... 3
5. Therefore from the second step the first Cube is . . . $aaa+3aa+3a+1$
6. And from the third step, the second Cube is . . . . $-aaa+12aa-48a+64$
7. And from the fourth step, the third Cube is . . . . . $+27$
8. Then (according to the Question) subtract those three Cubes severally from the given squared Cube 64, so the three remainders will be these, to wit,

1.	$-aaa-3aa-3a+63$,
2.	$+aaa-12aa+48a$,
3.	$+37$.

9. The summ of which remainders is $-15aa+45a+100$.
10. Which summ is to be equated to a Square, but the side thereof must be so feigned that the value of a may be less than 3, to the end the side $a+1$ in the second step may be less than 4, because the Cube of $a+1$ must be subtracted from the Cube of 4. Now to cause that effect, the side of the said Square may be feigned $10+$ any number of a less than $2\frac{1}{4}a$, or $10-$ any number of a greater than $6\frac{2}{3}a$; (which limits may be found out by the method before delivered in *Queſt.* 13. of this Book:) Let therefore the said side be feigned $a+10$, then the Square thereof, to wit, $aa+20a+100$ being equated to the summ of the remainders in the ninth step, this Equation ariseth, *viz.*

$$aa+20a+100 = -15aa+45a+100.$$

11. Which Equation, after due Reduction, gives . . . $a = \frac{25}{16}$
12. Therefore from the eleventh, second, third and fourth steps, the sides of the three Cubes sought will be found $\frac{41}{16}$, $\frac{12}{16}$ and 3 these, to wit, ...
13. Therefore the Cubes themselves are . . . . . $\frac{68921}{4096}$, $\frac{12312}{4096}$ and 27

Which three Cubes will solve the Question, as will be evident by

The Proof.

By subtracting the said three Cubes severally from 64, (the squared Cube given in the Question,) the three remainders will be these,

$$\frac{193223}{4096}, \quad \frac{202825}{4096}, \quad \frac{151552}{4096}.$$

The summ of those remainders is . . . . . $\frac{547600}{4096}$
Which summ being reduced to its least terms by the common Divisor 16, will be . . . . . $\frac{34225}{256}$
Which $\frac{34225}{256}$ is a Square whose side is $\frac{185}{16}$, therefore the Question is solved.

Example. 2.

Let it be required to find three such Cube-numbers, that if every one of them be subtracted from 1, the summ of the three remainders may be a Square.

Resolution.

1. First, by the preceding *Queſt.* 69. find two such Cube-numbers that if each of them be subtracted from 1, to wit, the given squared Cube, the summ of the three remainders may be a Square; such are the Cubes $\frac{1}{64}$ and $\frac{27}{64}$, for if each of them be subtracted from 1, the summ of the remainders will be $\frac{25}{16}$, which is a Square.

2. Then for the ſide of the firſt of the three Cubes ſought put $a + \frac{1}{4}$, ſide of one of the two Cubes found out in the firſt ſtep, to wit, ... $a + \frac{1}{4}$
3. And for the ſide of the ſecond Cube put $-a + 1$, (1 being the ſide of the given ſquared Cube,) . . . . . $-a + 1$
4. And let the ſide of the third Cube be the ſide of the other of the two Cubes found out in the firſt ſtep, to wit, ... $\frac{3}{4}$
5. Then (according to the Queſtion) ſubtract ſeverally the Cubes of thoſe three ſides aſſumed in the three laſt foregoing ſteps from 1, (the given ſquared Cube,) and add the three remainders together, ſo the ſumm will be
$$-\tfrac{15}{4}aa + \tfrac{225}{16}a + \tfrac{100}{64} \text{ (or } \tfrac{25}{16}.)$$
6. Which ſumm muſt be equal to a Square, whoſe ſide muſt be ſo feigned that the value of a may be leſs than $\frac{3}{4}$, to the end the ſide $a + \frac{1}{4}$ in the ſecond ſtep of this ſecond Example may be leſs than 1, for then every one of the three remainders of the ſubtraction mentioned in the fifth ſtep will be greater than nothing. Now to cauſe that effect, the ſide of the ſaid Square may be feigned $\frac{1}{4} +$ any number of a leſs than $\frac{3}{8}a$, or elſe $\frac{1}{4} -$ any number of a greater than $\frac{10}{3}a$; therefore let the ſaid ſide be feigned $\frac{1}{2}a + \frac{1}{4}$, then the Square thereof being equated to the ſumm of the three remainders in the fifth ſtep, from that Equation you will find
$$a = \tfrac{25}{64}.$$
7. Therefore from the ſixth, ſecond, third and fourth ſteps, the ſides of the three Cubes ſought will be $\frac{41}{64}$, $\frac{32}{64}$ and $\frac{48}{64}$.
8. And conſequently the Cubes themſelves are
$$\tfrac{68921}{262144}, \tfrac{32768}{262144}, \tfrac{110592}{262144}.$$

Which three Cubes will ſolve the Queſtion before propounded in *Example* 2. as will be manifeſt by

The Proof.

By ſubtracting every one of the ſaid three Cubes in the eighth ſtep from 1, (the given ſquared Cube, the three remainders will be theſe, to wit,
$$\tfrac{193223}{262144}, \tfrac{202825}{262144}, \tfrac{151552}{262144}.$$
The ſumm of thoſe remainders is . . . . . $\frac{547600}{262144}$
Which ſumm being reduced to its leaſt terms by the common Diviſor 16, will be . . . . . $\frac{34225}{16384}$
Which is a Square, whoſe ſide is $\frac{185}{128}$; therefore the Queſtion is ſolved.

Example 3.

1. The ſame things being ſuppoſed as in the firſt ſtep of *Example* 2. we may vary the ſides aſſumed in the ſecond, third and fourth ſteps, in this manner, *viz.*
2. Inſtead of . . . $\left\{\begin{matrix} a + \frac{1}{4}, \\ -a + 1, \\ \frac{3}{4}, \end{matrix}\right\}$ we may aſſume $\left\{\begin{matrix} a + \frac{3}{4}. \\ -a + 1. \\ \frac{1}{4}. \end{matrix}\right.$
3. Then by ſubtracting ſeverally the Cubes of the three ſides above-expreſt on the right hand from 1, and adding the three remainders together, the ſumm will be
$$-\tfrac{15}{4}aa + \tfrac{225}{16}a + \tfrac{225}{16}.$$
4. Which ſumm muſt be equal to a Square, whoſe ſide muſt be ſo feigned that the value of a may be leſs than $\frac{1}{4}$, to the end the ſide $a + \frac{3}{4}$ may be leſs than 1. Now to cauſe that effect we may feign the ſaid ſide to be $\frac{1}{4} +$ any number of a leſs than $\frac{15}{40}a$, or $\frac{3}{4} -$ any number of a greater than $10a$; ſuppoſe therefore the ſaid ſide be feigned $\frac{1}{2}a + \frac{3}{4}$, then the Square thereof being equated to the ſumm of the three remainders before mentioned in the third ſtep, from that Equation you will find
$$a = \tfrac{1}{88}.$$
5. Therefore from thoſe three aſſumed ſides which are placed on the right hand in the ſecond ſtep of this third Example, the ſides of the three Cubes ſought will be theſe, to wit,
$$\tfrac{67}{88}, \tfrac{87}{88} \text{ and } \tfrac{22}{88}.$$
6. And conſequently the Cubes themſelves are
$$\tfrac{300263}{681472}, \tfrac{658503}{681472}, \tfrac{-10648}{681472}.$$

Which three Cubes will ſolve the Queſtion before-propos'd in *Example* 2. for if every one of them be ſubtracted from the given ſquared Cube 1, the ſumm of the three remainders in its leaſt terms will be $\frac{488481}{309976}$, which is a Square whoſe ſide is $\frac{221}{176}$.

QUEST. 72.

QUEST. 72.

To find four cube-numbers, ſuch, that if they be ſeverally ſubtracted from a given ſquared cube-number, ſuppoſe 64, the ſumm of the four remainders may be a ſquare number.

RESOLUTION.

1. By the foregoing *Queſt.* 71. find three ſuch Cubes, that if they be ſeverally ſubtracted from the given ſquared Cube 64, the ſumm of the three remainders may be a Square; ſuch are the Cubes $\frac{68921}{4096}$, $\frac{52312}{4096}$ and 27, whoſe ſides are $\frac{41}{16}$, $\frac{32}{16}$ and 3, (found out in *Example* 1. of *Queſt.* 71.) for if the ſaid Cubes be ſeverally ſubtracted from 64, the ſumm of the three remainders in its leaſt terms will be $\frac{34225}{256}$, which is a Square, whoſe ſide is $\frac{185}{16}$.
2. Then for the ſide of the firſt of the four Cubes ſought put $a + 3$, (which 3 is the ſide of one of the three Cubes found out in the firſt ſtep,) . . . . . $a + 3$
3. For the ſide of the ſecond Cube put $-a + 4$, (4 being the Cube-root of the given ſquared Cube 64,) . . . . . $-a + 4$
4. Let the ſide of the third Cube be another of the ſides of the ſaid three Cubes found out in the firſt ſtep, *viz.* . . . . . $\frac{41}{16}$
5. And let the ſide of the fourth Cube be the ſide of the remaining Cube in the firſt ſtep, to wit, ... $\frac{32}{16}$
6. Therefore (from the ſecond ſtep) the firſt Cube ſought is . . . $aaa + 9aa + 27a + 27$
7. The ſecond Cube is . . . . . $-aaa + 12aa - 48a + 64$
8. The third Cube is . . . . . $\frac{68921}{4096}$
9. And the fourth Cube is . . . . . $\frac{52312}{4096}$
10. Then thoſe four Cubes being ſeverally ſubtracted from the given ſquared Cube 64, the four remainders will be theſe, to wit,

1.	$-aaa - 9aa - 27a + \frac{211552}{4096}$,
2.	$+aaa - 12aa + 48a$,
3.	. . . . . $\frac{121225}{4096}$,
4.	. . . . . $\frac{202825}{4096}$.

11. The ſumm of thoſe remainders is . . . $-21aa + 21a + \frac{34225}{256}$.
12. Which ſumm muſt be equated to a Square, the ſide whereof muſt be ſo feigned that the value of a may be leſs than 1, to the end the ſide $a + 3$ in the firſt ſtep may be leſs than 4, (becauſe the Cube of the ſaid $a + 3$ muſt be ſubtracted from the Cube of 4,) and if the value of a be leſs than 1, it will be much leſs than 4, as the third ſtep requires. Now to cauſe a to be leſsthan 1, the ſide of the feigned Square may be $\frac{185}{16} +$ any number of a leſs than $11\frac{13}{18}a$, or elſe $\frac{185}{16} -$ any number of a greater than $23\frac{1}{8}a$, (which limits may be diſcovered by the method in *Queſt.* 13. of this Book,) therefore we may feign the ſaid ſide to be $\frac{1}{2}a + \frac{185}{16}$, whoſe Square being equated to the ſumm of the four remainders in the eleventh ſtep, this Equation ariſeth, *viz.*
$$\tfrac{1}{4}aa + \tfrac{185}{16}a + \tfrac{34225}{256} = -21aa + 21a + \tfrac{34225}{256}.$$
13. Which Equation duly reduced will give . . . . . $a = \frac{221}{340}$
14. Therefore from the thirteenth, ſecond, third, fourth and fifth ſteps, the ſides of the four Cubes ſought are diſcovered to be theſe, to wit,
$$\tfrac{4684}{1360}, \tfrac{4836}{1360}, \tfrac{3485}{1360}, \tfrac{3315}{1360}.$$
15. And conſequently the four Cubes are theſe, to wit,
$$\tfrac{102766285504}{2515456000}, \tfrac{11309902956}{2515456000}, \tfrac{42326109125}{2515456000}, \tfrac{36429280875}{2515456000}.$$

Which four Cubes will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

By ſubtracting thoſe four Cubes ſeverally from the given ſquared Cube 64 (or $\frac{160282184000}{2515456000}$) the four remainders will be
$$\tfrac{58222898496}{2515456000}, \tfrac{47890154944}{2515456000}, \tfrac{11866307487 5}{2515456000}, \tfrac{12455990312 5}{2515456000}.$$
The ſumm of thoſe four remainders is . . . . . $\frac{349216021440}{2515456000}$
Which ſumm being reduced to its leaſt terms by the common Diviſor 1360, will be . . . . . $\frac{2568647229}{18496000}$
Which is a Square, whoſe ſide is . . . . . $\frac{16032}{1160}$

Therefore

Therefore the Queſtion is ſolved; and if the method of reſolving this and the preceding *Queſt.* 71. be well examined, it will not be difficult to apprehend how to find out as many Cubes as ſhall be deſired, which being ſeverally ſubtracted from a given ſquared cube-number, or from the Cube of a given Square, the ſumm of the remainders may be a Square.

QUEST. 73.

To find two numbers, that if each of them be ſubtracted from the Cube of their ſumm, the remainders may be Cubes.

RESOLUTION.

Firſt, making choice of ſome ſquare number as 4, I put $4a$ for the ſumm of the two numbers ſought, the Cube whereof will be $64aaa$; then for the firſt number I put $56aaa$, and for the other number $37aaa$, for theſe two being ſeverally ſubtracted from $64aaa$, the remainders will be $8aaa$ and $27aaa$, which are manifeſtly Cubes, whereby one part of the Queſtion is ſatisfied: It remains that the ſumm of the ſaid aſſumed numbers $56aaa$ and $37aaa$ be equal to $4a$, *viz.* $93aaa = 4a$, whence by dividing each part by a, there ariſeth $93aa = 4$. Now if the ſaid 93 were a ſquare number, then the value of a would be a rational number, and conſequently the Queſtion ſolved.

But 93 not being a Square, we muſt enquire whence it ariſeth, and by examining the Operation it will appear, that the two Cubes 8 and 27 having been ſubtracted ſeverally from 64, the remainders are 56 and 37, the ſumm whereof makes 93 before mentioned. So that our firſt ſcope muſt be to find two ſuch Cubes that if each of them be ſubtracted from the ſquared Cube 64, the ſumm of the remainders may be a Square: But ſuch are the Cubes 1 and 27, (found out by *Canon* 1. of the foregoing *Queſt.* 69.) for if each of them be ſubtracted from 64, the ſumm of the remainders 63 and 37 makes the Square 100: therefore I begin the Reſolution a-new thus; *viz.*

1. For the ſumm of the two numbers ſought I put . . . . . $4a$
2. The Cube thereof is . . . . . $64aaa$
3. Then for one of the two numbers ſought I put . . . . . $63aaa$
4. And for the other number ſought I put . . . . . $37aaa$
5. Which two numbers being ſeverally ſubtracted from $64aaa$ the remainders will be Cubes, to wit, ... aaa and $27aaa$
6. But the ſumm of the numbers aſſumed in the third and fourth ſteps muſt be equal to $4a$ in the firſt ſtep, therefore . . . . . $100aaa = 4a$
7. Which Equation duly reduced, gives . . . . . $a = \frac{1}{25}$
8. Therefore from the ſeventh, third and fourth ſteps the two numbers ſought are . . . . . $\frac{63}{125}$ and $\frac{37}{125}$

Which numbers will ſolve the Queſtion propos'd, as will be manifeſt by

The Proof.

9. The ſumm of the two numbers found out, to wit, $\frac{63}{125}$ and $\frac{37}{125}$, is . $\frac{4}{5}$
10. Therefore the Cube of their ſumm is . . . . . $\frac{64}{125}$
11. From which $\frac{64}{125}$ ſubtract each of the numbers $\frac{63}{125}$ and $\frac{37}{125}$, ſo the remainders are Cubes, to wit, ... $\frac{1}{125}$ and $\frac{27}{125}$

Another Example.

1. Firſt, I take ſome ſquare number, as 1, then I ſearch out two ſuch Cubes that each of them being ſubtracted from 1, (the Cube of the Square firſt taken,) the ſumm of the remainders may be a Square; ſuch are the Cubes $\frac{1331}{6859}$ and $\frac{512}{6859}$, whoſe ſides are $\frac{11}{19}$ and $\frac{8}{19}$, found out in the Example of *Canon* 2. *Queſt.* 69.) for if each of thoſe Cubes be ſubtracted from 1, the ſumm of the remainders $\frac{5828}{6859}$ and $\frac{6347}{6859}$ will be $\frac{12175}{6859}$, or in its leaſt terms $\frac{625}{361}$, which is a Square whoſe ſide is $\frac{25}{19}$; then by the help of thoſe Cubes and remainders I form the Reſolution as before in the firſt Example, *viz.*

2. For the ſumm of the two numbers ſought I put a or $1a$, (1 being the ſquare number firſt taken,) . . . . . $1a$
3. The Cube of the ſaid ſumm is . . . . . $1aaa$
4. Then for one of the two numbers ſought I put . . . . $\frac{5828}{6859}aaa$
5. And for the other number ſought, ... $\frac{6347}{6859}aaa$
6. Which two numbers being ſeverally ſubtracted from $1aaa$, the remainders will be Cubes, to wit, ... $\frac{1331}{6859}aaa$ and $\frac{512}{6859}aaa$

7. But

7. But the ſumm of the two numbers in the fourth and fifth ſteps muſt be equal to $1a$ in the ſecond ſtep, therefore . . . . . $\frac{61009}{20736}aaa = 1a$
8. Which Equation duly reduced gives . . . . . $a = \frac{144}{247}$
9. Therefore from the eighth, fourth and fifth ſteps, the two numbers ſought, in their leaſt terms, will be found . . . . . $\frac{85328}{151625}$ and $\frac{63147}{151625}$

Which numbers will ſolve the Queſtion propos'd, as will be evident by

The Proof.

The ſumm of the two numbers found out in the ninth ſtep, to wit, of $\frac{85328}{151625}$ and $\frac{63147}{151625}$ is in the leaſt terms . . . . . $\frac{19}{25}$
The Cube of that ſumm is . . . . . $\frac{6859}{15625}$
From which Cube if you ſubtract ſeverally the ſaid numbers found out, the remainders will be the Cubes of theſe ſides, to wit, . . . $\frac{11}{25}$ and $\frac{8}{25}$

QUEST. 74. (Quæſt. 19. Lib. 5. *Diophant.*)

To find three ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the three remainders may be Cubes.

[*The text of* Diophantus *in the Reſolution of this Queſtion is ſo obſcure, that it affords not any ſatisfactory Anſwer; I ſhall therefore ſhew how to ſolve it by two different ways of my own, by the latter of which this Queſtion may be extended to four, five, or as many numbers as ſhall be deſired.*]

RESOLUTION.

1. Firſt, take any ſquare number, as 1, then ſearch out three ſuch Cubes that if they be ſeverally ſubtracted from the Cube of the ſaid ſquare number 1, the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{148877}{2985984}$, $\frac{19683}{2985984}$ and $\frac{4096}{2985984}$, whoſe ſides are $\frac{53}{144}$, $\frac{27}{144}$ and $\frac{16}{144}$; (found out in *Example* 1. of the preceding *Queſt.* 70.) for if thoſe Cubes be ſeverally ſubtracted from 1 (or unity,) the ſumm of the three remainders $\frac{2837107}{2985984}$, $\frac{2946301}{2985984}$ and $\frac{2981888}{2985984}$ will be $\frac{61009}{20736}$, which is a Square, whoſe ſide is $\frac{247}{144}$. Now by the help of thoſe three preparatory Cubes and remainders, I proceed thus,

2. For the ſumm of the three numbers ſought I put a, or $1a$, (1 being the ſquare number firſt taken,) . . . . . $1a$
3. Therefore the Cube of the ſaid ſumm is . . . . . $1aaa$
4. Then for the firſt of the three numbers ſought I put $\frac{2837107}{2985984}aaa$, (the ſaid $\frac{2837107}{2985984}$ being one of the three remainders before mentioned in the firſt ſtep,) . . . . . $\frac{2837107}{2985984}aaa$
5. In like manner having multiplied the ſecond remainder into aaa, I put the Product for the ſecond number ſought, to wit, ... $\frac{2966301}{2985984}aaa$
6. Likewiſe multiplying the third remainder into aaa, I put the Product for the third number ſought, to wit, ... $\frac{2981888}{2985984}aaa$
7. Which three numbers in the three laſt ſteps being ſeverally ſubtracted from $1aaa$, the three remainders will (by the Conſtruction in the firſt ſtep) be Cubes, to wit,

$$\frac{148877}{2985984}aaa, \quad \frac{19683}{2985984}aaa, \quad \frac{4096}{2985984}aaa.$$

8. But the ſumm of the three numbers in the fourth, fifth and ſixth ſteps muſt be equal to $1a$ in the ſecond ſtep, whence this Equation ariſeth,

$$\frac{61009}{20736}aaa = 1a.$$

9. Which Equation duly reduced gives . . . . . $a = \frac{144}{247}$
10. Therefore from the ninth, ſecond, fourth, fifth and ſixth ſteps, the three numbers ſought will be made known, to wit, theſe,

$$\frac{2837107}{15069223}, \quad \frac{2966301}{15069223}, \quad \frac{2981888}{15069223}.$$

Which three numbers will ſolve the Queſtion, as will be evident by

The Proof.

The ſumm of the ſaid three numbers is $\frac{8285296}{15069223}$, which reduced to it ſmalleſt terms by the common Diviſor 61009, makes . . . . . $\frac{144}{247}$
The Cube of the ſaid ſumm is . . . . . $\frac{2985984}{15069223}$

From

From which Cube ſubtracting ſeverally the three numbers found out in the tenth ſtep, the remainders will be theſe three Cubes, to wit,

$\frac{148877}{15069223}$, $\frac{19683}{15069223}$, $\frac{4096}{15069223}$.

The ſides of which Cubes are theſe, *viz.*

$\frac{53}{247}$, $\frac{27}{247}$, $\frac{16}{247}$. Therefore the Queſtion is ſolved.

But becauſe the operation in finding out the three numbers, as alſo the three Cubes with their ſides as aforeſaid will be exceeding laborious by reaſon of long fractions, unleſs ſome Compendiums be uſed, I ſhall give a Canon deducible from the premiſſes to leſſen the work, reſpect being firſt had to theſe following

Preparatory Directions.

1. Firſt, the Rules for multiplying and dividing Fractions in *Sect.* 22, 26. of *Chap.* 6. *Book* 1. muſt be diligently obſerved, that the Products and Quotients may come out in the ſmalleſt terms.

2. Secondly, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be ſeverally divided by the ſame number, that multiplication and diviſion may be quite omitted, for the numbers firſt propos'd to be multiplied will be the ſame with the Quotients that ariſe by the ſaid multiplication and diviſion. Moreover, when one, two or more numbers are to be ſeverally multiplied by ſome number, and the Products are to be divided by ſome number greater or leſs than that multiplying number, reduce the ſaid Multiplicator and Diviſor into the leaſt terms (when they are not ſuch already) by their greateſt common Diviſor, and take the Quotients for a new Multiplicator and Diviſor inſtead of thoſe firſt preſcribed: As, if 41, 39 and 48 be to be ſeverally multiplied by 32, and the Products be to be ſeverally divided by 16, I firſt reduce the ſaid 32 and 16 to the ſmalleſt terms in the ſame Reaſon by the common Diviſor 16, ſo the Quotients or new terms will be 2 and 1; then multiplying 41, 39 and 48 ſeverally by 2, (inſtead of 32,) and dividing the Products ſeverally by 1, (inſtead of 16,) the Quotients will be 82, 78 and 96, which are found out much ſpeedier and in ſmaller terms than thoſe that would be found out by multiplying the ſaid 41, 39 and 48 by 32, and dividing the Products by 16 as was firſt preſcribed. This Rule will oftentimes be very uſeful in the fourth branch of the following Canon.

3. Thirdly, let the ſquare number firſt taken in the firſt ſtep of the foregoing Reſolution of this *Queſt.* 74. be called *bb*, and its ſide *b*.

4. Fourthly, let the other ſquare number which is equal to the ſumm of the three remainders found out in the ſaid firſt ſtep of the Reſolution be called *cc*, and its ſide *c*.

Theſe things premiſed, I proceed to the

CANON.

1. Divide the known number *b* by the known number *c*, and call the Quotient *a*, which is now a known number.

2. Divide the Cube of *b* by *c*, and let the Quotient be called *d*, which known number is the ſumm of the numbers ſought by the Queſtion.

3. Reduce the numbers *a* and *d* to their ſmalleſt common Denominator.

4. Reduce likewiſe the ſides of the preparatory Cubes (found out in the firſt ſtep of the Reſolution) to their ſmalleſt common Denominator, then multiply ſeverally the Numerators of thoſe ſides by the Numerator of the number *a*, and divide the Products ſeverally by the ſaid common Denominator of the ſides of the ſaid preparatory Cubes, and reſerve the Quotients for Dividends.

5. Divide ſeverally thoſe Dividends reſerved, by the Denominator of *a* or *d*, (for theſe were above reduced to a common Denominator,) ſo ſhall the Quotients be the ſides of the Cubes ſought.

6. Laſtly, by ſubtracting ſeverally the Cubes of the ſides laſt found out, from the Cube of the ſumm of the numbers ſought, (which ſumm was above found by the ſecond ſtep of the Canon,) the remainders ſhall be the numbers ſought, and the ſmalleſt that have a common Denominator with the Cubes found out in the fifth ſtep of the Canon.

This Canon with the preceding preparatory Directions may be practically illuſtrated by the Examples of the preceding *Queſt.* 73. and of this and the following 75 and 76 Queſtions.

Example 2.

Example 2.

Let it be required to find three ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the three remainders may be Cubes.

1. Firſt take ſome ſquare number, as 9, then find three ſuch Cubes, that if they be ſeverally ſubtracted from 729 (the Cube of the ſaid Square 9) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{148877}{4096}$, $\frac{19683}{4096}$ and $\frac{4096}{4096}$, (or 1,) whoſe ſides are $\frac{53}{16}$, $\frac{27}{16}$ and 1, (found out in the ſecond Example of *Queſt.* 70.) for if thoſe Cubes be ſeverally ſubtracted from 729, (or $\frac{2985984}{4096}$,) the ſumm of the three remainders $\frac{2837107}{4096}$, $\frac{2966101}{4096}$ and $\frac{2981888}{4096}$ being reduced to its leaſt terms will be $\frac{1485481}{256}$, which is a ſquare number whoſe ſide is $\frac{1241}{16}$.

2. Then by proceeding according to the foregoing preparatory Directions and Canon, the numbers and Cubes ſought will be found to be the ſame as were before found out in the firſt Example of this 74th Queſtion.

Example 3.

1. Taking again the ſame ſquare number 9 as in the ſecond Example, I ſeek three other Cubes, that every one of them being ſubtracted from 729 (the Cube of the ſaid Square 9) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{1222727}{12167}$, $\frac{-1718}{12167}$ and 1, whoſe ſides are $\frac{101}{23}$, $\frac{18}{23}$ and 1, (or $\frac{23}{23}$,) found out in the third Example of *Queſt.* 70. for if thoſe three Cubes be ſeverally ſubtracted from the ſaid 729, the ſumm of the three remainders in its leaſt terms will be $\frac{11281802}{529}$, which is a Square, whoſe ſide is $\frac{1051}{23}$.

2. Then by proceeding according to the preparatory Directions and the Canon, (which follow the firſt Example of this 74th Queſtion,) the ſides of the three Cubes ſought will be found $\frac{101}{351}$, $\frac{-18}{351}$ and $\frac{-251}{351}$, and the three numbers ſought are theſe, to wit, $\frac{-7111016}{43243551}$, $\frac{-8868051}{43243551}$ and $\frac{-8817776}{43243551}$, whoſe ſumm in its leaſt terms is $\frac{207}{351}$, from the Cube whereof if the ſaid three numbers be ſeverally ſubtracted, the three remainders will be Cubes, whoſe ſides are thoſe above found out. Therefore the Queſtion is ſolved.

QUESI. 75. (*Another way of ſolving the preceding* Queſt. 74.)

To find three ſuch cube-numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the remainders may be Cubes.

RESOLUTION.

1. Firſt take ſome ſquare number, as 4, then find three ſuch Cubes that if they be ſeverally ſubtracted from 64, (the Cube of the ſaid Square 4,) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{42241}{4096}$, $\frac{12112}{4096}$ and 27, whoſe ſides are $\frac{41}{16}$, $\frac{12}{16}$ and 3, (found out in the firſt Example of *Queſt.* 71.) for if thoſe Cubes be ſeverally ſubtracted from the ſaid 64, the three remainders will be theſe, to wit, $\frac{1212211}{4096}$, $\frac{2028241}{4096}$ and $\frac{1211152}{4096}$, whoſe ſumm in its leaſt terms is $\frac{14221}{256}$, which is a Square whoſe ſide is $\frac{121}{16}$.

2. Then by proceeding according to the preparatory Directions and Canon which follow the firſt Example of the preceding *Queſt.* 74. the ſides of the three Cubes ſought will be found theſe, to wit,

$\frac{82}{185}$, $\frac{78}{185}$, $\frac{96}{185}$.

3. And conſequently the Cubes themſelves are

$\frac{551368}{6331625}$, $\frac{474552}{6331625}$, $\frac{884736}{6331625}$.

4. And the three numbers ſought are theſe,

$\frac{1545784}{6331625}$, $\frac{1622600}{6331625}$, $\frac{1212416}{6331625}$.

Which will ſolve the Queſtion, as will be manifeſt by

The Proof.

5. The ſumm of the ſaid three numbers is $\frac{42202002}{6331625}$, which reduced to its ſmalleſt terms by the common Diviſor 34225, gives . . . . . } $\frac{1231}{185}$

6. The Cube of the ſaid ſumm is . . . . . } $\frac{2022151231}{6331625}$

From which Cube if you ſubtract ſeverally the three numbers before found out in the fourth ſtep, the remainders will be the three Cubes above expreſt in the third ſtep.

Another Example.

1. First take ſome ſquare number, as 1, then find three ſuch Cubes that if they be ſeverally ſubtracted from 1 (the Cube of the Square firſt taken) the ſumm of the three remainders may be a Square: But three ſuch Cubes are $\frac{1002361}{681472}$, $\frac{618101}{681472}$ and $\frac{-320648}{681472}$, whoſe ſides are $\frac{97}{88}$, $\frac{87}{88}$ and $\frac{22}{88}$, (found out in the third Example of *Queſt.* 71.) for if thoſe Cubes be ſeverally ſubtracted from 1, (the Cube of the Square firſt taken,) the ſumm of the three remainders $\frac{1320792}{681472}$, $\frac{-322262}{681472}$ and $\frac{670284}{681472}$ being reduced to its leaſt terms will be $\frac{42841}{30976}$, which is a Square whoſe ſide is $\frac{221}{176}$.

2. Then by proceeding according to the preparatory Directions and Canon (which follow the firſt Example of the preceding *Queſt.* 74.) the ſides of the three Cubes ſought will be found theſe, to wit, $\frac{134}{221}$, $\frac{174}{221}$ and $\frac{-44}{221}$; and the three numbers ſought will be theſe, to wit, $\frac{-1024164}{1079386}$, $\frac{1817752}{1079386}$ and $\frac{-1146128}{1079386}$, which will ſolve the ſaid 75th Queſtion, as will appear by the Proof.

QUEST. 76.

To find four ſuch numbers, that if every one of them be ſubtracted from the Cube of their ſumm, the four remainders may be Cubes.

RESOLUTION.

1. First take ſome ſquare number, as 4, then find four ſuch Cubes that if they be ſeverally ſubtracted from 64 (the Cube of the ſaid Square 4) the ſumm of the four remainders may be a Square: But four ſuch Cubes are theſe $\frac{10271668811504}{251545456000}$, $\frac{11909902290016}{251545456000}$, $\frac{42126102112}{251545456000}$ and $\frac{2642292810875}{251545456000}$, whoſe ſides are $\frac{4684}{1360}$, $\frac{4816}{1360}$, $\frac{3485}{1360}$ and $\frac{1115}{1360}$, (found out in the Example of the ſaid fifth Queſtion;) for if thoſe Cubes be ſeverally ſubtracted from the ſaid 64, the four remainders will be theſe, to wit, $\frac{1823222284296}{251545456000}$, $\frac{472890154944}{251545456000}$, $\frac{11866120748872}{251545456000}$ and $\frac{12451229031121}{251545456000}$; which remainders being added together, the ſumm in its leaſt terms is $\frac{2568641229}{1849600}$, which is a Square whoſe ſide is $\frac{16027}{1360}$.

2. Then by proceeding according to the preparatory Directions and Canon, (which follow the firſt Example of the foregoing *Queſt.* 74.) the ſides of the four Cubes ſought will be found theſe, to wit,

$$\frac{9368}{16027},\ \frac{9672}{16027},\ \frac{6970}{16027} \text{ and } \frac{6630}{16027}.$$

3. Therefore the four Cubes ſought are theſe,

$$\frac{822130284032}{4116771011683},\ \frac{904792232448}{4116771011683},\ \frac{33860887\,3000}{4116771011683},\ \frac{2914342\,47000}{4116771011683}.$$

4. Which Cubes being ſeverally ſubtracted from the Cube of $\frac{10880}{16027}$, (which by the ſecond branch of the Canon will be found out for the ſumm of the four numbers ſought,) the remainders will be the numbers ſought, to wit, theſe,

$$\frac{4657831\,87968}{4116771011683},\ \frac{9493045\,99000}{4116771011683},\ \frac{3831212\,39552}{4116771011683},\ \frac{996479\,225000}{4116771011683}.$$

Which four numbers ſolve the Queſtion, as will appear by the Proof.

From the manner of ſolving this and the preceding 75th Queſtion, it is eaſie to apprehend, how five, ſix or as many numbers as ſhall be deſired, may be found out, which being ſeverally ſubtracted from the Cube of their ſumm, may leave as many Cubes: But ſo many numbers as are deſired, ſo many preparatory Cubes muſt be firſt found out, ſuch, that if they be ſeverally ſubtracted from the Cube of ſome ſquare number choſen at pleaſure, the ſumm of the remainders may be a Square; which preparatory Cubes may be found out by the method before delivered in *Queſt.* 71, and 72.

QUEST. 77. (Quæſt. 20. Lib. 5. *Diophant.*)

To find three ſuch numbers that if the Cube of their ſumm be ſubtracted from every one of them, the remainders may be Cubes.

RESOLUTION.

1. For the ſumm of the three numbers ſought put . . . . . $\}$ a
2. And let the three numbers be . . . . . $\}$ $2aaa$, $9aaa$, $28aaa$
3. It remains that their ſumm $39aaa$ be equated to a, whence $39aa = 1$; where if 39 were a Square the Queſtion would be ſolved by Rational numbers. But 39 is not a Square, whence therefore is it produced? Examine the Poſitions, and you will find that 1 being added ſeverally to the three Cubes 1, 8 and 27, the ſumm of thoſe three additions makes 39. We muſt therefore ſearch out three Cubes whoſe ſumm increaſed with 3 may make a Square, to which end

4. For the ſides of the three Cubes put . . . . . $\}$ e; $-e+3$; and 1
5. Then the ſumm of the Cubes of thoſe three ſides increaſed with 3, makes . . . . . $\}$ $9ee-27e+31$
6. Which ſumm is to be equated to a Square, but the ſide thereof muſt be ſo feigned that the value of e may be leſs than 3; now to cauſe that effect the ſide may be variouſly feigned within limits eaſie to be diſcovered from the method in divers preceding Queſtions of this Book, let it be $3e-7$, then the Square of $3e-7$ being equated to $9ee-27e+31$ will give $e = \frac{6}{5}$; therefore the ſides of the three Cubes are $\frac{6}{5}$, $\frac{3}{5}$ and 1, and the Cubes themſelves are $\frac{216}{125}$, $\frac{27}{125}$ and 1, by the help whereof the work is to be renewed thus, *viz.*
7. Add 1 to every one of the three Cubes before found, and the ſumms will be $\frac{341}{125}$, $\frac{152}{125}$ and $\frac{250}{125}$; then inſtead of $2aaa$, $9aaa$ and $28aaa$ (in the ſecond ſtep) put for the three numbers ſought,

$$\frac{341}{125}aaa\ ,\ \frac{152}{125}aaa\ ,\ \frac{250}{125}aaa.$$

8. Then the ſumm of thoſe three numbers being equated to a, (which in the firſt ſtep was put for the ſumm of the three numbers ſought,) gives this Equation, to wit, ... $\}$ $\frac{743}{125}aaa = a$
9. Whence, after due Reduction, ... $\}$ . . $a = \frac{5}{17}$

Therefore from the ninth and ſeventh ſteps the three numbers ſought are $\frac{341}{4913}$, $\frac{152}{4913}$ and $\frac{250}{4913}$; for if from every one of them, the Cube of their ſumm, to wit, $\frac{125}{4913}$ be ſubtracted, there will remain the Cubes $\frac{216}{4913}$, $\frac{27}{4913}$, $\frac{-125}{4913}$, whoſe ſides are $\frac{6}{17}$, $\frac{3}{17}$, $\frac{-5}{17}$.

From the premiſſes it is evident that the Queſtion is capable of innumerable Anſwers, and may eaſily be extended to four, five, or as many numbers as you pleaſe.

QUEST. 78. (Quæſt. 21. Lib. 5. *Diophant.*)

To find three ſuch numbers that their ſumm may be a Square, and that if to the Cube of the ſaid ſumm the three numbers be ſeverally added, the three ſumms may be ſquare numbers.

RESOLUTION.

1. For the ſumm of the three numbers ſought, that it may be a Square, put . . . . . $\}$. aa
2. Then for the firſt number put . . . . . $\}$ $3aaaaaa$
3. For the ſecond, ... $\}$ $8aaaaaa$
4. And for the third, ... $\}$ $15aaaaaa$
5. Whence it is evident that every one of them added to the Cube of their ſumm makes a Square; But the ſumm of the three numbers (in the ſecond, third and fourth ſteps) muſt be equal to aa which was firſt put for their ſumm; therefore $26aaaaaa = aa$, and conſequently, (by dividing each part by aa,) $26aaaa = 1$. In which laſt Equation if 26 were a ſquared ſquare number, the value of a would be a rational number: Whence therefore comes 26? Examine the Poſitions, and you will find that 'tis the ſumm of the three numbers 3, 8 and 15, every one of which increaſed with 1 makes a Square; therefore the ſcope of our ſearch muſt be to find three numbers, every one of which increaſed with 1 may make a Square, and that the ſumm of the ſaid three numbers may be a Biquadrate: To which end let the three numbers be $aaaa-2aa$; $aa+2a$ and $aa-2a$; for every one of theſe increaſed with 1 makes a Square, and their ſumm makes a Biquadrate, to wit, $aaaa$; and 'tis evident the value of a may be any number greater than 2, (for the third number $aa-2a$ ſhews that aa muſt be greater than $2a$, and conſequently a greater than 2.) Suppoſe therefore $a = 3$, whence the three numbers $aaaa-2aa$; $aa+2a$; $aa-2a$, will be 63, 15 and 3; now with the help of theſe three numbers the work may be renewed thus, *viz.*
6. Let the ſumm of the three numbers ſought be (as before) . $\}$ aa
7. And the firſt number . . . . . $\}$ $63aaaaaa$
8. The ſecond . . . . . $\}$ $15aaaaaa$
9. And the third . . . . . $\}$ $3aaaaaa$

10. Therefore the ſumm of the three numbers is . . . . . } 81 *aaaaaa*
11. Which ſumm muſt be equal to *aa*, *viz.* . . . . . } 81 *aaaaaa* = *aa*
12. Which Equation after due Reduction gives . . . . . } . . . *a* = $\frac{1}{3}$

Therefore from the twelfth, ſeventh, eighth and ninth ſteps the three numbers ſought are $\frac{63}{729}$, $\frac{135}{729}$ and $\frac{243}{729}$, which will ſolve the Queſtion; for their ſumm is a Square, to wit, $\frac{1}{9}$. Alſo the Cube of the ſaid ſumm is $\frac{1}{729}$, to which if the three numbers be ſeverally added there will come forth three ſquare numbers, $\frac{64}{729}$, $\frac{136}{729}$ and $\frac{244}{729}$, for their ſides are $\frac{8}{27}$, $\frac{4}{27}$ and $\frac{2}{27}$.

QUEST. 79. (Quæſt. 22. Lib. 5. *Diophant.*)

To divide a given number, ſuppoſe 2, into three ſuch numbers, that every one of them ſubtracted from the Cube of their ſumm, *viz.* from 8, may leave a Square.

RESOLUTION.

Foraſmuch as every one of the three numbers ſought being leſs than 2 is to be ſubtracted from 8, each remainder ſhall be greater than 6, but leſs than 8; and the ſumm of the three remainders, to wit, the three Squares ſought makes 22: for the ſumm of the three deſired numbers, to wit, 2, ſubtracted from three times 8 leaves 22; we muſt therefore divide 22 into three ſuch Squares that every one of them may be greater than 6, but leſs than 8. But 22 is compoſed of three Squares 9, 9 and 4; therefore, firſt, (by *Queſt.* 4. of this Book) let 13 (the ſumm of 9 and 4) be divided into two Squares, that one may be between 6 and 8, ſuch are the Squares $\frac{1521}{625}$ and $\frac{1264}{625}$, whoſe ſides are $\frac{69}{25}$ and $\frac{58}{25}$; then the leſſer of thoſe Squares, to wit, $\frac{1264}{625}$ added to 9 makes $\frac{7889}{625}$, which muſt alſo be divided into two Squares that each may be between 6 and 8; but by the ſaid *Queſt.* 4. the ſides of two ſuch Squares will be found $\frac{12328}{4525}$ and $\frac{11861}{4525}$, which with $\frac{69}{25}$, (that is, $\frac{12489}{4525}$) before found are the ſides of the three Squares ſought; therefore the three Squares themſelves are $\frac{155710404}{20475625}$, $\frac{140778225}{20475625}$ and $\frac{151271121}{20475625}$, whoſe ſumm makes 22, and every one of them is greater than 6, but leſs than 8; therefore thoſe three Squares ſeverally ſubtracted from 8, leave the three deſired parts of 2, to wit, $\frac{100245504}{20475625}$, $\frac{31026771}{20475625}$ and $\frac{28229812}{20475625}$.

QUEST. 80. [*This is the* 12th *of the* 4th *Book of* Vieta's *Zeteticks.*]

To find three right-angled Triangles in rational numbers, that the Solid of the Perpendiculars may be to the Solid of the Baſes as a ſquare number to a ſquare number.

Note. By the Solid of three numbers is meant the Product made by their multiplication one into another, as, the Solid of 2, 3 and 4 is 24, that is, 2 × 3 × 4.

RESOLUTION.

1. Let *h*, *b*, *p* repreſent the Hypothenuſal, Baſe and Perpendicular of any right-angled Triangle in numbers given or found out by the Canon in *Obſervat.* 8. *Reſolut.* 2. *Queſt.* 2. of this Book,

viz. { Hypoth. *h*, Baſe *b*, Perp. *p*

2. Then from *h* and *b* (by the Canon above-mentioned) form a ſecond right-angled Triangle, and let 2*hb* be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. *hh* + *bb*. Baſe 2*bb*. Perp. *hh* − *bb*

3. Again, from *h* and *p* form a third right-angled Triangle, and let 2*hp* be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. *hh* + *pp*. Baſe 2*hp*. Perp. *hh* − *pp*

4. I ſay the Solid of the Perpendiculars of thoſe three right-angled Triangles is to the Solid of their Baſes as a Square to a Square, to wit, as *pp* to 4*hh*; which I prove thus,

The Perpendicular of the firſt right-angled Triangle is *p*; the ſecond Perpendicular is *hh* − *bb*, that is, *pp*, (for by Conſtruction in the firſt ſtep, *hh* = *bb* + *pp*, whence *hh* − *bb* = *pp*;) and the third Perpendicular is *hh* − *pp*, that is, *bb*. So that the three Perpendiculars are *p*, *pp* and *bb*, which multiplied one into another will produce

pppbb = the Solid of the three Perpendiculars.

Again,

Again, the Baſes of the ſame three right-angled Triangles are *b*, 2*hb*, 2*hp*, which multiplied one into another will produce

4*hhbbp* = the Solid of the three Baſes.

Now becauſe *bbp* is a common Factor in thoſe two Solids, they ſhall be in ſuch proportion one to another as the Quotients that ariſe by dividing the ſaid Solids by *bbp*, viz.

pppbb . 4*hhbbp* :: *pp* . 4*hh*.

Which was to be proved.

An Example in Numbers.

Let the firſt right-angled Triangle be . . . . . { *h*, *b*, *p* / 5, 3, 4
Then by Conſtruction in the ſecond ſtep, the ſecond right-angled Triangle ſhall be . . . . . { 34, 30, 16
And by Conſtruction in the third ſtep the third right-angled Triangle is } 41, 40, 9

Which three Triangles will ſolve the Queſtion; for the Solid of the Perpendiculars 4, 16, 9 hath ſuch proportion to the Solid of the Baſes 3, 30, 40 as the Square of 4 to the Square of 10.

Note. Inſtead of any one of the Triangles thus found out, you may take another like Triangle, as inſtead of 34, 30, 16, you may take 17, 15, 8, which with the other two Triangles will ſolve the Queſtion.

QUEST. 81. (Quæſt. 13. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, leſs by the Product of the Baſes, may be a Square.

RESOLUTION.

1. Let *h*, *b*, *p* repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that 2*p* may be greater than *b*,

viz. { Hypoth. *h*. Baſe *b*. Perp. *p*

2. Then from 2*p* and *b* let a ſecond right-angled Triangle be formed, and let 4*pb* be called the Perpendicular, ſo the three ſides will be theſe,

viz. { Hypoth. 4*pp* + *bb*. Baſe 4*pp* − *bb*. Perp. 4*pb*

3. Then divide the ſides of the ſaid ſecond right-angled Triangle ſeverally by *b* the Baſe of the firſt, ſo there will ariſe theſe following ſides of a third right-angled Triangle,

viz. { Hypoth. $\frac{4pp+bb}{b}$. Baſe $\frac{4pp-bb}{b}$. Perp. 4*p*

I ſay the firſt and third right-angled Triangles will ſolve the Queſtion; for if the Product of their Perpendiculars *p* and 4*p*, to wit, 4*pp* be leſſened by the Product of their Baſes *b* and $\frac{4pp-bb}{b}$, that is, by 4*pp* − *bb*, the remainder will be a Square, to wit, *bb*; which was required.

An Example in Numbers.

Let the firſt right-angled Triangle be . . . . . { *h*, *b*, *p* / 5, 4, 3
Then the other ſhall be . . . . . } 13, 5, 12

Which Triangles will ſolve the Queſtion; for the Product of the Perpendiculars 3 and 12, to wit, 36, exceeds 20 the Product of the Baſes, by the Square 16.

Note. If the two Triangles found out by this Queſtion be ſeverally multiplied or divided by the ſame number, they will produce two other Triangles to perform the ſame effect.

QUEST. 82. (Quæſt. 14. Lib. 4. Zetet. *Viet.*)

To find two right-angled Triangles in rational numbers, that the Product made by the mutual multiplication of the Perpendiculars, together with the Product of the Baſes, may make a Square.

RESO-

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that p may exceed $2b$,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from p and $2b$ form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe; ſo the three ſides will be theſe,

viz. { Hypoth. $pp + 4bb$. Baſe, $4pb$. Perp. $pp - 4bb$

3. Then divide the ſides of the ſaid ſecond right-angled Triangle ſeverally by p the Perpendicular of the firſt, ſo there will ariſe theſe following ſides of a third right-angled Triangle,

viz. { Hpyoth. $\frac{pp+4bb}{p}$. Baſe, $4b$. Perp. $\frac{pp-4bb}{p}$

4. I ſay the firſt and third right-angled Triangles will ſolve the Queſtion; for if to $pp - 4bb$ the Product of the Perpendiculars p and $\frac{pp-4bb}{p}$, you add $4bb$, to wit, the Product of the Baſes b and $4b$, the ſumm will be the Square pp. Which was required.

An Example in Numbers.

The firſt right-angled Triangle may be . . . . . } 13, 5, 12
Then the other Triangle ſought ſhall be . . . . . } $6\frac{1}{3}$, 20, $\frac{14}{3}$

Which Triangles will ſolve the Queſtion; for the Product of the Perpendiculars 12 and $\frac{14}{3}$, to wit, 44 increaſed with 100 the Product of the Baſes 5 and 20, makes the Square 144.

Note. If the two Triangles found out by this Queſtion be multiplied or divided by the ſame number, they will produce two other Triangles performing the ſame effect. So, if 13, 5, 12 and $6\frac{1}{3}$, 20, $\frac{14}{3}$ be multiplied ſeverally by 3, there will be produced 39, 15, 36 and 61, 60, 11; where 396 the Product of the Perpendiculars, with 900 the Product of the Baſes makes the Square 1296, whoſe Root is 36.

QUEST. 83. (Quæſt. 15. Lib. 4. Zeter. *Vict.*)

To find three right-angled Triangles in rational numbers, that the Solid of the Hypothenuſals may be to the Solid of the Baſes, as a ſquare number to a ſquare number.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, in numbers ſo given or found out that $2b$ may exceed p,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from $2b$ and p form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $4bb + pp$. Baſe, $4bp$. Perp. $4bb - pp$

3. Then by the help of thoſe two right-angled Triangles, find out a third, by the Canon in *Obſervat.* 4. upon *Reſolut.* 2, and 3. of *Queſt.* 2. *viz.* for the Hypothenuſal of the third right-angled Triangle take the Product of the Hypothenuſals of the firſt and ſecond; the Baſe ſhall be the Product of the Baſes of the firſt and ſecond, leſs by the Product of the Perpendiculars; and the Perpendicular ſhall be equal to the ſumm of the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Perpendicular of the firſt into the Baſe of the ſecond; ſo the three ſides of the third right-angled Triangle will be theſe,

viz. { Hypoth. $4bbh + pph$. Baſe, ppp . Perp. $4bbb + 3bpp$

I ſay thoſe three right-angled Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes as the Square of $4bbh + pph$ is to the Square of $2bpp$.

An

An Example in Numbers.

The firſt right-angled Triangle may be . . . . . } 5, 3, 4
Then the ſecond in its leaſt terms will be found . . . . . } 13, 12, 5
And the third in its leaſt terms is . . . . . } 65, 16, 63

Which three Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes, as the Square of 65 to the Square of 24.

Otherwiſe thus,

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, ſo given or found out that p may exceed $2b$,

viz. { Hypoth. h . Baſe, b . Perp. p

2. Then from p and $2b$ form a ſecond right-angled Triangle, and let $4bp$ be called the Baſe, ſo the three ſides will be theſe,

viz. { Hypoth. $pp + 4bb$. Baſe, $4pb$. Perp. $pp - 4bb$

3. Then by the help of thoſe two right-angled Triangles find out a third by the latter Canon in *Obſervat.* 4. before mentioned, *viz.* for the Hypothenuſal of the third right-angled Triangle take the Product of the multiplication of the Hypothenuſals of the firſt and ſecond; for the Baſe, take the ſumm of the Product of the Baſes and the Product of the Perpendiculars; and for the Perpendicular take the difference of theſe two Products, to wit, the Product of the Baſe of the firſt into the Perpendicular of the ſecond, and the Product of the Perpendicular of the firſt into the Baſe of the ſecond. So the three ſides of the third right-angled Triangle will be found theſe,

viz. { Hypoth. $pph + 4bbh$. Baſe, ppp . Perp. $3bpp + 4bbb$

An Example in Numbers.

Let the ſides of the firſt right-angled Triangle be . . . . } 13, 5, 12
Then the ſecond in its leaſt terms will be found . . . . . } 61, 60, 11
And the third in its leaſt terms ſhall be . . . . . } 793, 432, 665

Which three Triangles will ſolve the Queſtion; for the Solid of the Hypothenuſals is to the Solid of the Baſes, as the Square of 793 to the Square of 360.

QUEST. 84. (Quæſt. 24. Lib. 5. *Diophant.*)

To find three ſquare numbers, that if they be ſeverally added to the ſolid Product made by their continual multiplication, the three ſumms may be Squares.

RESOLUTION.

1. For the Solid of the three Squares ſought put . . . . . } aa
2. Then find three Squares, every one of which increaſed with unity may make a Square; which Squares may eaſily be found out by the help of three unlike right-angled Triangles; for if the Square of one of the ſides about the right-angle be divided by the Square of the other ſide, the Quotient will be a Square, which increaſed with 1 will make a Square: As, if there be expoſed theſe three unlike right-angled Triangles, to wit, 3, 4, 5. | 5, 12, 13. | 8, 15, 17; then in the firſt Triangle the Square of the Baſe 3 divided by the Square of the Perpendicular 4 gives the Square $\frac{9}{16}$, which increaſed with 1, that is, $\frac{16}{16}$, makes the Square $\frac{25}{16}$; the reaſon whereof is manifeſt, for by Conſtruction the Numerators 9 and 16 added together make a Square, wherefore the whole Fraction $\frac{25}{16}$ ſhall be a Square. In like manner in the ſecond Triangle the Square of the Baſe 5 divided by the Square of the Perpendicular 12 gives the Square $\frac{25}{144}$, which increaſed with 1, that is, $\frac{144}{144}$, makes the Square $\frac{169}{144}$. And in the third Triangle the Square of the Baſe 8 divided by the Square of the Baſe 15, gives the Square $\frac{64}{225}$, which increaſed with 1, makes the Square $\frac{289}{225}$. Thus three Squares are found out, to wit, $\frac{9}{16}$, $\frac{25}{144}$ and $\frac{64}{225}$, every one of which increaſed with 1 makes a Square. Now multiply thoſe Squares ſeverally by aa, and take the Products $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ for the three Squares ſought; for every one of them added to aa, (which was firſt put for the Solid made by their continual multiplication,) makes a Square: It remains that the Solid of the ſaid $\frac{9}{16}aa$, $\frac{25}{144}aa$ and $\frac{64}{225}aa$ be equal to aa; But their Solid by continual multiplication is $\frac{1}{5}\frac{4}{1}\frac{4}{8}\frac{2}{4}\frac{2}{0}\frac{0}{0}aaaaaa$,

$\frac{1}{518400}aaaaaa$, this equated to aa gives $\frac{1}{518400}aaaaaa = aa$; whence by dividing each part by aa there ariſeth $\frac{1}{518400}aaaa = 1$, and by extracting the ſquare Root out of each part it gives $\frac{1}{720}aa = 1$; in which laſt Equation if $\frac{1}{720}$ were a Square, then the value of a would be expreſſible by a rational number, and conſequently the Queſtion were ſolved. Whence therefore comes $\frac{1}{720}$? Examine the work, and you will find that the Numerator 120 is the Solid of the Baſes 3, 5 and 8 of the three Triangles firſt expoſed; (for 14400 is the Solid of the three Squares 9, 25 and 64, and therefore the ſquare Root of 14400, to wit, 120 is the Solid of the ſides of thoſe three Squares;) and the Denominator 720 is the Solid of the three Perpendiculars 4, 12 and 15; (for 518400 is the Solid of the three Squares 16, 144 and 125, and therefore the ſquare Root of 518400, to wit, 720 is the Solid of the three Roots of thoſe Squares.) We muſt therefore find three ſuch right-angled Triangles, that the Solid of their Perpendiculars may be to the Solid of their Baſes as a Square to a Square. But by the precedent *Queſt.* 80. three ſuch Triangles may be found out, as theſe, 4, 3, 5. | 8, 15, 17. | 9, 40, 41; the Solid of whoſe Baſes 4, 8, 9, to wit, 288, is to the Solid of their Perpendiculars 3, 15, 40, to wit, 1800, as 144 to 900; that is, as a Square to a Square; then with theſe Triangles let the work be renewed as before, *viz.*

3. For the Solid of the three Squares ſought put . . . . . } aa
4. Then divide the Squares of the Baſes of the three right-angled Triangles laſt found out, by the Squares of the Perpendiculars, and multiplying the Quotients ſeverally by aa, put the Products for the three Squares ſought, to wit, ... } $\frac{16}{9}aa$, $\frac{64}{225}aa$, $\frac{81}{1600}aa$
5. The Solid of thoſe three Squares equated to aa, gives . . } $\frac{82944}{3240000}aaaaaa = aa$
6. Which Equation, after due Reduction, gives . . . . . } $a = \frac{5}{2}$
7. Wherefore from the ſixth and fourth ſteps the three Squares ſought are $\frac{100}{9}$, $\frac{16}{9}$ and $\frac{81}{256}$, which mutually multiplied one into another make the Solid $\frac{81}{4}$, to which if the three Squares themſelves be ſeverally added, the ſumms will alſo be Squares, to wit, $\frac{625}{36}$, $\frac{289}{36}$ and $\frac{1681}{256}$; for their ſides are $\frac{25}{6}$, $\frac{17}{6}$ and $\frac{41}{16}$. Therefore the Queſtion is ſolved, and manifeſtly capable of innumerable Anſwers.

QUEST. 85. (Quæſt. 25. Lib. 5. *Diophant.*)

To find three ſuch Squares, that if they be ſeverally ſubtracted from the ſolid Product made by their continual multiplication, the three remainders may be Squares.

RESOLUTION.

1. For the ſolid Product of the three Squares ſought put . . } aa
2. Then ſearch out three Squares, every one of which ſubtracted from unity may leave a Square; but three ſuch Squares may be found out by the help of three unlike right-angled Triangles; for if the Square of one of the ſides about the right-angle be divided by the Square of the Hypothenuſal, the Quotient ſhall be a Square, which ſubtracted from 1 will leave a Square: Let therefore three unlike right-angled Triangles be expoſed, as 3, 4, 5. | 12, 5, 13. | 15, 8, 17; then by dividing the Squares of 4, 5 and 8, which I ſhall here call Baſes, (for it matters not which of the ſides about the right-angle be called the Baſe,) by the Squares of the Hypothenuſals 5, 13 and 17, the Quotients will be the Squares $\frac{16}{25}$, $\frac{25}{169}$ and $\frac{64}{289}$, every one of which ſubtracted from 1 leaves a Square. Then multiply every one of theſe Squares by aa and aſſume the Products to be the three Squares ſought, to wit, $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$; for every one of theſe ſubtracted from aa (which was firſt put for the Solid of the three deſired Squares) leaves a Square. It remains that the ſolid Product of the ſaid $\frac{16}{25}aa$, $\frac{25}{169}aa$ and $\frac{64}{289}aa$ be equal to aa; but the ſaid Solid by continual multiplication will be found $\frac{26214400}{12221025}aaaaaa$, therefore $\frac{26214400}{12221025}aaaaaa = aa$, whence after due Reduction there will ariſe $\frac{1600}{1105}aa = 1$; in which laſt Equation if $\frac{1600}{1105}$ were a Square, then the value of a would be expreſſible by a rational number. Whence therefore comes $\frac{1600}{1105}$? Examine the work, and you will find that the Numerator 1600 is the ſolid Product of the Perpendiculars 4, 5 and 8 of the three Triangles firſt expoſed; and the Denominator 1105 is the Solid of the Hypothenuſals 5, 13 and 17. We muſt therefore find three ſuch right-angled Triangles that the Solid of the Hypothenuſals may be to the Solid of their Baſes as a Square to a Square: But three ſuch right-angled Triangles may be found out by the preceding *Queſt.* 83. ſuppoſe theſe, 5, 3, 4. | 13, 12, 5. | 65, 16, 63; here I ſhall call 3, 12 and 16 the Baſes, by the help whereof the work may be renewed thus, *viz.*

3. For the Solid of the three Squares ſought put . . . . } aa
4. Then divide the Squares of the Baſes of the three Triangles laſt found out, by the Squares of the Hypothenuſals, and multiply the Quotients ſeverally by aa, and put the Products for the three Squares ſought, to wit, ... } $\frac{9}{25}aa$, $\frac{144}{169}aa$, $\frac{256}{4225}aa$
5. The Solid of thoſe three Squares being equated to aa, gives } $\frac{1179810625}{1}aaaaaa = aa$
Which Equation, after due Reduction, will give . . . . } $a = \frac{65}{24}$
6. According to which value of a, the Poſitions in the fourth ſtep being reſolved, the three Squares ſought will be found $\frac{1369}{64}$, $\frac{25}{4}$, $\frac{4}{9}$; for theſe mutually multiplied make the Solid $\frac{4225}{576}$, from which if every one of the ſaid three Squares be ſubtracted, the remainders will be Squares, to wit, $\frac{1369}{36}$, $\frac{4225}{576}$, $\frac{441}{64}$, whoſe ſides are $\frac{37}{6}$, $\frac{65}{24}$, $\frac{21}{8}$.

QUEST. 86. (Quæſt. 26. Lib. 5. *Diophant.*)

To find three Squares, that the Solid or Product made by their continual multiplication being ſubtracted from every one of them, the three remainders may be Squares.

RESOLUTION.

The Reſolution of this Queſtion depends upon the *Lemma* uſed in the laſt preceding Queſtion, for as there, ſo here, three right-angled Triangles are firſt to be found out, that the Solid of their Hypothenuſals may be to the Solid of their Baſes as a Square to a Square. Then inſtead of $\frac{9}{25}aa$, $\frac{144}{169}aa$ and $\frac{256}{4225}aa$, aſſumed in the fourth ſtep of the preceding *Queſt* 85. to be the three Squares ſought, let $\frac{25}{9}aa$, $\frac{1369}{144}aa$ and $\frac{42025}{256}aa$ be put for the three Squares ſought in this 86th Queſtion; where obſerve, that the Numerators of the three latter Squares which are multiplied into aa, are the ſame with the Denominators of the former, and the Denominators of the latter the ſame with the Numerators of the former; by which converſion it will follow, that if from every one of the latter Squares, *viz.* from $\frac{25}{9}aa$, $\frac{1369}{144}aa$ and $\frac{42025}{256}aa$ you ſubtract aa, which (as before in *Queſt.* 85.) is to be taken for the Solid of the three Squares ſought, the three remainders will be Squares: Then the Solid of the three aſſumed Squares, that is, of $\frac{25}{9}aa$, $\frac{1369}{144}aa$ and $\frac{42025}{256}aa$, being equated to aa, gives $\frac{1787562025}{331776}aaaaaa - aa$; whence, after due Reduction, $a = \frac{24}{65}$, according to which value of a, the Poſitions being reſolved will give $\frac{625}{169}$, $\frac{4}{25}$, $\frac{9}{4}$ for the three Squares ſought; for the Solid made by their continual multiplication is $\frac{2256}{42225}$, which ſubtracted from every one of the three Squares, leaves the Squares $\frac{1024}{42225}$, $\frac{144}{169}$ and $\frac{41209}{16900}$, whoſe ſides are $\frac{32}{65}$, $\frac{12}{13}$, $\frac{203}{130}$.

QUEST. 87. (Quæſt. 30. Lib. 5. *Diophant.*)

To find three Squares, that if to the ſumm of every two of them, a given number, ſuppoſe 15, be added, the three ſumms may be Squares.

RESOLUTION.

1. For one of the Squares ſought take any ſquare number at pleaſure, as . . . . . } 9
2. Then we muſt find two other Squares, ſuch, that each of them added to 24 may make a Square; for ſince 9 one of the three Squares added to the given number 15 makes 24, it will not be difficult to conceive from the tenor of the Queſtion, that each of the other two Squares taking to it 24 muſt make a Square. Now to find out thoſe two Squares, divide 24 by each of two ſides about the right-angle of ſome right-angled Triangle, as 3 and 4; ſo the Quotients 8 and 6 ſhall alſo be the ſides about the right-angle of a right-angled Triangle, becauſe they are in the ſame proportion with the former; for by Conſtruction $8 \times 3 = 4 \times 6 = 24$, therefore $3 . 4 :: 6 . 8$; and conſequently the halves of 3 and 4, to wit, $\frac{3}{2}$ and 2, and the halves of 8 and 6, to wit, 4 and 3 ſhall be alſo the ſides about the right-angle of a right-angled Triangle. Theſe things premiſed,
3. Let the ſide of one of the two Squares be the difference between $3a$ and $\frac{2}{a}$, to wit, ... } $3a - \frac{2}{a}$, or $\frac{2}{a} - 3a$
4. And the ſide of the other Square the difference between $4a$ and $\frac{\frac{1}{2}}{a}$, to wit, ... } $4a - \frac{\frac{1}{2}}{a}$, or $\frac{\frac{1}{2}}{a} - 4a$

5. Therefore the Square of the ſide in the third ſtep is . . . $9aa - 12 + \frac{4}{aa}$

6. And the Square of the ſide in the fourth ſtep is . . . $16aa - 12 + \frac{2}{4}\frac{4}{aa}$

7. The former of thoſe two Squares increaſed with 24 makes a Square, to wit, ... $9aa + 12 + \frac{4}{aa}$

8. And the latter Square in the ſixth ſtep increaſed with 24 makes a Square, to wit, ... $16aa + 12 + \frac{2}{4}\frac{4}{aa}$

9. The ſides of which two Squares in the two laſt ſteps are . . . $3a + \frac{2}{a}$ and $4a + \frac{\frac{1}{2}}{a}$

10. It remains that the ſumm of the Squares in the fifth and ſixth ſteps, together with the given number 15 may make a Square, but it makes . . . . . $25aa - 9 + \frac{2\frac{1}{4}}{aa}$

11. Which ſumm muſt be equated to a Square, *viz.* either to $25aa$, or to $\frac{2\frac{1}{4}}{aa}$, let it firſt be equated to $25aa$, *viz.* ſuppoſe . . . . . $25aa - 9 + \frac{2\frac{1}{4}}{aa} = 25aa$

12. Which Equation, after due Reduction, gives . . . . $a = \frac{1}{6}$

13. Therefore from the twelfth, third and fourth ſteps the ſides of the ſecond and third Squares ſought are $\frac{1}{10}$ and $\frac{2\frac{1}{2}}{15}$, and conſequently 9, $\frac{1}{100}$ and $\frac{1\frac{1}{2}2}{225}$ are the three Squares ſought; for every two with 15 make Squares, to wit, $\frac{2\frac{1}{2}01}{100}$, $\frac{2\frac{1}{2}25}{225}$, $\frac{625}{36}$, whoſe ſides are $\frac{48}{10}$, $\frac{32}{15}$, $\frac{25}{6}$.

But if $25aa - 9 + \frac{2\frac{1}{2}}{aa}$ be equated to $\frac{2\frac{1}{2}}{aa}$, then the value of a will be $\frac{1}{5}$, yet the ſame Solution will be found as before, becauſe here we muſt conceive $\frac{2}{a} - 3a$ and $\frac{1}{2} - 4a$ to be the ſides of the two Squares; which ſides being reſolved according to the latter value of a, to wit, $a = \frac{1}{5}$, there will come forth (as before) $\frac{24}{15}$ and $\frac{1}{10}$.

This Queſtion is capable of innumerable Anſwers upon a double ground; for firſt, the firſt ſquare may be any known ſquare number at pleaſure; then the ſides of the ſecond and third Squares may be variouſly feigned from divers numbers, which may be the ſides about the right-angle of unlike right-angled Triangles; as inſtead of 3 and 4 we may take 8 and 15, 5 and 12, and innumerable others.

QUEST. 88. (Quæſt. 32. Lib. 5. *Diophant.*)

To find three Squares that the ſumm of their Squares may make a Square; or, (which is the ſame thing) to find three numbers that the ſumm of their ſquared Squares may be a Square.

RESOLUTION.

1. For one of the ſquare numbers ſought put . . . . aa
2. And for the two others put . . . . . bb and cc
3. Then the ſumm of the Squares of thoſe three Squares is $aaaa + bbbb + cccc$
4. Which ſumm muſt be equated to a Square, let its ſide be $aa - dd$, the Square whereof equated to the ſaid ſumm gives
$$aaaa + bbbb + cccc = aaaa - 2ddaa + dddd.$$
5. Which Equation, after due Reduction, gives . . . $aa = \frac{dddd - bbbb - cccc}{2dd}$
6. In which laſt Equation, becauſe the Numerator and Denominator are not perfect Squares, the value of a is not expreſſible by a rational number; but to cauſe it to be rational, we may diſcover by the ſaid Fraction that a certain number and two Squares muſt be found out, ſuch, that if from the Square of the number the ſumm of the two Squares be ſubtracted, the remainder may be to the double of the ſaid number as a ſquare number to a ſquare number: Now to find out ſuch a number and two Squares, let $rr + ss$ be put for the number, and rr for one Square, and ss for the other; then from the Square of $rr + ss$, that is, from $rrrr + 2rrss + ssss$ ſubtract the ſumm of the Squares of rr and ss, that is, $rrrr + ssss$, and the remainder $2rrss$ muſt be to the double of the number $rr + ss$, that is, to $2rr + 2ss$, as a Square to a Square; therefore alſo the halves of $2rrss$ and $2rr + 2ss$, that is, $rrss$ and $rr + ss$ ſhall be as a Square to a Square, for the Proportion is not changed: And ſince $rrss$ is a Square, whoſe ſide is rs, it remains only to make $rr + ss$ a Square; but ſuch it will be if r and s repreſent the ſides about the right-angle of a right-angled Triangle whoſe three ſides are expreſſible by rational numbers, for then $rr + ss$ will be the Square of the Hypothenuſal. Therefore from the premiſſes the following Canon is deducible to ſolve the Queſtion propoſed.

CANON.

Take for two of the three Squares ſought the Squares of the ſides about the right-angle of ſome right-angled Triangle in rational numbers; then divide the Product made by the mutual multiplication of thoſe two Squares, by the Square of the Hypothenuſal, and there will come forth the third Square ſought.

As, for example, let there be expoſed the right-angled Triangle 3, 4, 5, then two of the Squares ſought ſhall be 9 and 16, (to wit, the Squares of 3 and 4 the ſides about the right-angle;) and if the Product of 9 into 16, that is, 144, be divided by 25 (the Square of the Hypothenuſal,) the Quotient $\frac{144}{25}$ ſhall be the third Square ſought. I ſay 9, 16 and $\frac{144}{25}$ are three Squares, which will ſolve the Queſtion, for the ſumm of their Squares makes $\frac{2\frac{1}{2}1\frac{1}{2}\frac{1}{4}}{625}$, which is a Square; whoſe ſide is $\frac{2\frac{1}{2}1}{25}$.

QUEST. 89. (Quæſt. 33. Lib. 5. *Diophant.*)

A certain Vintner made a mixture of two ſorts of Wines, whereof one coſt eight pence the quart, and the other five pence; at which prices the whole mixture was worth a ſquare number of pence, unto which 60 being added the ſumm would alſo make a ſquare number, whoſe ſide was the number of quarts contained in the mixture. The Queſtion is, to find the number of quarts of each ſort of Wine in the mixture.

RESOLUTION.

1. For the price of the whole mixture put . . . . . $aa - 60$
2. Then if 60 be added to that total price the ſumm will be the Square aa, whoſe Root (as the Queſtion requires) muſt be equal to the number of quarts of both ſorts of Wine in the mixture, to wit, ... a
3. From the premiſſes we may rightly inferr, that $aa - 60$ (the total coſt of the mixture) is greater than $5a$, but leſs than $8a$; (that is, greater than the Product of the multiplication of the total number of mixed quarts by the price of the worſer ſort of Wine in the mixture, but leſs than the Product of the ſame number of quarts multiplied into the dearer ſort of Wine:) But if $aa - 60$ be greater than $5a$, and leſs than $8a$, then the value of a (by *Queſt* 10. of this *Book*) is greater than $10\frac{6\frac{1}{2}}{1\frac{1}{2}}$, *&c.* but leſs than $12\frac{11}{100}\frac{12}{000}\frac{12}{000}$, *&c.* Therefore $aa - 60$ (which the Queſtion requires to be a Square) muſt be equated to ſome Square whoſe ſide muſt be ſo feigned that the value of a may be within thoſe limits: Now to cauſe that effect, the ſide of the ſaid Square may be feigned $-a + $ any abſolute number between $17\frac{2\frac{1}{2}}{1000}$, *&c.* and $22\frac{8\frac{1}{2}}{1000}$, *&c.* or $a -$ any abſolute number between $2\frac{6\frac{1}{2}}{1000}$, *&c.* and $3\frac{1\frac{1}{2}}{1000}$, *&c.* (as hath been ſhewn in *Queſt.* 11. of this *Book.*) Let then the ſaid ſide be feigned $-a + 12$, whoſe Square $aa - 24a + 144$ equated to $aa - 60$ will give $a = 12\frac{3}{11}$ for the deſired number of quarts in the whole mixture.
4. Then from $\frac{1\frac{1}{2}2\frac{1}{2}6}{121}$, the Square of the ſaid $12\frac{3}{11}$, ſubtract 60, and the remainder $\frac{11\frac{1}{2}16}{121}$ is the price of the whole mixture, which is a Square whoſe ſide is $\frac{10\frac{1}{2}6}{11}$; and becauſe $\frac{11\frac{1}{2}16}{121}$ is the number of pence expreſſing the value of the mixture, it muſt be equal to the Product of 8 multiplied by a certain part of $12\frac{3}{11}$, (the number of quarts in the mixture,) together with the Product of 5 multiplied by the remaining part of $12\frac{3}{11}$; we muſt therefore divide $12\frac{3}{11}$ into two ſuch parts, that if the one be multiplied by 5, and the other by 8, the ſumm of the two Products may make $\frac{11\frac{1}{2}16}{121}$; but that may be done thus,
5. For one of the deſired parts of $12\frac{3}{11}$ put . . . . . e
6. Then the other ſhall be . . . . . $12\frac{3}{11} - e$
7. And if the former part be multiplied by 5, and the latter by 8, the ſumm of the Products will be . . . . . $98\frac{10}{11} - 3e$
8. Which ſumm muſt be equated to $\frac{11\frac{1}{2}16}{121}$, *viz.* . . . . $98\frac{10}{11} - 3e = \frac{11\frac{1}{2}16}{121}$

9. Which Equation, after due Reduction, makes known one of the deſired parts, to wit, ... } $e = \frac{\pm 44}{121}$
10. Which ſubtracted from $12\frac{4}{11}$, leaves the other part, to wit, . . $\frac{\pm 132}{121}$
11. I ſay the total mixture of Wine might be compoſed of $\frac{\pm 44}{121}$ quarts of five pence the quart, and $\frac{\pm 132}{121}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $12\frac{4}{11}$ quarts is $\frac{1452}{121}$ pence, which is a ſquare number whoſe ſide is $\frac{\pm 36}{11}$; and if to the ſaid Square $\frac{1452}{121}$ you add 60, the ſumm is alſo a Square, to wit, $\frac{\pm 8436}{121}$, whoſe ſide $12\frac{4}{11}$ is the number of quarts in the mixture.
12. But becauſe the ſide of the Square to be equated to $aa - 60$ may be feigned $a -$ any abſolute number between $2\frac{\pm 1}{1000}$, *&c.* and $3\frac{\pm 1}{1000}$, *&c.* let the ſaid ſide be $a - 3$, the Square whereof equated to $aa - 60$ will give $a = 11\frac{1}{2}$ for the number of quarts in the mixture; then the Square of $11\frac{1}{2}$ is $\frac{529}{4}$, from which ſubtracting 60, the remainder $\frac{289}{4}$ is the ſquare number of pence expreſſing the value of the mixture. Now the ſaid $11\frac{1}{2}$ is to be divided into two ſuch numbers that if one of them be multiplied by 5, and the other by 8, the ſumm of the Products may make the Square $\frac{289}{4}$; but two ſuch numbers (by working as before) will be found $\frac{23}{12}$ and $\frac{22}{12}$.

I ſay again, the mixture may be compos'd of $\frac{23}{12}$ quarts of five pence the quart, and $\frac{22}{12}$ quarts of eight pence the quart; whence the value of the whole mixt quantity, to wit, of $11\frac{1}{2}$ quarts, is $\frac{289}{4}$ pence, which is a Square, to which if you add 60 the ſumm is alſo a Square, to wit, $\frac{529}{4}$, whoſe ſide $11\frac{1}{2}$ is the number of quarts in the whole mixture. From the premiſſes 'tis evident that the Queſtion is capable of innumerable Anſwers in rational numbers.

QUEST. 90.

To find a right-angled Triangle in rational numbers, that one of the ſides about the right-angle may be to the Area in a given Reaſon, ſuppoſe as r to s.

RESOLUTION.

1. For the Triangle ſought let a right-angled Triangle be formed from two numbers, *viz.* a the greater and e the leſſer, ſo the three ſides will be theſe, to wit, ... } $aa + ee$; $aa - ee$; $2ae$
2. The Area of that Triangle is . . . . . } $aaae - aeee$
3. Then (according to the Queſtion) let theſe four quantities be ſuppoſed to be Proportionals, *viz.* $r . s :: aa - ee . aaae - aeee$.
4. And becauſe if the two latter terms of that Analogy be ſeverally divided by $aa - ee$, the Quotients are 1 and ae, therefore that Analogy may be reduced to this, *viz.* $r . s :: 1 . ae$.
5. And by comparing the Product of the extremes to the Product of the means, this Equation ariſeth, *viz.* . . . . . } $rae = s$
6. And by dividing each part of the laſt Equation by ra, this ariſeth } $e = \frac{s}{ra}$
Hence this

CANON.

7. Take any number at pleaſure, which may be called a, then divide s the latter term of the given Reaſon, by the Product of the firſt term r multiplied into the number a, and call the Quotient the number e; laſtly, from the ſaid numbers a and e form a right-angled Triangle, and it ſhall be that which is ſought.

An Example in Numbers.

Let it be required to find a right-angled Triangle, ſuch, that one of the ſides about the right-angle may be to the Area as 1 to 10.

Suppoſe, ... } $r = 1$, $s = 10$ } the terms of the given Reaſon;

Then by the Canon, } $a = 2$ } taken at pleaſure, $e = 5 = \frac{s}{ra}$.

Laſtly, from 5 and 2 form a right-angled Triangle, and the three ſides will be 29, 21 and 20, which Triangle will ſolve the Queſtion; for 21, one of the ſides about the right-angle, is to the Area 210, as 1 to 10. Which was required.

Likewiſe

Likewiſe by the Canon, this right-angled Triangle, to wit, 101, 99 and 20 will be found to ſolve the Queſtion, for 99 is to the Area 990, as 1 to 10; and innumerable right-angled Triangles in Fractions may be found to perform the ſame effect.

QUEST. 91.

To find a right-angled Triangle in rational numbers, that the Hypothenuſal may be to the Area in a given Reaſon, ſuppoſe as r to s.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . . } a
2. And for the other ſide about the right-angle put . . . . } e
3. Then the ſquare Root of the ſumm of the Squares of thoſe ſides ſhall be the Hypothenuſal, to wit, ... } $\sqrt{:aa + ee:}$
4. And the Area of the ſaid Triangle is . . . . . } . . $\frac{1}{2}ae$
5. Now according to the Queſtion, the Hypothenuſal muſt be to the Area as r to s, therefore from the third and fourth ſteps this Analogy ariſeth, *viz.* $r . s :: \sqrt{:aa + ee:} . \frac{1}{2}ae$.
6. But the Squares of thoſe Proportionals are alſo Proportionals, therefore $rr . ss :: aa + ee . \frac{1}{4}aaee$.
7. And from the laſt Analogy, by comparing the Product of the multiplication of the extremes to the Product of the means, this Equation ariſeth, *viz.* $\frac{1}{4}rraaee = ssaa + ssee$.
8. From which Equation, by tranſpoſition of $ssee$, this ariſeth, $\frac{1}{4}rraaee - ssee = ssaa$.
9. And by dividing each part of the laſt Equation by $\frac{1}{4}rraa - ss$, there will ariſe . . . . . } $ee = \frac{ssaa}{\frac{1}{4}rraa - ss}$
10. In which laſt Equation the Numerator $ssaa$ is a Square whoſe ſide is sa, and if the Denominator were a Square, then the whole Fraction would be alſo a Square, and conſequently the ſide thereof, to wit, the number e would be rational, it remains therefore to equate the Denominator $\frac{1}{4}rraa - ss$ to a Square, to which end, let the ſide thereof be feigned $\frac{1}{2}ra - b$; then the Square of $\frac{1}{2}ra - b$ being equated to $\frac{1}{4}rraa - ss$, this Equation ariſeth, *viz.* $\frac{1}{4}rraa - ss = \frac{1}{4}rraa - rba + bb$.
11. Whence, after due Reduction, you will find . . . . . } $a = \frac{ss + bb}{rb}$
12. Now if we ſuppoſe r, s and b to repreſent known rational numbers, then a, e and $\sqrt{:aa + ee:}$ which in the three firſt ſteps were put for the three ſides of the right-angled Triangle ſought, will alſo (from the eleventh, ninth and tenth ſteps,) be expreſſible by rational numbers, to wit, theſe, $\frac{ss + bb}{rb}$, $\frac{2sss + 2sbb}{rss - rbb}$, $\frac{ssss + 2ssbb + bbbb}{rssb - rbbb}$.
13. Or the two firſt of the three ſides laſt expreſt may be reduced to the ſame Denominator with the third, and then the three ſides of the right-angled Triangle ſought will be theſe, to wit, $\frac{ssss - bbbb}{rssb - rbbb}$, $\frac{2sssb + 2sbbb}{rssb - rbbb}$, $\frac{ssss + 2ssbb + bbbb}{rssb - rbbb}$.
Which three ſides, if they be expreſt by words, will give this

CANON.

14. Take for b any number leſs than s the latter term of the given Reaſon; then from the numbers s and b form a right-angled Triangle, and multiply the three ſides ſeverally by the Hypothenuſal; laſtly, divide thoſe three Products ſeverally by the Product made by the multiplication of the difference of the Squares of the two numbers s and b, (which formed the ſaid Triangle,) into the Product of b the leſſer of the ſame two numbers and r the firſt term of the given Reaſon, ſo ſhall the Quotients be the three ſides of a right-angled Triangle, which will ſolve the Queſtion propoſed.

An

An Example in Numbers.

Let it be required to find out a right-angled Triangle whose Hypothenusal may be to the Area as 2 to 3.

Suppose . . . . . $\left\{\begin{array}{l} r = 2 \\ s = 3 \\ b = 1 \end{array}\right\}$ the Terms of the given Reason, $\}$ less than 3, (or s.)

Then form a right-angled Triangle from 3 and 1, (to wit, s and b,) and the three sides will be 10, 8 and 6; these multiplied severally by the Hypothenusal 10 will produce 100, 80 and 60, which divided severally by the Product which answers to $ss - bb$ into rb, that is, by 16, will give $\frac{25}{4}$, $\frac{20}{4}$ and $\frac{15}{4}$ for the Triangle sought; for the Hypothenusal $\frac{25}{4}$, or $\frac{100}{16}$, is to the Area $\frac{150}{16}$, as 2 to 3. Which was required.

In like manner, if it were desired to find a right-angled Triangle whose Hypothenusal might be to the Area as 3 to 2; then by the Canon, the three sides will be found $\frac{15}{9}$, $\frac{20}{9}$ and $\frac{12}{9}$.

QUEST. 92.

To find a right-angled Triangle in rational numbers, that the summ of all the three sides may be to the Area in a given Reason, suppose as r to s.

RESOLUTION.

1. For the right-angled Triangle sought let a Triangle be formed from any two numbers, suppose from a the greater and e the lesser, so the three sides will be these, $\}$ $aa + ee$, $aa - ee$, $2ae$
2. Then the Area will be . . . . . $\}$ $aaae - aeee$
3. And the summ of the three sides is . . . . . $\}$ $2aa + 2ae$
4. Now (according to the Question) the summ of the three sides must be to the Area, as r to s, therefore,

   As r . s :: $2aa + 2ae$. $aaae - aeee$.
5. Or, by dividing each of the two latter terms of that Analogy by a, this ariseth; *viz.*

   As r . s :: $2a + 2e$. $aae - eee$.
6. Whence, by comparing the Product of the extremes to the Product of the means, this Equation ariseth, *viz.* . $\}$ $raae - reee = 2sa + 2se$
7. Therefore by due transposition, ... $\}$ $raae - 2sa = reee + 2se$
8. And by dividing all in the last Equation by re, this ariseth, ... $\}$ $aa - \frac{2s}{re}a = \frac{ree + 2s}{r}$
9. Which last Equation being resolved by the Canon in *Sect.* V. *Chap.* 15. *Book* 1. the value of a will be discovered, *viz.* . . . . . $\}$ $\therefore a = e + \frac{2s}{re}$

Hence this

CANON.

10. Take any number at pleasure, which may be called e; then to the number e add the Quotient that ariseth by dividing the double of the latter term of the given Reason, by the Product of the first term multiplied into the number e, and call the summ the number a; lastly, from the said numbers a and e form a right-angled Triangle, and it shall be that which is sought.

An Example in Numbers.

Let it be required to find out a right-angled Triangle, that the summ of all the three sides may be to the Area as 1 to 5.

Suppose . . . . $\left\{\begin{array}{l} r = 1 \\ s = 5 \end{array}\right\}$ the Terms of the given Reason, then

By the Canon, . . $\left\{\begin{array}{l} e = 2 \\ a = 7 = e + \frac{2s}{re}. \end{array}\right.$ $\}$ taken at pleasure,

Then form a right-angled Triangle from 7 and 2, and the three sides will be 53, 45, 28, which will solve the Question, for the summ of all the three sides, to wit, 126, is to the Area 630, as 1 to 5. Which was required to be done.

Likewise if a right-angled Triangle be formed from 11 and 1, (to wit, a and e, found out by the Canon,) the three sides will be 122, 120, 22, whose summ 264 is to the Area 1320, as 1 to 5.

Again,

Again, if a right-angled Triangle be formed from 11 and 10, the three sides will be 221, 21 and 220, whose summ 462 is to the Area 2310, as 1 to 5.

Lastly, you may find out as many right-angled Triangles in Fractions as you please to solve the Question. Also, upon the same ground it will not be difficult to find out innumerable Isosceles-Triangles, in every one of which the Perimeter shall be to the Area in a given Reason.

QUEST. 93.

To find a right-angled Triangle in rational numbers, that as well the Hypothenusal as the difference of the sides about the right-angle may be a Square.

RESOLUTION.

1. For the Triangle sought let a right-angled Triangle be formed from two numbers, suppose from a and e, and let a be the greater, so the three sides will be these, *viz.* $\}$ $aa + ee$, $aa - ee$, $2ae$
2. Now (according to the Question) as well the Hypothenusal as the difference of the sides about the right-angle must be a Square, so we are faln upon this Duplicate equality, *viz.* . . . . . $\}$ $aa + ee = \square$ / $2ae - aa + ee = \square$
3. The difference of those two quantities is . . . . $\}$ $2aa - 2ae$
4. Which difference is equal to the Product of these two quantities, to wit, ... $\}$ $2a - 2e$ and a
5. The half-summ of those two Factors in the last step is $\}$ $\frac{1}{2}a - e$
6. Then the Square of the said half-summ being equated to the greater of the two quantities in the second step, this Equation ariseth, *viz.* . . . . . $\}$ $aa + ee = \frac{1}{4}aa - 3ae + ee$
7. Whence after due Reduction there will arise . . . $\}$ $12e = 5a$
8. And by reducing the last Equation into Proportionals, it shall be

   Hence this As 12 . 5 :: a . e.

CANON.

9. If from 12 and 5; or any two numbers in that proportion, a right-angled Triangle be formed, it will solve the Question.

As, for example, in the right-angled Triangle 169, 119 and 120, which is formed from 12 and 5, the Hypothenusal 169 is a Square; also the difference of the sides about the right-angle, to wit, 1 is a Square. The same effect will be produced in a right-angled Triangle formed from any two numbers which have such proportion one to another as 12 to 5.

QUEST. 94

To find a right-angled Triangle, that one of the sides about the right-angle may be a Square, which added to a given multiple, suppose the triple, of the Square of the difference of the sides about the right-angle may make a Square.

RESOLUTION.

1. For one of the sides about the right-angle, that it may be a Square, put . . . . . $\}$ 4
2. And for the other put . . . . . $\}$ a
3. The summ of their Squares must make a Square, to wit, the Square of the Hypothenusal, therefore . . . . . $\}$ $aa + 16 = \square$
4. The difference of the sides about the right-angle is . . $\}$ $a - 4$
5. The triple of the Square of that difference is . . . . $\}$ $3aa - 24a + 48$
6. To which (according to the Question) add 4, the square number first assumed for one of the sides about the right-angle, and the summ must be equal to a Square, *viz.* . . $\}$ $3aa - 24a + 52 = \square$
7. So in the third and sixth steps we are faln upon a Duplicate equality, but the numbers prefixt to aa in the quantities to be equated, are not Squares, neither are the two known numbers in the same quantities both Squares, for 52 is not a Square, whereby the said Duplicate equality is inexplicable; but if the said 52 were a Square, then the Duplicate equality

equality might be resolved; therefore instead of the square number 4 which in the first step was assumed for one of the sides about the right-angle, we must seek such a Square that if it be added to the triple of its Square the summ may be a Square. Suppose therefore that Square sought to be ee, this added to the triple of its Square makes $3eeee + ee$ to be equated to a Square, the side whereof may be variously feigned, let it be $ee + e$, then the Square of $ee + e$, to wit, $eeee + 2eee + ee$ being equated to $3eeee + ee$, after due Reduction the value of e will be found 1, and ee is also 1. So we have found a Square, to wit, 1, which added to the triple of its Square makes the Square 4; therefore now the Resolution may be renewed thus, *viz.*

8. For one of the sides about the right-angle put the Square . . . 1
9. And for the other side . . . . . a
10. The summ of their Squares must be equal to a Square, *viz.* $aa + 1 = \square$
11. The difference of the sides about the right-angle is . . $a \backsim 1$
12. The triple of the Square of that difference is . . . . $3aa - 6a + 3$
13. To which adding the side 1 in the eighth step, the summ must be equal to a Square, *viz.* . . . . . $3aa - 6a + 4 = \square$
14. Also from the tenth step, ... $aa + 1 = \square$
15. So in the two last steps we have a new Duplicate equality which may be resolved thus; first, to the end there may be one and the same known square number in each of the two quantities to be equated to Squares, I multiply the quantity in the fourteenth step, to wit, $aa + 1$ by 4, and it makes $4aa + 4$; now each of these quantities is to be equated to a Square, *viz.* $3aa - 6a + 4 = \square$ and $4aa + 4 = \square$
16. The difference of those two quantities is . . . . . $aa + 6a$
17. Which difference is equal to the Product of these two Factors, to wit, ... $\frac{2}{3}a + 4$, and $\frac{1}{2}a$
18. Half the summ of those Factors is . . . . . $\frac{11}{12}a + 2$
19. The Square of the said half-summ is . . . . . $\frac{169}{144}aa + \frac{11}{3}a + 4$
20. Which Square equated to $4aa + 4$, (the greater of the two quantities in the fifteenth step,) will after due Reduction give $a = \frac{624}{407}$.
21. Therefore from the twentieth, ninth and eighth steps the sides about the right-angle are $\frac{624}{407}$ and 1, the summ of whose Squares is $\frac{1115049}{165649}$, whose square Root $\frac{745}{407}$ is the Hypothenusal sought.

I say $\frac{624}{407}$, 1 and $\frac{745}{407}$ are the sides of a right-angled Triangle, which will solve the Question; for one of the sides about the right-angle is a Square, to wit, 1, and if this be added to the triple of the the Square of the difference of the sides about the right-angle, it makes the Square $\frac{1065249}{165649}$, whose side is $\frac{745}{407}$. From the premisses it is evident that innumerable right-angled Triangles may be found to solve the Question.

QUEST. 95.

To find out a right-angled Triangle in rational numbers, that the Square of one of the sides about the right-angle may be equal to the other of the same sides.

[*This is* Problem. 15 *in* pag. 80. *of the Introduction to* Algebra *before cited in* Quest. 58. *but I shall resolve it after another manner.*]

RESOLUTION.

1. For one of the sides about the right-angle put ra, (r representing some known number, and a some number unknown,) . . . ra
2. Then (according to the Question) the Square of that side must be the other of the sides about the right-angle, to wit, . . . $rraa$
3. The summ of the Squares of those sides is . . . . . $rrrraaaa + rraa$
4. Which summ must be equal to the Square of the Hypothenusal, and therefore it remains to equate the said $rrrraaaa + rraa$ to a Square, to which end take some known number s greater than r, and then the side of the said Square may be feigned $sa - rraa$, the Square whereof being equated to $rrrraaaa + rraa$, and due Reduction made, you will find

$$a = \frac{ss - rr}{2sr}.$$

5. But

5. But s and r were assumed to represent two known numbers whereof s is the greater, therefore from the premisses the three sides of the Triangles sought shall be known also, and may be expresst thus,

$\frac{ss - rr}{2sr}$, or $\frac{2sssr - 2srrr}{4ssrr}$, } The sides about the right-angle.
$\frac{ssss - 2ssrr + rrrr}{4ssrr}$, }

$\frac{ssss - rrrr}{4ssrr}$, The Hypothenusal.

6. Moreover, because the Square of $\frac{2sr}{ss - rr}$ is $\frac{4ssrr}{ssss - 2ssrr + rrrr}$; and the summ of the Squares of the two last Fractions is equal to the Square of $\frac{2sssr + 2srrr}{ssss - 2ssrr + rrrr}$, (as will easily appear by Multiplication and Addition;) therefore the three quantities last expresst shall also be the sides of a right-angled Triangle to solve the Question proposed, *viz.*

$\frac{2sr}{ss - rr}$, or $\frac{2sssr - 2srrr}{ssss - 2ssrr + rrrr}$, } The sides about the right-angle.
$\frac{4ssrr}{ssss - 2ssrr + rrrr}$, }

$\frac{2sssr + 2srrr}{ssss - 2ssrr + rrrr}$, The Hypothenusal.

7. If the three sides of the Triangle sought, as they be above expresst in the fifth and sixth steps be compared together, it will be easie to deduce from thence this following

CANON.

First, form a right-angled Triangle from any two unequal numbers; then multiply the three sides of that Triangle severally by either of the sides about the right-angle; lastly, divide severally those three Products by the Square of the other of the sides about the right-angle; so the three Quotients shall be the sides of the right-angled Triangle sought.

Examples in Numbers.

First, find out three numbers to expres the sides of a right-angled Triangle, suppose these, ...	3 , 4 , 5
Then multiply those three sides severally by 4, the greater of the sides about the right-angle, and the three Products will be these, *viz.*	12 , 16 , 20
Lastly, divide those three Products severally by 9, the Square of the lesser of the sides about the right-angle; so the Quotients will be the sides of a right-angled Triangle to solve the Question proposed, to wit, these, ...	$\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$

Again,

Let the right-angled Triangle first found out be here repeated, to wit, ...	4 , 3 , 5
Then multiply those three sides severally by 3, the lesser of the sides about the right-angle, and the Products will be these, .	12 , 9 , 15
Lastly, divide those three Products severally by 16, the Square of the greater of the sides about the right-angle; so the Quotients will give these three sides of a right angled Triangle to solve the Question, to wit, ...	$\frac{3}{4}$, $\frac{9}{16}$, $\frac{15}{16}$

QUEST. 96. (Quæst. 1. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the sides about the right-angle being severally subtracted from the Hypothenusal may leave cube-numbers.

RESOLUTION.

1. First, form a right-angled Triangle from two unequal numbers, suppose from a and e, and let a be the greater, so the three sides will be these, *viz.*	Hypoth. Base, Perp. $aa + ee$. $aa - ee$. $2ae$

2. Then by ſubtracting the Baſe from the Hypothenuſal, the remainder is $2ee$, which ſhould be a Cube, but it is not; yet it ſhews that e (to wit, one of the numbers from which the deſired right-angled Triangle is to be formed,) muſt be ſuch that the double of its Square may make a Cube. Now that we may chuſe the number e with that condition, let $2ee$ be equated to the Cube $bbbeee$, *viz.* ſuppoſe $2ee = bbbeee$, (where bbb may repreſent any known cube-number,) whence after due Reduction the value of e will be made known, *viz.*

$$e = \frac{2}{bbb}.$$

3. It remains, that the Perpendicular ſubtracted from the Hypothenuſal may leave a Cube; but the remainder is the Square $aa + ee - 2ae$, which is not a Cube, but if its Root $a - e$ were a Cube, then that Square would be a Cube; (for a Cube multiplied into a Cube produceth a Cube, by *Prop.* 3, & 4. *Elem.* 9. *Euclid.*) Let therefore the ſaid Root $a - e$ be equated to ſome Cube, *viz.* ſuppoſe $a - e = ddd$; hence, and from the third ſtep it follows, that

$$a = e + ddd = \frac{2}{bbb} + ddd.$$

Now, if from $\frac{2}{bbb} + ddd$ and $\frac{2}{bbb}$ (the values of a and e) a right-angled Triangle be formed, it will ſolve the Queſtion. Hence this

CANON.

4. Divide 2 by any cube-number, and reſerve the Quotient; then to the ſaid Quotient add any cube-number; laſtly, from the ſumm and Quotient form a right-angled Triangle and it ſhall ſolve the Queſtion propoſed.

As, for example, I divide 2 by the Cube 8, and reſerve the Quotient $\frac{1}{4}$ for one of the numbers by which the right-angled Triangle is to be formed; then to the Quotient $\frac{1}{4}$ I add ſome Cube, as 1, and the ſumm is $\frac{5}{4}$; laſtly, from $\frac{5}{4}$ and $\frac{1}{4}$ I form a right-angled Triangle and find the ſides $\frac{1}{8}\frac{3}{8}$, $\frac{1}{8}\frac{2}{8}$ and $\frac{5}{8}$, which will ſolve the Queſtion; for the ſides about the right-angle, to wit, $\frac{1}{8}\frac{2}{8}$ and $\frac{5}{8}$ being ſeverally ſubtracted from the Hypothenuſal $\frac{1}{8}\frac{3}{8}$, the remainders are the Cubes $\frac{1}{8}$ and 1.

But after one right-angled Triangle is found out to ſolve the Queſtion, if you multiply or divide every one of the ſides thereof by one and the ſame cube-number, the Products or Quotients will give another right-angled Triangle to ſolve the Queſtion: As, if the three ſides before found out, to wit, $\frac{1}{8}$, $\frac{1}{8}$ and $\frac{5}{8}$ be ſeverally multiplied by the Denominator 8, they will produce 13, 12 and 5, which will ſolve the Queſtion. Likewiſe, if 13, 12 and 5 be ſeverally multiplied by 8, there will be produced the right-angled Triangle 104, 96 and 40, where the differences between the Hypothenuſal and the other two ſides are Cubes, to wit, 8 and 64. The reaſon is evident; for,

First, by Conſtruction $\frac{1}{8}\frac{3}{8}$, $\frac{1}{8}\frac{2}{8}$, $\frac{5}{8}$ are the ſides of a right-angled Triangle that will ſolve the Queſtion, *viz.* . . . $\left\{\begin{array}{l} \frac{1}{8}\frac{3}{8} - \frac{1}{8}\frac{2}{8} = \text{a Cube,} \\ \frac{1}{8}\frac{3}{8} - \frac{5}{8} = \text{a Cube.} \end{array}\right.$

And becauſe a Cube multiplied by a Cube produceth a Cube, thoſe two differences or Cubes multiplied by the Cube 8 ſhall neceſſarily produce theſe differences or Cubes, *viz.* $\left\{\begin{array}{l} 13 - 12 = \text{a Cube,} \\ 13 - 5 = \text{a Cube.} \end{array}\right.$

Wherefore the right-angled Triangle 13, 12, 5 ſhall neceſſarily ſolve the Queſtion, as well as $\frac{1}{8}\frac{3}{8}$, $\frac{1}{8}\frac{2}{8}$, $\frac{5}{8}$.

QUEST. 97. (This is a *Lemma*, uſed by *Dioph.* in reſolving the following *Queſt.* 98.)

To find a right-angled Triangle and a Square in rational numbers, that if the Area of the Triangle be ſubtracted from the Square, the remainder multiplied by a given number (d) may produce a ſquare number.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $\frac{1}{a}$; ſo the three ſides will be . . . . . $\left\{\begin{array}{l} aa + \frac{1}{aa} = \text{the Hypoth.} \\ aa - \frac{1}{aa} = \text{the Perpend.} \\ 2 = \text{the Baſe.} \end{array}\right.$

2. Therefore the Area, (by multiplying the Perpendicular into half the Baſe,) is . . . . . $aa - \frac{1}{aa}$

3. Then

3. Then feign the ſide of the Square ſought to be . . . . . $a + \frac{2d}{a}$

4. Therefore the Square it ſelf is . . . . . $aa + 4d + \frac{4dd}{aa}$

5. From which Square, the Area above expreſt being ſubtracted, there will remain . . . . . $4d + \frac{4dd + 1}{aa}$

6. That is, ... $\frac{4daa + 4dd + 1}{aa}$

7. Then the ſaid remainder being multiplied into the given number (d) produceth . . . . . $\frac{4ddaa + 4ddd + d}{aa}$

8. Which Product muſt (according to the Queſtion) be a Square: But the Denominator aa is a Square, it remains therefore to equate the Numerator to a Square, *viz.* $4ddaa + 4ddd + d$ muſt be equated to a Square, the ſide whereof may be variouſly feigned, let it be $2da + d$; and then the Square of $2da + d$ being equated to the ſaid $4ddaa + 4ddd + d$, this Equation ariſeth, to wit,

$$4ddaa + 4dda + dd = 4ddaa + 4ddd + d.$$

9. Which Equation after due Reduction gives . . . . . $a = \frac{4dd + 1 - d}{4d}$

From the ninth, firſt and third ſteps ariſeth this

CANON.

10. From $\frac{4dd + 1 - d}{4d}$ and $\frac{4d}{4dd + 1 - d}$ form a right-angled Triangle, which ſhall be that ſought by the Queſtion; and the ſide of the Square ſought ſhall be $\frac{4dd + 1 - d}{4d} + \frac{8dd}{4dd + 1 - d}$.

An Example in Numbers.

Suppoſe $5 = d$ the number given; then form a right-angled Triangle from $\frac{24}{5}$ and $\frac{7}{24}$, ſo the Hypothenuſal will be $\frac{24}{5}\frac{24}{24}\frac{24}{24}\frac{1}{1}$, the Perpendicular $\frac{24}{5}\frac{7}{24}\frac{7}{24}\frac{24}{1}$, and the Baſe 2; moreover, the ſide of the Square ſought will be $\frac{41}{60}$, and the Square it ſelf $\frac{1681}{3600}$, or (in the ſame Denominator with the ſaid Hypothenuſal and Perpendicular) $\frac{672}{1400}$: which Square and Triangle will ſolve the Queſtion; for if the Area of the Triangle, to wit, $\frac{24}{5}\frac{7}{24}\frac{7}{24}\frac{24}{1}$ be ſubtracted from the ſaid Square $\frac{672}{1400}$, the remainder $\frac{24}{5}\frac{24}{24}\frac{24}{24}\frac{1}{1}$, that is, $\frac{1}{176}$ multiplied into the given number 5, produceth the Square $\frac{25}{576}$, whoſe Root is $\frac{24}{24}$.

Another Example.

Suppoſe $3 = d$ the number given in the Queſtion; then let a right-angled Triangle be formed from $\frac{12}{6}$ and $\frac{6}{17}$, (which numbers are diſcovered by the preceeding Canon,) ſo the Hypothenuſal will be $\frac{2}{1}\frac{4}{0}\frac{4}{4}\frac{1}{0}\frac{7}{4}$, the Perpendicular $\frac{2}{1}\frac{2}{0}\frac{2}{4}\frac{1}{0}\frac{7}{4}$, and the Baſe 2: moreover, the ſide of the Square ſought will be found $\frac{101}{102}$, and the Square it ſelf $\frac{1}{1}\frac{1}{0}\frac{2}{4}\frac{1}{0}\frac{7}{4}$; from which Square ſubtracting the Area of the Triangle, to wit, $\frac{2}{1}\frac{2}{0}\frac{2}{4}\frac{1}{0}\frac{7}{4}$, the remainder $\frac{1}{1}\frac{2}{0}\frac{8}{4}\frac{0}{0}\frac{0}{4}$, or in its leaſt terms $\frac{200}{289}$, multiplied into the given number 3, produceth the Square $\frac{144}{289}\frac{00}{}$, whoſe ſide is $\frac{120}{17}$.

Note. Inſtead of 2 which is prefixt to d in the Numerator of the Fraction $\frac{2d}{a}$ in the third ſtep of the preceding Reſolution, you may take the half of any ſquare number and prefix it to d for a Numerator, over the Denominator a; as, $4\frac{1}{2}d$, $8d$, $12\frac{1}{2}d$, &c. and then by proſecuting the work as before from the third ſtep to the end of the Reſolution, various Anſwers to the Queſtion from one and the ſame given number will be diſcovered.

QUEST. 98. (Quæſt. 3. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with a given number, ſuppoſe 5, may make a Square.

RESOLUTION.

1. Let the ſides of ſome known right-angled Triangle, as 5, 4 and 3 be ſeverally multiplied by a, and take the Products to repreſent the ſides of the Triangle ſought, to wit, ... $\left\{\begin{array}{l} 5a, \; 4a, \; 3a \end{array}\right.$

2. Then

2. Then the Area thereof increaſed with 5 (the number given in the Queſtion) makes . . . . . $6aa + 5$
3. Which ſumm muſt be equal to a Square, ſuppoſe it be $9aa$, therefore . . . . . $6aa + 5 = 9aa$
4. Whence by equal ſubtraction of $6aa$, there remains . . . $5 = 3aa$
5. Let each part of the laſt Equation be multiplied by 5, and it produceth . . . . . $25 = 15aa$
6. Now if 15 which is prefix'd to aa were a Square, then the value of a would be rational: Whence therefore comes 15? Examin the work, and you will find, that by ſubtracting 6 the Area of the Triangle 5, 4, 3 from the ſquare number 9, and then by multiplying the remainder 3 into the given number 5, there is produced 15; whereby it is manifeſt that the ſcope of our ſearch muſt be to find a right-angled Triangle and a ſquare number, that the Area of the Triangle being ſubtracted from the Square and the remainder multiplied by the given number 5 may make a Square. But the preceding 97th Queſtion ſhews how to find out ſuch a Triangle and Square; take if you pleaſe thoſe there found in the firſt Example, to wit, the right-angled Triangle whoſe Hypothenuſal is $\frac{221}{120}$, Perpendicular $\frac{119}{120}$, and Baſe 2; and the Square $\frac{48841}{14400}$, whoſe ſide is $\frac{221}{120}$: Then renew the Reſolution thus,
7. For the three ſides of the Triangle ſought put . . . $\frac{221}{120}a$, $\frac{119}{120}a$, $2a$
8. The Area thereof increaſed with 5 makes . . . $\frac{119}{14400}aa + 5$
9. Which ſumm muſt be equated to the Product of $\frac{48841}{14400}$ (the Square before found) multiplied into aa, ſo this Equation ariſeth, ... $\frac{119}{14400}aa + 5 = \frac{48841}{14400}aa$
10. Therefore by ſubtracting $\frac{119}{14400}aa$ from each part of that Equation this remains, to wit, ... $5 = \frac{14281}{14400}aa = \frac{169}{176}aa$
11. And by multiplying each part of the laſt Equation into 5 it produceth . . . . . $25 = \frac{2197}{176}aa$
12. And by extracting the ſquare Root out of each part of the laſt Equation, there will ariſe . . . $5 = \frac{221}{24}a$
13. Whence by dividing each part by $\frac{221}{24}$, the value of a will be made known, *viz.* . . . . . $a = \frac{24}{13}$
14. Wherefore from the thirteenth and ſeventh ſteps the three ſides of the Triangle ſought will be found theſe, to wit, $\frac{312}{313}$, $\frac{120}{313}$, $\frac{48}{13}$, the Area whereof is $\frac{2340}{169}$, to which adding 5, the ſumm will be the Square $\frac{48841}{70225}$, whoſe Root is $\frac{221}{265}$. Therefore the Queſtion is ſolved.

Vieta, in the 9th of the 5th Book of his *Zeteticks*, ſhews how to find out a right-angled Triangle whoſe Area increaſed with a number compos'd of two Squares may make a Square; whereby 'tis probable he thought this Queſtion to be applicable only to a number compos'd of two Squares, becauſe *Diophantus* propos'd the given number 5, which is compos'd of two Squares; but 'tis evident from the precedent Reſolution that the Queſtion may be extended to any given number whatſoever: And for greater illuſtration, let it be required to find out a right-angled Triangle whoſe Area increaſed with 3 may make a Square.

Firſt a right-angled Triangle is to be ſought, and alſo a Square, that the Area of the right-angled Triangle being ſubtracted from the Square and the remainder multiplied by the given number 3, the Product may be a ſquare number: But by the latter Example of the preceding 97th Queſtion ſuch a Triangle and Square are found out, *viz.* the Triangle whoſe Hypothenuſal is $\frac{481}{104}$, the Perpendicular $\frac{425}{104}$, and the Baſe 2; and the Square $\frac{231361}{10816}$, whoſe ſide is $\frac{481}{104}$: then for the three ſides of the right-angled Triangle ſought let there be put $\frac{481}{104}a$, $\frac{425}{104}a$ and $2a$, the Area whereof is $\frac{425}{10816}aa$, to which adding 3 it makes $\frac{425}{10816}aa + 3$; which ſumm equated to $\frac{231361}{10816}aa$, ($\frac{231361}{10816}$ being the Square above found,) will give $\frac{17}{40}$ for the value of a, by which $\frac{17}{40}$ the three ſides above put being reſolved, you will find the three ſides of the right-angled Triangle ſought to be theſe, to wit, $\frac{481}{480}$, $\frac{425}{480}$ and $\frac{34}{40}$; the Area of which Triangle is $\frac{2040}{2304}$, to which adding 3 it makes the Square $\frac{9409}{2304}$, whoſe ſide is $\frac{97}{48}$.

QUEST. 99.

QUEST. 99. (Quæſt. 6. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof increaſed with one of the ſides about the right-angle may make a given number, ſuppoſe n.

RESOLUTION.

1. For the ſides about the right-angle of the Triangle ſought put . . . a and e
2. Then is the Area . . . . . $\frac{1}{2}ae$
3. Which increaſed with one of the ſides about the right-angle muſt make the given number n, hence this Equation, ... $\frac{1}{2}ae + a = n$
4. From which Equation, after due Reduction, you will find . . . $e = \frac{n-a}{\frac{1}{2}a}$
5. And becauſe $aa + ee$ muſt be equal to the Square of the Hypothenuſal, it follows from the firſt and fourth ſteps, that the Square of the Hypothenuſal muſt be equal to the ſumm of the Squares of a and $\frac{n-a}{\frac{1}{2}a}$, that is,
$$\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}.$$
6. Of which fractional quantity the Denominator $\frac{1}{4}aa$ is a Square, whoſe ſide is $\frac{1}{2}a$; if therefore the Numerator were a Square the whole Fraction would be a Square: It remains then to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to ſome Square, to which end, let its ſide be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the ſaid Numerator, this Equation ariſeth, *viz.*
$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$
7. Whence the value of a will be made known, *viz.* . . . . $a = \frac{2n}{n+1}$
8. And according to that value of a, the value of e will alſo be be made known by the fourth ſtep, *viz.* . . . . . $e = n - 1$
9. And becauſe the ſquare Root of $aa + ee$ is equal to the Hypothenuſal, therefore the ſquare Root of the ſumm of the Squares of the two quantities in the latter parts of the Equations in the ſeventh and eighth ſteps, will give for the Hypothenuſal ſought $\frac{nn+1}{n+1}$
10. Therefore from the ſeventh, eighth and ninth ſteps the three ſides of the right-angled Triangle ſought are theſe, to wit,
$$\frac{2n}{n+1},\ n-1,\ \left(\text{or } \frac{nn-1}{n+1},\right) \text{ and } \frac{nn+1}{n+1}.$$
Which three ſides if they be expreſt by words will give the following Canon, which is the ſame with that delivered by *Fermat* in his Obſervation upon the ſixth Queſtion of the ſixth Book of *Diophantus*.

CANON.

11. When the given number is greater than unity let a right-angled Triangle be formed from thoſe two numbers, and then divide the three ſides ſeverally by the ſumm of the given number and unity; ſo ſhall the Quotients be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

Let it be required to find a right-angled Triangle whoſe Area increaſed with one of the ſides about the right-angle may make 7.

Firſt (as the Canon directs) let a right-angled Triangle be formed from the given number 7 and 1, ſo the three ſides will be theſe, 50, 48, 14

Then divide thoſe three ſides ſeverally by 8, (to wit, $7+1$,) ſo the Quotients ſhall be the ſides of the right-angled Triangle ſought, to wit, ... $6\frac{1}{4}$, 6, & $1\frac{1}{4}$

I ſay $6\frac{1}{4}$, 6, and $1\frac{1}{4}$ are the ſides of a right-angled Triangle, which will ſolve the Queſtion; for the Area $\frac{25}{4}$ increaſed with $\frac{1}{4}$ (one of the ſides about the right-angle) makes $\frac{28}{4}$, that is, 7, as was required.

But becauſe the ſaid Canon takes not place unleſs the given number exceed unity, I ſhall in the next place explain *Diophantus*'s Reſolution of this Queſtion, by which way, whatever the given number be, a right-angled Triangle may be found out to ſolve the Queſtion propoſed.

Another

Another way of reſolving the foregoing 99th Queſtion.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers, then multiply thoſe three ſides ſeverally by a, (which repreſents a number unknown,) and take the Products for the three ſides of the Triangle ſought, to wit, ... ha , ba , pa
2. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with ba, muſt be equal to the given number n, therefore . . . . . $\frac{1}{2}bpaa + ba = n$
3. Which Equation divided by $\frac{1}{2}bp$ gives . . . . . $aa + \frac{b}{\frac{1}{2}bp}a = \frac{n}{\frac{1}{2}bp}$
4. Now to the end that the value of a in the laſt Equation may be a rational number, the Square of half the Coefficient which is drawn into a, together with the abſolute quantity which poſſeſſeth the latter part of the Equation muſt make a Square, (as is evident by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1.) therefore . . . . . $\frac{\frac{1}{4}bb + \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$
5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, *viz.* . . . . $\frac{1}{4}bb + \frac{1}{2}bpn = \square$
6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb + \frac{1}{2}bpn$ be multiplied by 4, and then . . . . . $bb + 2bpn = \square$
7. Which laſt Equation ſhews, that in order to the ſolving of the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that the Square of one of the ſides about the right-angle, together with the Product of the quadruple Area multiplied by the given number n, may make a Square. Now to find out ſuch a Triangle,
8. For one of the ſides about the right-angle put . . . . . e
9. And for the other ſide put ſome ſquare number, as, ... 1
10. Then the quadruple of the Area is . . . . . $2e$
11. Which multiplied by the given number n, ſuppoſe by $\frac{1}{2}$, makes e
12. To which Product add the Square of one of the ſides about the right-angle, to wit, the Square of 1, which is alſo 1, and the ſumm muſt be equal to a Square, *viz.* . . . . . $e + 1 = \square$
13. Alſo the ſumm of the Squares of the ſides about the right-angle muſt be equal to a Square, to wit, the Square of the Hypothenuſal, therefore . . . . . $ee + 1 = \square$
14. Now in order to reſolve the Duplicate equality in the two laſt ſteps, firſt the difference of the two quantities which are to be equated to Squares is . . . . . $ee - e$
15. Which difference is equal to the Product of the multiplication of theſe two quantities, or Factors, to wit, . . . $2e - 2$ and $\frac{1}{2}e$
16. The half-ſumm of thoſe two Factors is . . . . . $\frac{1}{4}e - 1$
17. The Square of which half-ſumm being equated to $ee + 1$, will give . . . . . $e = \frac{40}{9}$
18. And becauſe e and 1 were put for the ſides about the right-angle, therefore the ſquare Root of the ſumm of the Squares of $\frac{40}{9}$ (that is e) and 1, ſhall be the Hypothenuſal, to wit, $\frac{41}{9}$; ſo we have found out a right-angled Triangle whoſe three ſides are $\frac{41}{9}$, $\frac{40}{9}$ and 1, which are fit for renewing the ſearch of the right-angled Triangle ſought by the Queſtion, in this manner, *viz.*
19. For the three ſides of the right-angled Triangle ſought put . $\frac{41}{9}a$, $\frac{40}{9}a$, and a
20. The Area of which Triangle increaſed with one of the ſides about the right-angle, ſuppoſe with a, muſt be equal to the given number $\frac{1}{2}$, therefore . . . . . $\frac{20}{9}aa + a = \frac{1}{2}$
21. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give . . . . . $a = \frac{1}{10}$
22. According to which value of a, the three ſides in the nineteenth ſtep being reſolved, there will be produced the three ſides of the right-angled Triangle ſought, to wit, . . . $\frac{41}{30}$, $\frac{40}{30}$, $\frac{9}{30}$

I ſay $\frac{41}{30}$, $\frac{40}{30}$ and $\frac{9}{30}$ are the three ſides of a right-angled Triangle which will ſolve the Queſtion;

Queſtion; for the Area $\frac{6}{30}$ increaſed with $\frac{9}{30}$, (one of the ſides about the right-angle,) makes $\frac{15}{30}$, that is, $\frac{1}{2}$; as was required.

QUEST. 100.

To find a right-angled Triangle in rational numbers, that the Area ſubtracted from one of the ſides about the right-angle may leave a given number; let the given number be n.

RESOLUTION.

1. For the ſides about the right-angle of the Triangle ſought put . . . . . a and e
2. Then is the Area . . . . . $\frac{1}{2}ae$
3. Which ſubtracted from one of the ſides about the right-angle, ſuppoſe from a, leaves . . . . . $a - \frac{1}{2}ae = n$
4. Whence, after due Reduction, ... $e = \frac{a - n}{\frac{1}{2}a}$
5. And becauſe $aa + ee$ muſt be equal to the Square of the Hypothenuſal, it follows from the firſt and fourth ſteps, that the Square of the Hypothenuſal muſt be equal to the ſumm of the Squares of a and $\frac{a - n}{\frac{1}{2}a}$, that is, $\frac{\frac{1}{4}aaaa + aa - 2na + nn}{\frac{1}{4}aa}$
6. And ſince the Denominator $\frac{1}{4}aa$ is a Square, whoſe ſide is $\frac{1}{2}a$, it remains only to equate the Numerator $\frac{1}{4}aaaa + aa - 2na + nn$ to a Square, whoſe ſide may be feigned either $\frac{1}{2}aa - n$ or $\frac{1}{2}aa + n$; firſt then, let the ſide of the ſaid Square be feigned $\frac{1}{2}aa - n$, and then the Square of $\frac{1}{2}aa - n$ being equated to the ſaid $\frac{1}{4}aaaa + aa - 2na + nn$, this Equation ariſeth, *viz.*

$$\frac{1}{4}aaaa + aa - 2na + nn = \frac{1}{4}aaaa - naa + nn.$$

7. From which Equation the value of a will be made known, *viz.* . . . . . $a = \frac{2n}{1 + n}$
8. According to which value of a, the number e will be diſcovered from the fourth ſtep, *viz.* . . . . . $e = 1 - n$
9. And the ſquare Root of the ſumm of the Squares of $\frac{2n}{1 + n}$ and $1 - n$ ſhall be the Hypothenuſal, to wit, . . . $\therefore \frac{1 + nn}{1 + n}$
10. Therefore from the three laſt preceding ſteps, the three ſides of the right-angled Triangle ſought are theſe, to wit,

$$\frac{2n}{1 + n}\ ,\ 1 - n,\ \left(\text{ or } \frac{1 - nn}{1 + n}\ ,\right) \text{ and } \frac{1 + nn}{1 + n}.$$

11. But if inſtead of $\frac{1}{2}aa - n$, which in the ſixth ſtep was feigned for the ſide of a Square, we aſſume $\frac{1}{2}aa + n$, and equate the Square of this ſide to the before-mentioned $\frac{1}{4}aaaa + aa - 2na + nn$, there will ariſe $a = \frac{2n}{1 - n}$; according to which value, the ſides of the Triangle ſought will be found theſe, to wit,

$$\frac{2n}{1 - n}\ ,\ \frac{1 - nn}{1 - n}\ ,\ \frac{1 + nn}{1 - n}.$$

The two laſt ſteps give this

CANON.

12. When the given number is leſs than unity, let a right-angled Triangle be formed from unity and the given number; then divide the three ſides ſeverally by the ſumm or difference of unity and the given number, ſo ſhall the Quotients be the ſides of the Triangle ſought.

As, for example, if it be deſired to find out a right-angled Triangle, that the Area ſubtracted from one of the ſides about the right-angle may leave $\frac{4}{7}$, the Canon will diſcover the ſides of two Triangles, to wit, $\frac{63}{77}$, $\frac{22}{77}$, $\frac{56}{77}$ and $\frac{63}{21}$, $\frac{11}{21}$, $\frac{16}{21}$, each of which Triangles will ſatisfie your deſire; for in the firſt Triangle the Area $\frac{126}{77}$ ſubtracted from $\frac{56}{77}$, (one of the ſides about the right angle,) leaves the given number $\frac{4}{7}$; likewiſe in the latter Triangle, the Area $\frac{44}{21}$ ſubtracted from $\frac{16}{21}$ leaves $\frac{4}{7}$.

But how to ſolve this Queſtion when the given number is any number whatever, I ſhall hereafter ſhew by *Fermat*'s method, in *Queſt.* 130. of this Book.

QUEST. 101.

QUEST. 101. (Quæſt. 12. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that as well the difference of the ſides about the right-angle as the greater of the ſame ſides may be a Square; and that the Area, with the leſſer of the ſides about the right-angle may make a Square.

The Reſolution of this Queſtion depends upon three *Lemma's*, which I ſhall firſt explain.

LEMMA 1.

1. If a right-angled Triangle be formed from two numbers whereof the greater is the double of the leſſer, as well the difference of the ſides about the right-angle as the greater of the ſame ſides ſhall be a Square. Moreover, if the Area of the ſaid Triangle be multiplied by the Square of a Fraction having unity for its Numerator, and the leſſer of the two numbers by which the ſaid Triangle was formed for a Denominator, the Product increaſed with the leſſer of the ſides about the right-angle, will make a Square containing nine times the Square of the leſſer of the two numbers by which the ſaid Triangle was formed.

To make this manifeſt, let a right-angled Triangle be formed from a and $2a$, ſo the three ſides will be theſe, to wit, . . } $5aa$, $3aa$, $4aa$

Whence 'tis evident, firſt, that the difference of the ſides about the right-angle is a Square, to wit, aa; ſecondly, that the greater of the ſides about the right-angle is a Square, to wit, $4aa$; and laſtly, if the Area $6aaaa$ be multiplied by the Square of $\frac{1}{a}$, that is, by $\frac{1}{aa}$, the Product $6aa$ increaſed with $3aa$, (that is, the leſſer of the ſides about the right-angle,) makes the Square $9aa$, which contains nine times the Square of the leſſer of the two numbers by which the ſaid right-angled Triangle was formed. Which was to be ſhewn.

LEMMA 2.

2. Two numbers being given whoſe ſumm is a Square, to find innumerable Squares, every one of which being multiplied by one of the given numbers, and taking to the Product the other number, may make a Square.

Let there be two given numbers 6 and 3, and let it be deſired to find a Square, ſuch that if it be multiplied by 6, and 3 be added to the Product, the ſumm may make a Square.

Let the ſide of the Square ſought be . . . . . } . . . $a+1$
Whence the Square it ſelf is . . . . . } $aa+2a+1$
Which multiplied by 6 produceth . . . . . } $6aa+12a+6$
To which Product add 3 and it makes . . . . . } $6aa+12a+9$

Which $6aa+12a+9$ is to be equated to a Square, whoſe ſide, (becauſe the abſolute number 9 is a Square,) may be variouſly feigned; let it then be $3a-3$, the Square whereof, to wit, $9aa-18a+9$ being equated to the ſaid $6aa+12a+9$ will give $a=10$; wherefore $a+1$ the ſide of the Square ſought is 11, and the Square it ſelf is 121, which multiplied by 6 produceth 726, to which adding 3 it makes 729, which is a Square whoſe ſide is 27. It is alſo evident, that inſtead of 121 innumerable other Squares may be found out to perform the like effect, becauſe the ſide of the Square to be equated to $6aa+12a+9$ may be feigned infinitely.

In like manner if it were deſired to find a Square which multiplied into 3, and taking 6 to the Product, may make a Square; let the Square ſought be feigned $aa+2a+1$, this multiplied by 3 and the Product increaſed with 6, gives $3aa+6a+9$ to be equated to a Square whoſe ſide may be feigned $3a-3$, whence $a=4$, and therefore $a+1$ the ſide of the Square ſought is 5.

LEMMA 3.

3. If two numbers b and c whoſe ſumm makes not a Square be given, ſuch, that by multiplying one of them, ſuppoſe b, by a given Square dd, and by adding to the Product the other number c, the ſumm $bdd+c$ makes a Square; we may find innumerable other Squares inſtead of the given Square dd to produce the like effect.

Suppoſe . . . $\left\{\begin{array}{l} b=2, \\ c=8, \\ d=2, \\ dd=4; \end{array}\right\}$ whence $bdd+c=16$.

Now

Now it is required to find another Square inſtead of dd, or 4, that if the Square found out be multiplied by 2, to wit, b; and the Product be increaſed with 8, to wit, c, the the ſumm may be a Square.

For the ſide of the Square ſought put $a+$ the ſide of the given Square dd, viz. . . . . } . . . $a+2$
Whence the Square ſought is . . . . . } $aa+4a+4$
Which multiplied by 2, (to wit, b,) produceth . . . . . } $2aa+8a+8$
To which Product add 8, (to wit, c,) and the ſumm is . . } $2aa+8a+16$

Which ſumm is to be equated to a Square, the ſide whereof (in regard 16 is a Square) may be variouſly feigned, let it then be $2a-4$, the Square whereof being equated to the ſaid $2aa+8a+16$ will give $a=12$; wherefore $a+2=14$ the ſide of the Square ſought, and the Square it ſelf is 196, which multiplied by 2, and taking 8 to the Product makes the Square 400, whoſe ſide is 20. I ſhall now proceed to

The RESOLUTION of the preceding QUEST. 101.

4. Let a right-angled Triangle be formed from two ſuch numbers, that the greater may be the double of the leſſer, as from a and $2a$, ſo the three ſides will be theſe, to wit, ... } $5aa$, $3aa$, $4aa$

Whence it is evident that the two firſt parts of the Queſtion are ſatisfied, for as well the difference of the ſides about the right-angle as the greater of them is a Square. It remains that we ſee whether the Area of the Triangle, with the leſſer of the ſides about the right-angle makes a Square: But it makes $6aaaa+3aa$, and by dividing all by aa there ariſeth $6aa+3$ to be equated to a Square, which is poſſible to be done, becauſe the ſumm of 6 and 3 is a Square; for if a be 1, then aa is alſo 1, and conſequently $6aa+3$ makes the ſame Square as $6+3$, to wit, 9. Since then a is found 1, a right-angled Triangle formed from 1 and 2 (that is, a and $2a$) will ſolve the Queſtion. Wherefore the ſides of the Triangles ſought are 5, 3, 4.

5. Having found out one Square for the value of aa, to wit, 1, we may by the help of the preceding *Lemma* 2. find out innumerable other Squares to perform the ſame effect; ſo inſtead of 1, the Square 121 is found out in the Example of the ſaid *Lemma*. And therefore let a Triangle be formed from 11 and 22, and the ſides will be 605, 363, 484, which will ſolve the Queſtion; for the difference of the ſides about the right-angle is the Square 121, and the greater of the ſides about the right-angle, to wit, 484, is a Square whoſe ſide is 22; alſo the Area added to the leſſer ſide makes 88209, which is a Square whoſe ſide is 297.

6. But for removing an Obſtruction which the Learner may meet with, let a right-angled Triangle be formed from $2a$ and $4a$, whence the ſides are $20aa$, $12aa$ and $16aa$; where as well the difference of the ſides about the right-angle, as the greater of them is a Square. But the Area increaſed with the leſſer ſide makes $96aaaa+12aa$, and by dividing all by aa it makes $96aa+12$, which muſt be equated to a Square. But how ſhall that be done, ſince 96 which is multiplied into aa is not a Square, neither is the abſolute number 12 a Square, nor is the ſumm of 96 and 12 a Square? This knot, though it ſeems to be a very hard one, may yet by the help of the laſt clauſe of the preceding *Lemma* 1. be eaſily untied: For ſince 96 is the Area of a right-angled Triangle formed from 2 and 4, and 12 is the leſſer of the ſides about the right-angle, it will appear by *Lemma* 1. that if 96 be multiplied by $\frac{1}{4}$, and the Product 24 be increaſed with 12 it ſhall neceſſarily make a Square, to wit, 36; and conſequently by *Lemma* 3. we may find out innumerable Squares inſtead of $\frac{1}{4}$, every one of which being multiplied by 96, and taking 12 to the Product will make a Square. As, for example,

For the ſide of a Square to be found out inſtead of $\frac{1}{4}$, let there be put $a-$ the ſquare Root of $\frac{1}{4}$, *viz.* . . . . . } . . . $a-\frac{1}{2}$
Then the Square of $a-\frac{1}{2}$ is . . . . . } . . . $aa-a+\frac{1}{4}$
Which multiplied by 96 produceth . . . . . } $96aa-96a+24$
To which Product adding 12 it makes . . . . . } $96aa-96a+36$

Which $96aa-96a+36$ muſt be equated to a Square, (the ſide whereof in regard 36 is a Square,) may be variouſly feigned, let it then be $4a-6$, the Square whereof being equated to $96aa-96a+36$ will give $a=\frac{1}{5}$; therefore $a-\frac{1}{2}$, (that is, $\frac{6}{10}-\frac{1}{10}$) will give $\frac{1}{10}$ for the ſide of the Square ſought and the Square it ſelf is $\frac{1}{100}$, by which if you

multiply 96, and to the Product add 12, it makes the Square $1\frac{2}{25}$, whose side is $\frac{12}{5}$. Now forasmuch as the said side $\frac{1}{10}$ is to be taken for the value of a in the $2a$ and $4a$ by which the Triangle was first formed, let a Triangle be now formed from $\frac{1}{5}$ and $\frac{2}{5}$, and the three sides are $\frac{1}{5}$, $\frac{3}{25}$, $\frac{4}{25}$; where 'tis evident that the difference of the sides about the right-angle, as also the greater of them is a Square: But the Area is $\frac{6}{625}$, to which if you add the lesser side $\frac{3}{25}$, it makes the Square $\frac{81}{625}$, whose side is $\frac{9}{25}$.

QUEST. 102.

To find a right-angled Triangle, and a square number, such, that if the Square be multiplied by the lesser of the sides about the right-angle, and to that Product there be added the Product made by the multiplication of the Area of the said Triangle into the difference of the sides about the right-angle, the summ may be a Square. Moreover, that the greater of the sides about the right-angle may be a Square.

CANON.

1. By the preceding *Quest.* 101. find a right-angled Triangle, that as well the difference of the sides about the right-angle, as the greater of the same sides, may be a Square; Moreover, that the Area increased with the lesser of the said sides may make a Square; so shall such Triangle be that which is sought by this *Quest.* 102. and the difference of the sides about the right-angle shall be the first Square sought.

But that this Canon will solve the Question proposed, I demonstrate thus;

2. Suppose . . . . . $\left\{\begin{array}{l} h = \text{the Hypothenusal;} \\ b = \text{the greater side,} \\ p = \text{the lesser side,} \end{array}\right\}$ about the right-angle.

3. Suppose also that by the preceding *Quest.* 101. h, b, p are found such, that . . . . . $\left\{\begin{array}{l} b-p, \\ b, \\ \frac{1}{2}bp+p, \end{array}\right\}$ are three square numbers.

4. I say $b-p$ is such a square number, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product made by the multiplication of the Area $\frac{1}{2}bp$, into $b-p$ the difference of the sides about the right-angle, the summ shall be a Square: For,

5. The Product of $b-p$ into p is . . . . . $bp-pp$

6. To which if you add the Product of the Area into the difference of the sides about the right-angle, to wit, . . $\frac{1}{2}bbp-\frac{1}{2}bpp$

7. The summ is . . . . . $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$

8. Which summ is a square number, for it is equal to the Product of $b-p$ multiplied into $\frac{1}{2}bp+p$: But each of these Factors $b-p$ and $\frac{1}{2}bp+p$ is a Square by Construction; wherefore the Product of their multiplication, to wit, $bp-pp+\frac{1}{2}bbp-\frac{1}{2}bpp$ is a Square.

An Example in Numbers.

9. Take any right-angled Triangle which will solve the preceding *Quest.* 101. as, ... 5, 3, 4

In which Triangle the difference of the sides about the right-angle, to wit, 1, is such a Square that if it be multiplied by 3 the lesser side, and the Product be increased with 6, to wit, the Product of the Area multiplied by the difference of the sides about the right-angle it makes the Square 9. Moreover, the greater side about the right-angle, to wit, 4, is a Square; as was required.

10. But the same right-angled Triangle 5, 3, 4 being retained, we may instead of the Square 1, (to wit, the difference of the sides about the right angle) find out innumerable Squares, (by the help of *Lemma* 2. in *Quest.* 101.) every one of which shall solve this Question, and be within given limits if need require. So if it were desired to find out a Square greater than 6, and such as together with the said right-angled Triangle 5, 3, 4 may solve this 102d Question, the said *Lemma* 2. will discover the Squares 25 and 361, (among innumerable others,) which are such, that if each of them be multiplied by 3, (the lesser of the sides about the right-angle of the said Triangle,) and the Products 75 and 1083 be severally increased with 6, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Squares 81 and 1089, whose sides are 9 and 33.

11. And

11. And in like manner, by the help of any other right-angled Triangle which will solve the preceding *Quest.* 101. as the Triangle 605, 363, 484, we may find out innumerable Answers to this *Quest.* 102. For, first, the difference of the sides about the right-angle, to wit, 121 is a Square, and such, that if it be multiplied by the lesser side 363, and to the Product 43923 there be added 10629366, to wit, the Product of the Area multiplied into the difference of the sides about the right-angle, it makes the Square 10673289, whose side is 3267. And therefore by the help of the third *Lemma* which belongs to the Resolution of the preceding *Quest.* 101. you may instead of the Square 121 find out infinite other Squares to perform the same thing, and each Square shall be within given limits if need require.

QUEST. 103. (Quæst. 13. Lib. 6. *Diophant.*)

To find a right-angled Triangle, that the Area thereof being increased severally with each of the sides about the right-angle may make Squares.

RESOLUTION.

1. Let the Hypothenusal, Base and Perpendicular of some right-angled Triangle in numbers be represented by . . $\left\{\begin{array}{l} h \\ b \\ p \end{array}\right.$

2. Then multiply those sides severally by a, and suppose the Products to be the sides of the right-angled Triangle sought, to wit, ... $\left\{\begin{array}{l} ha \\ ba \\ pa \end{array}\right.$

3. Then, (according to the Question,) the Area of the Triangle in the last step being increased with each of the sides about the right-angle must make a Square, hence this Duplicate equality ariseth to be resolved, *viz.* . . $\left\{\begin{array}{l} \frac{1}{2}bpaa+ba = \square \\ \frac{1}{2}bpaa+pa = \square \end{array}\right.$

4. Now suppose . . . . . $\frac{1}{2}bpaa+ba = eeaa$

5. Whence, after due Reduction, you will find . . . . . $a = \frac{b}{ee-\frac{1}{2}bp}$

6. According to which value of a, the latter of the two quantities in the third step being resolved, instead of that quantity this that follows ariseth to be equated to a Square, *viz.* $\frac{bpee+\frac{1}{2}bbbp-\frac{1}{2}bbpp}{eeee-bpee+\frac{1}{4}bbpp} = \square$

7. But because the Denominator of the Fraction last exprest is a Square, whose side is $ee-\frac{1}{2}bp$, it remains only to equate the Numerator to a Square: And because a Square divided by a Square gives the Quotient a Square, therefore if we suppose b in the said Numerator $bpee+\frac{1}{2}bbbp-\frac{1}{2}bbpp$ to be a square number, then the said Numerator divided by b gives $pee+\frac{1}{2}bbp-\frac{1}{2}bpp$, that is, $pee, -\frac{1}{2}bp \times b-p$ to be equated to a Square. So that the matter is reduced to this, we must find out a right-angled Triangle, such, that the greater of the sides about the right-angle, suppose b, may be a square number; and we must also find another square number, suppose ee, that may be greater than the Area of the said Triangle, (as may be inferr'd from the Denominator of the Fraction in the fifth step,) and such, that if it be multiplied by p the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the sides about the right-angle, the summ may make a Square: But such a right-angled Triangle and Square, the preceding *Quest.* 102. shews how to find out. Suppose then the Hypothenusal, Base and Perpendicular of the said Triangle so found out to be h, b, p; and the Square to be ee, all which being known in numbers, the number represented by a shall consequently be known from the fifth step: And lastly, if you multiply the numbers h, b, p severally by the number a, the Products shall be the three sides of the Triangle sought. From the premisses there ariseth the following

CANON.

First, by the Canon of the preceding *Quest.* 102. find out a right-angled Triangle, (whose three sides in the Resolution of this *Quest.* 103. are represented by h, b, p;) and besides, a Square, call it ee, that may be greater than the Area of the said Triangle, and such that if it be multiplied by the lesser of the sides about the right-angle, and to the Product there be added the Product of the Area multiplied into the difference of the same sides, the summ may be a Square; then divide the greater of the sides about the right-angle of the said Triangle, by the excess of the said Square ee above the Area, and by the Quotient multiply severally the three sides h, b, p, so shall the Products be the three sides of the Triangle sought.

An Example in Numbers.

It hath been ſhewn in *Sect.* 10. of the preceding *Queſt.* 102. that the right-angled Triangle 5, 3, 4 and the Square 25 will ſolve that Queſtion; and beſides, the ſaid Square 25 is greater than 6 the Area of the ſaid Triangle, therefore according to the directions of the Canon above-delivered, I divide 4 the greater of the ſides about the right-angle, by 19 the exceſs of the ſaid Square 25 above 6 the Area of the ſaid Triangle, and the Quotient is $\frac{4}{19}$, by which I multiply ſeverally 5, 3, 4 (the ſides of the Triangle firſt found) and there comes forth $\frac{20}{19}$, $\frac{12}{19}$, $\frac{16}{19}$, which ſhall be the ſides of the Triangle ſought; for the Area is $\frac{240}{361}$, to which if we add ſeverally $\frac{12}{19}$ and $\frac{16}{19}$, (the ſides about the right-angle,) there will be made the Squares $\frac{144}{361}$ and $\frac{400}{361}$, whoſe ſides are $\frac{12}{19}$ and $\frac{20}{19}$.

After the ſame manner, the ſame right-angled Triangle 5, 3, 4 and the Square 361, (found out alſo by the preceding *Queſt.* 102.) will diſcover $\frac{220}{355}$, $\frac{132}{355}$ and $\frac{176}{355}$ for the three ſides of another right-angled Triangle to ſolve this Queſtion, as may eaſily be proved. And becauſe innumerable right-angled Triangles and Squares may be found out to ſolve the ſaid *Queſt.* 102. infinite Anſwers may be given to this.

QUEST. 104. (Quæſt. 15. Lib. 6. *Diophant.*)

To find a right-angled Triangle, ſuch, that if from its Area the Hypothenuſal and one of the ſides about the right-angle be ſeverally ſubtracted, each remainder may be a Square.

The Reſolution of this and the following 105th Queſtion depends upon a *Lemma*, which I ſhall here premiſe and demonſtrate.

LEMMA.

1. If a right-angled Triangle be formed from two ſquare numbers, or from two numbers in proportion one to another as a ſquare number to a ſquare number; I ſay firſt, the Square of the difference of thoſe two ſquare numbers being multiplied by the Product of their multiplication will produce a Square leſs than the Area of the ſaid Triangle; ſecondly, if from the ſolid Product made by the multiplication of theſe three numbers, to wit, the ſquare number above produced leſs than the Area, the Hypothenuſal, and that ſide about the right-angle which is the double Product of the multiplication of the two ſquare numbers forming the Triangle, there be ſubtracted the ſolid Product made of theſe three numbers, to wit, the Area, the ſaid ſide about the right-angle, and the exceſs by which the Hypothenuſal exceeds the ſame ſide, the remainder ſhall be a ſquare number; thirdly and laſtly, the ſumm of thoſe two ſolid Products ſhall alſo be a ſquare number.

Demonſtration.

2. Let a right-angled Triangle be formed from two ſquare numbers, ſuppoſe *bb* the greater, and *cc* the leſſer, whoſe ſides are *b* and *c*, ſo the three ſides of the ſaid Triangle will be theſe,

viz. $bbbb + cccc$ = the Hypothenuſal,
$bbbb - cccc$ = the Baſe,
$2bbcc$ = the Perpendicular.

3. The Area of the ſaid Triangle is . . . . . $b^6cc - bbc^6$
4. The Product of the multiplication of *bb* and *cc*, to wit, *bbcc*, being multiplied by the Square of *bb* — *cc*, produceth this Square, to wit, . . $b^6cc - 2b^4c^4 + bbc^6$
5. Which Square is leſs than the Area . . . . $b^6cc - bbc^6$
6. For it is evident that . . . . . $bbbc - bccc \supset bbbc + bccc$
7. Therefore by multiplying each part into *bbbc* — *bccc*, it neceſſarily follows that . . . . $b^6cc - 2b^4c^4 + bbc^6 \supset b^6cc - bbc^6$.

Which was affirmed in the firſt part of the *Lemma*.

8. In the next place, if theſe three following quantities be multiplied one into another;

viz. $b^6cc - 2b^4c^4 + bbc^6$, (the Square in the fourth ſtep,)
$b^4 + c^4$, (the Hypothenuſal,)
$2bbcc$, (the Perpendicular;)

The ſolid Product of their multiplication will be . . . . . $2b^{12}c^4 - 4b^{10}c^6 + 4b^8c^8 - 4b^6c^{10} + 2b^4c^{12}$.

9. And if theſe three following quantities be multiplied one into another, to wit,

$b^6cc - bbc^6$, (the Area,)
$+ 2bbcc$, (the Perpendicular,)
$bbbb + cccc - 2bbcc$, (the exceſs of the Hypothenuſal above the Perpendicular;)

The ſolid Product of their multiplication will be $2b^{12}c^4 - 4b^{10}c^6 - 2b^8c^{12} + 4b^6c^{10}$.

10. Which latter ſolid Product ſubtracted from the former, leaves this Square, to wit, ... $4b^8c^8 - 8b^6c^{10} + 4b^4c^{12}$.
The ſide whereof is . . . . . $2b^4c^4 - 2bbc^6$.

Wherefore the ſecond part of the *Lemma* is manifeſt.

11. Laſtly, the ſumm of the two ſolid Products in the eighth and ninth ſteps makes this Square, to wit, . . . $4b^{12}c^4 - 8b^{10}c^6 + 4b^8c^8$.
The ſide whereof is . . . . . $2b^6cc - 2b^4c^4$.

Wherefore the *Lemma* is every way proved.

An Example in Numbers.

Suppoſe $4 = bb$, and $1 = cc$; then let a right-angled Triangle be formed from 4 and 1, ſo the three ſides will be 17, 15, 8. Now I ſay, firſt, that if 9 the Square of the difference between the ſaid 4 and 1, be multiplied by 4 the Product of 4 and 1, there will be produced the ſquare number 36, which is leſs than 60 the Area of the ſaid Triangle.

Secondly, if from 4896, which is the ſolid Product made by the multiplication of theſe three numbers, to wit, 36 the Square before found, the Hypothenuſal 17, and 8 the double Product of 4 and 1; there be ſubtracted 4320, which is the ſolid Product of theſe three numbers, to wit, the Area 60; the ſaid double Product 8, and 9 the exceſs of the Hypothenuſal above the ſaid 8; there will remain the Square 576, whoſe ſide is 24.

Thirdly and laſtly, the ſumm of the ſaid ſolid Products 4896 and 4320 makes the Square 9216, whoſe ſide is 96.

Now followeth the RESOLUTION of QUEST. 104. *before propoſed.*

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . . h ; b ; p
2. Then multiply thoſe ſides ſeverally by a, and ſuppoſe the Products to be the ſides of the Triangle ſought, to wit, ha , ba , pa
3. Then, (according to the Queſtion,) if the Area of the Triangle in the laſt ſtep be leſſened by the Hypothenuſal and one of the ſides about the right-angle ſeverally, each remainder muſt be a Square; hence this Duplicate equality ariſeth to be reſolved, *viz.* . . . . . $\frac{1}{2}bpaa - ha = \square$ and $\frac{1}{2}bpaa - pa = \square$
4. Now in order to reſolve that Duplicate equality, ſuppoſe . . . . . $\frac{1}{2}bpaa - pa = eeaa$
5. Whence after due Reduction you will find . . . . . $a = \frac{p}{\frac{1}{2}bp - ee}$
6. According to which value of a, if the former of the two quantities in the third ſtep be reſolved, inſtead of that quantity, (to wit, $\frac{1}{2}bpaa - ha$) this that follows ariſeth to be equated to a Square, *viz.* . . . . . $\frac{hpee, - \frac{1}{2}bpp \times h - p}{\frac{1}{4}bbpp - bpee + eeee} = \square$
7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, for its ſide is $\frac{1}{2}bp - ee$, it remains only to equate the Numerator to a Square. We muſt therefore inquire into the riſe of the Numerator, ſo we ſhall find that it imports the ſearch of a right-angled Triangle h, b, p in numbers, and a ſquare number ee leſs than the Area of the ſaid Triangle; (for the Denominator of the Fraction in the fifth ſtep of the Reſolution ſhews that $\frac{1}{2}bp$ muſt exceed ee.) Moreover, the ſaid Triangle and Square muſt be ſuch, that if from the ſolid Product made by the multiplication of the ſaid Square, the Hypothenuſal, and one of the ſides about the right-angle, there be ſubtracted the ſolid Product made by the multiplication of the Area, the ſaid ſide about the right-angle, and the exceſs of the Hypothenuſal above the ſame ſide, the remainder may be a Square. But the laſt preceding *Lemma* ſhews how to find out ſuch a Triangle and Square. Suppoſe then the Hypothenuſal, Baſe and Perpendicular of the ſaid Triangle ſo found out, to wit, h, b, p, and the Square ee, to be all known in numbers, then the number repreſented by a ſhall conſequently be known from the fifth ſtep of the Reſolution; and laſtly, if you multiply the numbers h, b, p ſeverally by the number a, the Products ſhall be the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

CANON.

8. First, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication and reſerve the Product, which is a ſquare number, and may be called *ee*; that done, divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the exceſs of its Area above the Square *ee* before reſerved, and by the Quotient multiply ſeverally the three ſides of the ſame Triangle, ſo ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

Firſt, I form a right-angled Triangle from the ſquare numbers 4 and 1, ſo the three ſides are 17, 15, 8; then I multiply 9 (the Square of the difference between 4 and 1) by 4, the Product of 4 into 1, and there comes forth the Square 36, (to wit, *ee*;) then I divide 8, (to wit, that ſide about the right-angle of the ſaid Triangle which is the double Product of 4 into 1,) by 24, which is the exceſs of 60 the Area of the ſaid Triangle above the Square 36, (to wit, *ee*,) and the Quotient is $\frac{1}{3}$, by which I multiply ſeverally 17, 15, 8, (the ſides of the right-angled Triangle firſt found,) and the Products $\frac{17}{3}$, $\frac{15}{3}$ and $\frac{8}{3}$ are the ſides of the right-angled Triangle ſought: For, if from the Area $\frac{20}{3}$, the Hypothenuſal $\frac{17}{3}$, and the ſide $\frac{8}{3}$ be ſeverally ſubtracted, there will remain the Squares 1 and 4.

This Queſtion is capable of innumerable Anſwers in a double reſpect; for firſt, inſtead of 4 and 1 we may take any two ſquare numbers, or any two numbers which are in ſuch proportion to one another as a ſquare number to a ſquare number, for the forming of a right-angled Triangle as the Canon directs: ſecondly, the ſame right-angled Triangle 17, 15, 8 being retained, we may inſtead of the Square 36, to wit, *ee*, find infinite others, every one of which ſhall be leſs than the Area 60; and ſuch, that if it be multiplied into 136, (to wit, *bp*,) and from the Product there be ſubtracted 4320, (to wit, $\frac{1}{2}bpp \times \overline{h-p}$,) the remainder ſhall be a ſquare number. (The finding out of which Squares may eaſily be deduced from *Lemma* 3. in the preceding *Queſt.* 101.)

QUEST. 105.

To find a right-angled Triangle, that if its Area be ſubtracted from the Hypothenuſal, and from one of the ſides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle in numbers be repreſented by . . . h ; b ; p
2. Then multiply thoſe ſides ſeverally by a, and aſſume the Products to be the ſides of the right-angled Triangle ſought, to wit, ... ha , ba ; pa
3. Then, (according to the Queſtion,) each of theſe two quantities muſt be equated to a Square, *viz.* . . . $ha - \frac{1}{2}bpaa = \square$; $pa - \frac{1}{2}bpaa = \square$
4. Now in order to reſolve that Duplicate equality, ſuppoſe $pa - \frac{1}{2}bpaa = eeaa$
5. Whence after due Reduction you will find . . . . . $a = \frac{p}{ee + \frac{1}{2}bp}$
6. According to which value of a, if the former of the two quantities in the third ſtep be reſolved, inſtead of that quantity, (to wit, $ha - \frac{1}{2}bpaa$,) this that follows ariſeth to be equated to a Square, *viz.* . . . . . $\frac{hpee, + \frac{1}{2}bpp \times \overline{h-p}}{eeee + bpee + \frac{1}{4}bbpp} = \square$
7. But becauſe the Denominator of the Fraction laſt expreſt is a Square, for its ſide is $ee + \frac{1}{2}bp$, it remains only to equate the Numerator to a Square; and the Numerator well examined, ſhews that we muſt find a right-angled Triangle h, b, p in numbers, and a ſquare number *ee*, ſuch, that if to the ſolid Product made by the multiplication of the ſaid Square, the Hypothenuſal and one of the ſides about the right-angle, there be added the ſolid Product made by the multiplication of the Area, the ſaid ſide about the right-angle, and the exceſs of the Hypothenuſal above the ſame ſide, the ſumm may be a Square: But the *Lemma* prefixt to the Reſolution of the preceding *Queſt.* 104. ſhews how to find out ſuch a Triangle and Square: Suppoſe then the Hypothenuſal, Baſe

Baſe and Perpendicular of the ſaid Triangle ſo found out, to wit, h, b, p, and the Square *ee* to be all known in numbers, then conſequently the number a ſhall be known alſo from the fifth ſtep of the Reſolution: And laſtly, the numbers h, b, p being multiplied ſeverally by the number a will give the three ſides of the Triangle ſought. From the premiſſes there ariſeth the following

CANON.

8. Firſt, let a right-angled Triangle be formed from two ſquare numbers, or from two numbers which have ſuch proportion to one another as a ſquare number to a ſquare number; then multiply the Square of the difference of thoſe two numbers by the Product of their multiplication, and reſerve the Product, which is a ſquare number, and may be called *ee*; then divide that ſide about the right-angle which is the double Product of the two numbers that formed the ſaid Triangle, by the ſumm of its Area and the Square *ee* before reſerved; laſtly, by the Quotient multiply ſeverally the three ſides of the Triangle firſt formed: So ſhall the Products be the three ſides of the right-angled Triangle ſought.

An Example in Numbers.

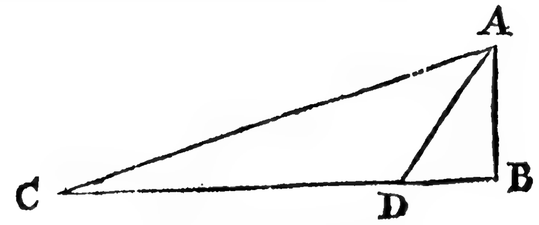
Firſt, a right-angled Triangle being formed from the ſquare numbers 4 and 1, the three ſides will be 17, 15, 8; then I multiply 9 the Square of the difference between 4 and 1, by 4 the Product of 4 into 1, and it produceth the Square 36, (to wit, *ee*;) then I divide the ſide 8, by 96 which is the ſumm of the Area 60 and the Square 36, (to wit, *ee*,) ſo the Quotient is $\frac{1}{12}$; laſtly, the ſides 17, 15, 8 being multiplied ſeverally by $\frac{1}{12}$ will give $\frac{17}{12}$, $\frac{15}{12}$ and $\frac{8}{12}$ for the right-angled Triangle ſought. For if from the Hypothenuſal $\frac{17}{12}$, and from the ſide $\frac{8}{12}$, the Area $\frac{5}{12}$ be ſubtracted, there will remain the Squares 1 and $\frac{1}{4}$.

After the ſame manner, among innumerable right-angled Triangles that might be found out by the ſaid Canon to ſolve this Queſtion, theſe three will be diſcovered, to wit, $\frac{41}{36}, \frac{40}{36}, \frac{9}{36}$ | $\frac{221}{845}, \frac{645}{845}, \frac{732}{845}$ | $\frac{2337}{2340}, \frac{2335}{2340}, \frac{1332}{2340}$; every one of which Triangles beſides that firſt found, to wit, $\frac{17}{12}, \frac{15}{12}, \frac{8}{12}$ is expreſt by ſmaller numbers than that Triangle found out by *Fermat*'s method, in the following *Queſt.* 127.

This Queſtion is alſo capable of innumerable Anſwers upon another ground, as may eaſily be collected from what hath been ſaid at the latter end of the preceding *Queſt.* 104.

QUEST. 106. (Quæſt. 18. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, ſuppoſe A B C right-angled at B, that one of its acute-angles B A C being cut into two equal parts by the line A D, the ſaid line A D may be alſo expreſſible by a rational number.



RESOLUTION.

1. Let the Hypothenuſal, Baſe and Perpendicular of ſome right-angled Triangle known in numbers be repreſented by . . . h , b , p
2. Then multiply thoſe ſides ſeverally by a, (which repreſents a number unknown,) and put . . . . . $ha = AD$; $ba = BD$; $pa = AB$; $b - ba = DC$
3. Now foraſmuch as (*per* 3. *Prop.* 6. *Elem.* Euclid.) theſe lines in the preceding Diagram are Proportionals, *viz.* . . . BD . AB :: DC . CA
4. Therefore from the premiſſes theſe numbers ſhall be alſo Proportionals, *viz.* . . . . . $ba . pa :: b - ba . p - pa$
5. And becauſe (*per* 47. *Prop.* 1. *Elem.* Euclid.) the Square of A B together with the Square of B C is equal to the Square of C A; therefore the Square of pa together with the

the Square of b, (for $ba + b - ba = b$,) ſhall be equal to the Square of $p - pa$, whence this Equation ariſeth, to wit,

$$ppaa + bb = ppaa - 2ppa + pp.$$

6. Which Equation, after due Reduction, gives . . . . . $a = \frac{pp - bb}{2pp}$

7. Therefore from the firſt, ſecond and ſixth ſteps, all the lines ſought ſhall now be known in rational numbers, *viz.* . .
$$\frac{ppp - pbb}{2pp} = AB$$
$$\frac{2bpp}{2pp} = BC$$
$$\frac{ppp + pbb}{2pp} = CA$$
$$\frac{bpp - bbb}{2pp} = AD$$
$$\frac{bpp - bbb}{2pp} = BD$$
$$\frac{bpp + bbb}{2pp} = DC$$

8. And by multiplying all the Fractions in the laſt ſtep by the common Denominator $2pp$, the Products will give theſe numbers which will alſo ſolve the Queſtion, and may ſerve as a Canon for that purpoſe, *viz.* . . . . . .
$$ppp - pbb = AB$$
$$2bpp = BC$$
$$ppp + pbb = CA$$
$$bpp - bbb = AD$$
$$bpp - bbb = BD$$
$$bpp + bbb = DC$$

An Example in Numbers.

Take any right-angled Triangle in numbers, as 5, 3, 4, then by putting $5 = h$, $3 = b$ and $4 = p$, (the greater of the ſides about the right-angle,) you will find

$ppp - pbb = 28 = AB$ the Perpendicular,
$2bpp = 96 = BC$ the Baſe,
$ppp + pbb = 100 = CA$ the Hypothenuſal,
$bpp - bbb = 35 = AD$ the line biſecting the acute-angle oppoſite to the Baſe,
$bpp - bbb = 21 = BD$, $bpp + bbb = 75 = DC$ the ſegments of the Baſe.

The Proof.

Firſt, theſe numbers are Proportionals, $\begin{cases} 21 . 28 :: 75 . 100 \\ BD . AB :: DC . CA \end{cases}$

Therefore (*per Prop.* 3. *Elem.* 6. Euclid.) the angle BAD is equal to the angle DAC. The reſt of the Proof is obvious.

QUEST. 107.

To find out an oblique-angled Triangle, whoſe three ſides, as alſo the Perpendicular, and a line cutting one of the angles into two equal parts may be expreſt ſeverally by rational numbers.

[Jac. de Billy *in* probl. 5. cap. 4. *of the latter part of his* Diophant. redivi. *printed at* Lyons *in* 1670. *reſolves this Queſtion briefly by Numeral* Algebra; *but to the end a Canon may be raiſed to ſolve the Queſtion propoſed, I ſhall form the Reſolution thereof by Literal* Algebra.]

Preparation.

Let there be an oblique-angled Triangle ABC; then ſuppoſing AD to be perpendicular to the Baſe BC, and the line AE to cut the angle BAC into two equal angles EAB

EAB and EAC; let it be required to find out rational numbers to expreſs the quantities of the ſides AB, BC, AC, as alſo of AD, AE, BE, ED, DC.



RESOLUTION.

1. Firſt, let b, p, h repreſent the Baſe, Perpendicular and Hypothenuſal of any right-angled Triangle known in numbers, and ſuppoſe the Perpendicular p to be greater than the Baſe b, then

Put . . . . . $\begin{cases} b = DE \text{ the Baſe,} \\ p = AD \text{ the Perp.} \\ h = AE \text{ the Hyp.} \end{cases}$ of $\triangle$ ADE.

2. Secondly, making p, (that is, AD) to be the Perpendicular of a ſecond right-angled Triangle, ſuppoſe ADB, find out the Baſe DB and the Hypothenuſal BA in rational numbers, which may be done thus, *viz.* Foraſmuch as the Square of the Perpendicular AD is equal to the difference of the Squares of the Hypothenuſal BA and the Baſe DB, let the Square of the given number p be eſteem'd to be the difference of two ſquare numbers, and find out the Squares themſelves, then put the ſide of the leſſer Square for DB, and the ſide of the greater for BA; but the ſaid Squares muſt be found out with this Caution, that DB the ſide of the leſſer Square may have greater proportion to DE, (that is, b) than $2pp$ hath to $pp - bb$; as may eaſily be inferr'd from the Canon of the preceding *Queſt.* 106. where it appears, that when the angle DAC in the Diagram belonging to that Queſtion is equal to the angle DAB, then the Baſe BC hath ſuch proportion to BD, as $2pp$ to $pp - bb$; but in the Diagram of of this Queſtion the angle EAB muſt be greater than the angle EAD, in regard by ſuppoſition the angle EAB is equal to the angle EAC. Now ſince by *Queſt.* 7. of this *Book*, innumerable pairs of Squares having one common difference may be found out, ſuch, that the ſide of one Square of each pair ſhall be greater or leſs than a given number, let us ſuppoſe the ſides of two ſquare numbers to be found out agreeable to the ſaid Caution,

viz. $\begin{cases} b + d = DE + EB = DB, \\ d = EB, \\ g = BA. \end{cases}$

3. Thirdly, the next ſcope is to find out EC and AC in rational numbers, which muſt have the ſame proportion one to another as EB and BA; (for by ſuppoſition the angle EAB is equal to the angle EAC, and therefore (*per Prop.* 3. *Elem.* 6. Euclid.) EB . BA :: EC . AC.) Moreover, the Square of $\overline{EC - ED}$, that is, of DC, together with the Square of AD muſt be equal to the Square of AC. But EB and BA were before found out in numbers, to wit, $d = EB$, and $g = BA$; now to find out two numbers in the ſame proportion as d and g, multiply theſe ſeverally by a (which repreſents a number yet unknown,) and put the Products da and ga for EC and AC; whence theſe are Proportionals,

viz. $\begin{cases} d . g :: da . ga, \\ EB . BA :: EC . AC. \end{cases}$

4. And becauſe the Square of $\overline{EC - ED}$, that is, the Square of DC, together with the Square of AD makes the Square of AC, therefore in the letters of the Reſolution, the Square of $da - b$ together with the Square of p muſt be equal to the Square of ga; hence this Equation ariſeth, *viz.*

$$ddaa - 2bda + bb + pp = ggaa.$$

5. And becauſe (by the firſt ſtep of the Reſolution,) . . . . . $hh = bb + pp$
6. Therefore . . . . . $ddaa - 2bda + hh = ggaa$
7. Whence, after due Reduction, this following Equation ariſeth, *viz.* . . . . . $aa + \frac{2bd}{gg - dd}a = \frac{hh}{gg - dd}$

8. Which laſt Equation being reſolved by the Canon in *Sect. 6. Chap. 15. Book 1.* the value of a will be made known, *viz.*

$$a = \sqrt{\frac{bbdd + gghh - ddhh}{gg - dd \text{ into } gg - dd}} : - \frac{bd}{gg - dd}.$$

9. Now if the Numerator $bbdd + gghh - ddhh$ be a rational ſquare number, then the value of a is manifeſtly rational, for the Denominator is a Square, whoſe ſide is $gg - dd$; but that the ſaid Numerator is a rational Square I prove thus: In the Triangle B A E obtuſe-angled at E, the Square of B A leſs by the ſumm of the Square of B E and twice the rectangle made of B E and E D is equal to the Square of A E, (*per* 12. *prop.* 2. *Elem.* Euclid.) therefore in the letters of the Reſolution,

$$gg - dd - 2db = hh.$$

10. But if $gg - dd - 2db$ inſtead of hh be multiplied into $gg - dd$, and to the Product there be added $bbdd$, then inſtead of the aforeſaid Numerator $bbdd + gghh - ddhh$, this following Square will ariſe, to wit,

$$gggg - 2ggdd - 2ggdb + dddd + 2ddbb + ddbb;$$

Whoſe ſide is . . . . . $> gg - dd - db.$

11. Therefore from the eighth and tenth ſteps, the value of a is expreſſible by a rational number, *viz.* $\left\{ a = \frac{gg - dd - 2db}{gg - dd} \right.$

12. Therefore from the third and eleventh ſteps, . . $\left\{ \frac{dgg - ddd - 2ddb}{gg - dd} = \text{EC} \right.$

13. And from the third and eleventh ſteps, . . . $\left\{ \frac{ggg - gdd - 2gdb}{gg - dd} = \text{AC} \right.$

14. And by ſubtracting b from the quantity in the twelfth ſtep, (*viz.* E D from E C,) it gives . $\left\{ \frac{dgg - ddd - bgg - bdd}{gg - dd} = \text{DC} \right.$

15. Laſtly, if the three Fractions in the three laſt preceding ſteps, as alſo b, p, h, d, g, be ſeverally multiplied by the Denominator $gg - dd$, there will come forth the following quantities in Integers, which may ſerve as a Canon to ſolve the Queſtion propoſed; provided that the numbers b, p, h, d, g be firſt found out agreeable to the Caution before preſcribed.

CANON.

$$\left.\begin{array}{rcl} ggd - ddd - 2ddb &=& \text{EC} \\ ggd - ddd &=& \text{EB} \\ 2ggd - 2ddd - 2ddb &=& \text{BC} \\ ggg - gdd &=& \text{BA} \\ ggg - gdd - 2gdb &=& \text{AC} \\ ggp - ddp &=& \text{AD} \\ ggh - ddh &=& \text{AE} \\ ggb - ddb &=& \text{DE} \\ ggd - ddd - ggb - bdd &=& \text{DC} \\ ggd + ggb - ddd - ddb &=& \text{DB} \end{array}\right\} \text{in the Diagram belonging to this Queſtion.}$$

An Example in Numbers.

Firſt, take any right-angled Triangle in whole numbers, as the Triangle 18, 24, 30; then

$$\text{Put} \ldots \left\{ \begin{array}{l} b = 18 \ \text{ED}, \\ p = 24 \ \text{AD}, \\ h = 30 \ \text{AE}. \end{array} \right.$$

Secondly, making 24, (to wit, $p = \text{AD}$) to be the Perpendicular of a ſecond right-angled Triangle, as well as of the firſt A D E, find out the Baſe D B, and the Hypothenuſal A B in rational numbers; but for the reaſon before given, the number of the Baſe D B muſt have greater proportion to the number of the Baſe D E, than $2pp$ to $pp - bb$, *viz.* in this Example, greater proportion than 32 to 7, and conſequently D B muſt exceed D E taken $4\frac{4}{7}$ times: But by ſuppoſition D E = 18; therefore D B muſt exceed $82\frac{2}{7}$. Now becauſe the Square of the Perpendicular A D is equal to the difference of the Squares of A B and D B, therefore 576 the Square of the Perpendicular 24 (= A D) being taken for the difference of two Squares, find out the Squares ſeverally, but with this condition, that the ſide of the leſſer of them may exceed $82\frac{2}{7}$: But two ſuch Squares (among innumerable other pairs of Squares that may be found out by the ſeventh Queſtion of

of this Book,) are 20449 and 21025, whoſe ſides are 143 and 145; therefore 143 = D B, | 145 = A B, and 125 = E B, (that is, D B — D E;) then

$$\text{Put} \ldots \left\{ \begin{array}{l} d = 125 = \text{EB}, \\ g = 145 = \text{BA}. \end{array} \right.$$

Laſtly, by the help of the numbers before found out for the values of b, p, h, d and g, the preceding Canon will diſcover rational numbers, which reduced to their leaſt terms by their greateſt common Diviſor, will give the whole numbers here-under expreſt, for the meaſures of the ſides of the oblique-angled Triangle ſought; as alſo of the Perpendicular, and the line cutting the angle oppoſite to the Baſe into two equal parts, and of the ſegments of the Baſe made as well by the Perpendicular as by the line biſecting the ſaid angle, *viz.*

$$\left.\begin{array}{rcl} \text{EC} &=& 125 \\ \text{EB} &=& 750 \\ \text{BC} &=& 875 \\ \text{BA} &=& 870 \\ \text{AC} &=& 145 \\ \text{AD} &=& 144 \\ \text{AE} &=& 180 \\ \text{DE} &=& 108 \\ \text{DC} &=& 17 \\ \text{DB} &=& 858 \end{array}\right\} \text{agreeable to the Diagram and Canon belonging to this Queſtion.}$$

The Proof.

Firſt, theſe numbers are Proportionals, *viz* $\left\{ \begin{array}{l} 750 . 870 :: 125 . 145 \\ \text{EB} . \text{BA} :: \text{EC} . \text{AC} \end{array} \right.$

Therefore, (*per* 3. *prop.* 6. *Elem.* Euclid.) the angle E A B is equal to the angle E A C. The reſt of the Proof is obvious.

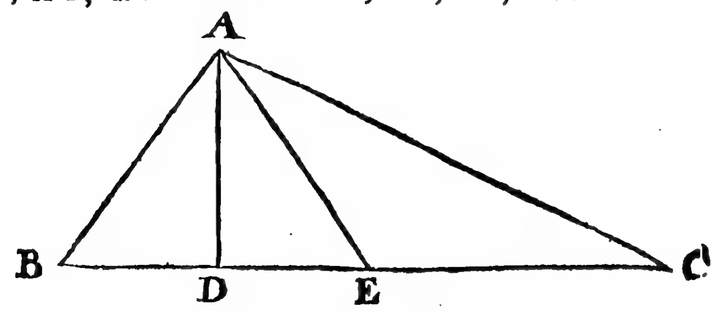
QUEST. 108.

To find out an oblique-angled Triangle, whoſe three ſides, as alſo the Perpendicular, and a line drawn from the angle oppoſite to the Baſe, and cutting the Baſe into two equal parts, may be expreſt ſeverally by rational numbers.

[Jac. de Billy *in his Appendix to the Problem cited in the preceding* 107th *Queſtion reſolves this alſo, but very briefly; I ſhall therefore form the Reſolution thereof at large by Numeral Algebra, by the ſteps whereof the more curious Analyſt may eaſily reſolve it by Specious Algebra, but the Canon thence ariſing will be exceeding tedious.*]

Preparation.

Let there be an oblique-angled Triangle A B C; then ſuppoſing A D to be perpendicular to the Baſe B C, and the line A E to cut the Baſe B C into two equal parts in the point E, let it be required to find out rational numbers to expreſs the quantities of the ſides A B, B C, A C; as alſo of the lines A D, A E, B D, D E.



RESOLUTION.

1. Firſt, take any right-angled Triangle in numbers, as 3, 4, 5; then,

$$\text{Put} \ldots \left\{ \begin{array}{l} 3 = \text{BD the Baſe}, \\ 4 = \text{AD the Perpendicular}, \\ 5 = \text{AB the Hypothenuſal}, \end{array} \right\} \text{of the right-angled Triangle BDA.}$$

2. Then for the diſtance between the foot of the Perpendicular and the middle of the Baſe B C, put a, *viz.* ſuppoſe . . $\left\{ a = \text{DE} \right.$

3. And becauſe by ſuppoſition, BE = EC, if to a you add 3, (to wit, BD,) the ſumm ſhall be equal to half the Baſe, *viz.* $a + 3 = BE = EC$

4. And becauſe DE + EC = DC, the ſumm of the two Equations in the ſecond and third ſteps ſhall be equal to the greater ſegment of the Baſe made by the Perpendicular, *viz.* $2a + 3 = DC$

5. And becauſe the Square of DC together with the Square of AD is equal to the Square of AC, the Square of $2a + 3$ together with the Square of 4 muſt make a Square, *viz.*

$$4aa + 12a + 25 = \square.$$

6. And becauſe the Square of DE together with the Square of AD is equal to the Square of AE, the Square of a together with the Square of 4 muſt be equated to a Square, *viz.*

$$aa + 16 = \square.$$

7. So in the two laſt ſteps we are faln upon a Duplicate equality, which, in regard the Squares 25 and 16 are unequal, I reduce to another that ſhall have equal known Squares, *viz.* (after the manner uſed in divers preceding Queſtions of this Book,) I divide the greater Square 25 by the leſſer 16, and by the Quotient $\frac{25}{16}$ I multiply the quantity in the ſixth ſtep, to wit, $aa + 16$, ſo the Product $\frac{25}{16}aa + 25$ is to be equated to a Square, and therefore this Duplicate equality comes now to be reſolved,

viz. $\begin{cases} 4aa + 12a + 25 = \square \\ \frac{25}{16}aa + 25 = \square \end{cases}$

8. Now in order to reſolve the Duplicate equality laſt expreſt, firſt, by ſubtracting the leſſer quantity from the greater, I find their difference to be

$$\frac{25}{16}aa + 12a.$$

9. Then I ſearch out two quantities, the Product of whoſe multiplication may make the ſaid difference $\frac{25}{16}aa + 12a$; and that as well in the difference as in the ſumm of the ſame quantities there may be found 10, (to wit, the double of the ſide of the Square 25,) ſo I find thoſe two quantities to be

$$\frac{25}{32}a + 10 \text{ and } \frac{6}{5}a.$$

10. Then the Square of half the difference of the two quantities laſt expreſt, *viz.* the Square of $\frac{1225}{3200}a + 5$ being equated to $\frac{25}{16}aa + 25$, gives

$$\frac{25}{16}aa + 25 = \frac{1225}{1024}aa + \frac{125}{32}a + 25.$$

11. Which Equation, after due Reduction, will diſcover the number a, *viz.* . . . . . $a = \frac{4000}{1225} = DE$

12. Then by adding the Square of the ſaid number a to 16, and extracting the ſquare Root of the ſumm, there ariſeth $\frac{2102}{1225} = AE$

13. And by adding 3 (to wit, BD) to the number a, (to wit, DE,) the ſumm ſhall be the meaſure of half the Baſe BC, *viz.* . . . . . $\frac{2327}{1225} = BE = EC$

14. Therefore the double of the number in the laſt ſtep is the meaſure of the Baſe BC, *viz.* . . . . . $\frac{1740}{1225} = BC$

15. And by adding the number a in the eleventh ſtep to half the Baſe in the fourteenth, the ſumm is the greater ſegment of the Baſe made by the Perpendicular, *viz.* . . . . $\frac{2327}{1225} = DC$

16. Then by adding the Square of the ſaid greater ſegment DC to 16 the Square of the Perpendicular AD, and extracting the ſquare Root of the ſumm, there comes forth $\frac{1393}{1225} = AC$

17. Laſtly, by multiplying the numbers in the firſt, eleventh, twelfth, thirteenth, fourteenth, fifteenth and ſixteenth ſteps ſeverally by the Denominator 142311, there will come forth theſe following whole numbers for the meaſures of the lines ſought, *viz.*

426933	= BD	in the Diagram belonging to this Queſtion.
569244	= AD	
711555	= AB	
425600	= DE	
710756	= AE	
852533	= BE = EC	
1705066	= BC	
1278133	= DC	
1399165	= AC	

The

The Proof is eaſie to be made, by comparing the ſumm of the Squares of the numbers anſwering to the ſides about the right-angle of every right-angled Triangle in the Diagram, to the number anſwering to the Square of the Hypothenuſal of ſuch Triangle reſpectively.

QUEST. 109. (Quæſt. 21. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Area thereof increaſed with one of the ſides about the right-angle may make a Square; and that the ſumm of all the three ſides may be a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and $a + 1$; then divide the three ſides ſeverally by $a + 1$, and take the Quotients for the ſides of the Triangle ſought, *viz.* . . . . . $\frac{2aa + 2a + 1}{a + 1}$ = the Hypoth. $\frac{2aa + 2a}{a + 1}$ = the Baſe, $\frac{2a + 1}{a + 1}$ = the Perpend.

2. The ſumm of thoſe three ſides is . . . . . $\frac{4aa + 6a + 2}{a + 1}$

3. Which ſumm reduced to its leaſt terms (by dividing the Numerator by the Denominator according to the general method of Diviſion in *Sect. 9. Chap. 5. Book 1.*) will be . . . . . $4a + 2$

Therefore (according to the Queſtion) the ſaid $4a + 2$ muſt be equal to a Cube, which in the following ninth ſtep I ſhall ſhew how to find out.

4. Moreover, the Area together with one of the ſides about the right-angle of the Triangle in the firſt ſtep muſt make a Square: But the Area (by multiplying half the Baſe into the Perpendicular) will be found $\frac{2aaa + 3aa + a}{aa + 2a + 1}$

5. And one of the ſides about the right-angle (to wit, the Perpendicular) is . . . . . $\frac{2a + 1}{a + 1}$

6. Which ſide reduced to the ſame Denominator with the Area, (by multiplying as well the Numerator $2a + 1$, as the Denominator $a + 1$, by $a + 1$,) will be $\frac{2aa + 3a + 1}{aa + 2a + 1}$

7. Now if the ſide in the laſt ſtep be added to the Area in the fourth, the ſumm will be . . . . $\frac{2aaa + 5aa + 4a + 1}{aa + 2a + 1}$

8. Which ſumm reduced to its leaſt terms, (by dividing the Numerator by the Denominator) will be . . $2a + 1$

9. Therefore (according to the Queſtion) $2a + 1$ muſt be equated to a Square, and before in the third ſtep it was found that $4a + 2$ muſt be equated to a Cube; now becauſe $4a + 2$ is the double of $2a + 1$, we muſt find out a Cube that may be the double of a Square; but the Cube 8 is the double of the Square 4, therefore let $4a + 2$ be equated to 8, or $2a + 1$ to 4; whence you will find $a = \frac{3}{4}$, and conſequently $a + 1 = \frac{7}{4}$. Wherefore according to the Poſitions in the firſt ſtep, let a right-angled Triangle be formed from $\frac{7}{4}$ and $\frac{1}{4}$, and divide the three ſides ſeverally by $\frac{1}{4}$, ſo there will come forth the ſides of the right-angled Triangle ſought, to wit, $\frac{17}{4}, \frac{15}{4}, \frac{8}{4}$, which ſolve the Queſtion; for the Area $\frac{15}{4}$ together with $\frac{8}{4}$ (to wit, one of the ſides about the right-angle) makes the Square 4; and the ſumm of all the three ſides makes a Cube, to wit, 8.

It is alſo evident by the premiſſes, that the Queſtion is capable of innumerable Anſwers; in regard you may find out as often as you pleaſe a Cube and a Square, ſuch, that the former ſhall be the double of the latter: As, by equating aaa to $8aa$, it will give the Cube 512, and the Square 256, the former of which is the double of the latter.

QUEST. 110. (Quæſt. 23. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be a Square; and that the ſaid ſumm increaſed with the Area may make a Cube.

RESOLUTION.

1. Let a right-angled Triangle be formed from a and 1, ſo the three ſides will be theſe, to wit, ... $aa + 1$, $aa - 1$, $2a$

2. The

2. The ſumm of all the three ſides is $2aa + 2a$, which muſt be equal to a Square, let it therefore be equated to $eeaa$, *viz.* ſuppoſe . . . . . $2aa + 2a = eeaa$

3. Whence, after due Reduction, you will find . . . . $a = \frac{2}{ee-2}$

4. Therefore by ſquaring the laſt Equation, there ariſeth . $aa = \frac{4}{eeee-4ee+4}$

5. Which laſt Equation doubled, is . . . . . $2aa = \frac{8}{eeee-4ee+4}$

6. And by adding the double of the Equation in the third ſtep, to wit, $2a = \frac{4}{ee-2}$ to the Equation in the fifth ſtep, the ſumm gives this Equation, to wit,

$$2aa + 2a = \frac{4ee}{eeee-4ee+4}.$$

7. The latter part of which Equation is manifeſtly a Square, and by Conſtruction 'tis equal to the ſumm of all the three ſides of the Triangle in the firſt ſtep. It remains that the ſaid ſumm with the Area make a Cube; but from the firſt ſtep the Area is $aaa - a$, and according to the value of a in the third ſtep, the Area $aaa - a$ will be reduced to this, to wit, $\frac{8ee-2eeee}{eeeeee-6eeee+12ee-8}$.

8. To which add the ſumm of all the three ſides, to wit, the latter part of the Equation in the ſixth ſtep, that is, $\frac{4ee}{eeee-4ee+4}$, (which, by multiplying the Numerator and Denominator ſeverally by $ee-2$, will be reduced to $\frac{4eeee-8ee}{eeeeee-6eeee+12ee-8}$,) and there will come forth this ſumm, to wit, $\frac{2eeee}{eeeeee-6eeee+12ee-8}$.

9. Which ſumm laſt produced muſt be equal to a Cube; and becauſe by Conſtruction the Denominator is a Cube, to wit, the Cube of $ee-2$, it remains that we equate the Numerator $2eeee$ to ſome Cube; or by dividing $2eeee$ by eee it gives $2e$ to be equated to a Cube, which is eaſie to be done, for we may put $2e$ equal to any known cube-number, as ddd;

10. Suppoſe therefore . . . . . $2e = ddd$

11. Then becauſe the Denominator of the Fraction in the third ſtep ſhews that . . . . . $e \sqsubset \sqrt{2}$

12. And conſequently by doubling each part, ... $2e \sqsubset \sqrt{8}$

13. It follows from the tenth and twelfth ſteps, that . . $ddd \sqsubset \sqrt{8}$

14. Again, one of the ſides about the right-angle is by Conſtruction in the firſt ſtep $aa - 1$, therefore . . $aa \sqsubset 1$

15. But by the fourth ſtep, ... $\frac{4}{eeee-4ee+4} = aa$

16. Therefore from the two laſt ſteps, ... $\frac{4}{eeee-4ee+4} \sqsubset 1$

17. And by multiplying each part in the ſixteenth ſtep by the Denominator $eeee-4ee+4$, 'tis manifeſt that . . $4 \sqsubset eeee-4ee+4$

18. Whence, by adding $4ee$ to each part, ... $4+4ee \sqsubset eeee+4$

19. And by ſubtracting 4 from each part in the laſt ſtep . . $4ee \sqsubset eeee$

20. And by dividing each part in the nineteenth ſtep by ee, . $4 \sqsubset ee$

21. And by extracting the ſquare Root out of each part in the 20th ſtep, 'tis evident that . . . . . $2 \sqsubset e$

22. And by doubling each part in the twenty-firſt ſtep, . . $4 \sqsubset 2e$

23. But by ſuppoſition in the tenth ſtep, ... $ddd = 2e$

24. Therefore from the two laſt ſteps, ... $4 \sqsubset ddd$

25. And conſequently, ... $ddd \sqsupset 4$

26. Thus in the 13th and 25th ſteps it is found that the cube-number repreſented by ddd muſt be greater than the ſquare Root of 8, but leſs than 4; and then the half of the cube-number taken within thoſe limits ſhall be the number e, which being known, the number a will alſo be diſcovered by the third ſtep: Laſtly, a right-angled Triangle formed from the number a and unity, ſhall be that which is ſought. From the premiſſes ariſeth this

CANON.

CANON.

27. Firſt, take any cube-number greater than the ſquare Root of 8, (*viz.* greater than $2\frac{828}{1000}$, *&c.*) but leſs than 4; then take the half of that cube-number, and call it e; that done, divide 2 by the exceſs of the Square of the number e above 2, and call the Quotient a; laſtly, let a right-angled Triangle be formed from the number a and 1, ſo ſhall this Triangle be that which is ſought.

An Example in Numbers.

Firſt, I take ſome cube-number within the limits preſcribed in the Canon, as $\frac{27}{8}$; whoſe half, to wit, $\frac{27}{16}$ is the number e; then I divide 2 by the exceſs of the Square of $\frac{27}{16}$ above 2, ſo the Quotient $\frac{512}{217}$ is the number a; laſtly, from $\frac{512}{217}$ and 1 I form a right-angled Triangle and find the three ſides to be theſe, to wit, $\frac{309233}{47089}$, $\frac{235095}{47089}$ and $\frac{233208}{47089}$, which ſolve the Queſtion: For the ſumm of thoſe three ſides make the Square $\frac{746496}{47089}$, whoſe ſide is $\frac{864}{217}$; and the ſaid ſumm of the ſides together with the Area $\frac{110108160}{10218313}$ makes the Cube $\frac{27202772232}{10218313}$, whoſe ſide is $\frac{648}{217}$.

QUEST. 111. (Quæſt. 24. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the ſumm of all the three ſides may be a Cube; and that the ſaid ſumm increaſed with the Area may make a Square.

RESOLUTION.

1. For the ſumm of all the three ſides of the Triangle ſought put . . . . . s
2. And for the Area . . . . . a
3. Then becauſe the double of the Area is equal to the Product made by the mutual multiplication of the ſides about the right-angle, the ſaid Product is . . . . . $2a$
4. For one of the ſides about the right-angle put . . . . . $\frac{1}{e}$
5. Then becauſe the Product of the ſaid ſides is $2a$, 'tis manifeſt that by dividing $2a$ by $\frac{1}{e}$, the Quotient ſhall be the other ſide, to wit, $2ae$
6. Therefore from the fourth and fifth ſteps the ſumm of the ſides about the right-angle is . . . . . $2ae + \frac{1}{e}$
7. Which ſumm ſubtracted from s the ſumm of all the three ſides, leaves the Hypothenuſal, to wit, ... $s - 2ae - \frac{1}{e}$
8. The Square of the ſaid Hypothenuſal is

$$ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e}.$$

9. And the ſumm of the Squares of the ſides about the right-angle, to wit, of the Squares of $\frac{1}{e}$ and $2ae$ is $\frac{1}{ee} + 4aaee$.

10. But the Square of the Hypothenuſal is equal to the ſumm of the Squares of the ſides about the right-angle, therefore from the eighth and ninth ſteps this Equation ariſeth, *viz.* $ss + 4aaee + \frac{1}{ee} + 4a - 4sae - \frac{2s}{e} = \frac{1}{ee} + 4aaee$.

11. From which Equation, after due Reduction in order to find out the value of e, by the letters s and a, there will ariſe this following Equation, *viz.*

$$\frac{ss+4a}{4sa}e - ee = \frac{2s}{4sa}.$$

12. And by reſolving the laſt Equation according to the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of e will be found theſe, to wit,

$$e = \frac{\frac{1}{2}ss+2a}{4sa} + \sqrt{:\frac{\frac{1}{4}ssss+4aa-6ssa}{16ssaa}};$$

$$e = \frac{\frac{1}{2}ss+2a}{4sa} - \sqrt{:\frac{\frac{1}{4}ssss+4aa-6ssa}{16ssaa}};$$

13. But 'tis evident that thoſe values of e will not be rational unleſs the Numerator $\frac{1}{4}ssss+4aa-6ssa$ be equal to a Square. Moreover, the Queſtion requires that the

the ſumm of all the three ſides with the Area, to wit, $s+a$ may be equal to a Square; ſo we are faln upon this Duplicate equality,

$$viz.\ \begin{cases} \frac{1}{4}ssss+4aa-6ssa = \square \\ s+a = \square. \end{cases}$$

14. Now if in that Duplicate equality any known ſquare number be taken for the value of s, then we may diſcover the number a. But the Queſtion requires s, (that is, the ſumm of all the three ſides of the Triangle ſought,) to be a Cube, therefore ſome number which is both a Square and a Cube muſt be taken for the value of s; let therefore the ſquared Cube 64 be put equal to s, and then the ſaid Duplicate equality will be reſolved into this,

$$viz.\ \begin{cases} 4aa-24576a+4194304 = \square \\ a+64 = \square. \end{cases}$$

15. Or you may divide the firſt of thoſe two quantities by 4, then inſtead of that firſt quantity the Quotient is to be equated to a Square, and ſo this following Duplicate equality ariſeth,

$$viz.\ \begin{cases} aa-6144a+1048576 = \square \\ a+64 = \square. \end{cases}$$

16. But becauſe in the Duplicate equality laſt expreſt, the ſquare numbers 1048576 and 64 are unequal, I divide the greater of them by the leſs, and by the Quotient 16384 I multiply $a+64$, and then the Product $16384a+1048576$ is to be equated to a Square inſtead of $a+64$; ſo at length this Duplicate equality remains to be reſolved

$$viz.\ \begin{cases} aa-6144a+1048576 = \square \\ 16384a+1048576 = \square. \end{cases}$$

17. Now in order to reſolve that Duplicate equality, firſt, ſuppoſing the former of the two quantities to be equated to be the greater, their difference is $aa-22528a$, then according to the method uſed in divers preceding Queſtions of this Book, two ſuch numbers are to be found out that the Product of their multiplication may make the ſaid difference $aa-22528a$, and that as well in the half of the ſumm as in the half of the difference of the ſaid two numbers there may be found 1024, which is the ſide of the Square 1048576. But two ſuch numbers are $11a$ and $\frac{1}{11}a-2048$; the half-difference of theſe is $\frac{10}{11}a+1024$, and the Square of $\frac{10}{11}a+1024$ being equated to $16384a+1048576$ will give $a=\frac{12424}{225}$, to wit, the Area of the Triangle ſought.

Since then the value of a is found to be $\frac{12424}{225}$, and s was before aſſumed to be 64; according to theſe values of a and s, the twelfth ſtep will diſcover the two values of e to be $\frac{25}{448}$ and $\frac{2}{176}$; by either of which if you reſolve the poſitions in the fourth and fifth ſteps, you will find the ſides about the right-angle to be $\frac{4012}{225}$ and $\frac{44002}{225}$; therefore the Hypothenuſal is $\frac{12682}{225}$, and the ſumm of all the three ſides is the Cube 64, to which if you add the Area $\frac{119424}{225}$ it makes the Square $\frac{121824}{225}$, whoſe ſide is $\frac{242}{15}$.

QUEST. 112. (Quæſt. 25. Lib. 6. *Diophant.*)

To find a right-angled Triangle in rational numbers, that the Square of the Hypothenuſal may be compoſed of ſome Square and its ſide; and that the Square of the ſaid Hypothenuſal being divided by one of the ſides about the right-angle may give a number compoſed of a Cube and its ſide.

RESOLUTION.

1. For one of the ſides about the right-angle put . . . . . } a
2. And for the other, ... } aa
3. The ſumm of their Squares is equal to the Square of the Hypothenuſal, to wit, ... } $aaaa+aa$
4. Now 'tis evident that $aaaa+aa$ is compos'd of the Square $aaaa$ and its ſide aa; but if the ſaid $aaaa+aa$ be divided by a, (which was put for one of the ſides about the right-angle,) it gives the Quotient $aaa+a$, which is compos'd of a Cube with its ſide; ſo that it remains only to equate $aaaa+aa$ to a Square, that is, to find out a right-angled Triangle that one of the ſides about the right-angle may be equal to the Square of the other of the ſame ſides: But the preceding 95th Queſtion ſhews how to find out ſuch a Triangle; take if you pleaſe that in the firſt Example of the ſaid Queſtion, to wit, $\frac{4}{3}$, $\frac{16}{9}$, $\frac{20}{9}$, which will ſolve this 112th Queſtion; for the Square of the Hypothenuſal $\frac{20}{9}$, *viz.* $\frac{400}{81}$, is compos'd of the Square $\frac{256}{81}$ and its ſide $\frac{16}{9}$. Moreover, if the

if the ſaid $\frac{400}{81}$ be divided by $\frac{4}{3}$, (one of the ſides about the right-angle,) it gives the Quotient $\frac{120}{27}$, which is compos'd of the Cube $\frac{64}{27}$ and its ſide $\frac{4}{3}$.

A Proſpect of all Diophantus's *kinds of Duplicate Equality, ſhewing alſo at firſt ſight in which of the preceding Queſtions they are reſolved.*

I. The firſt kind of Duplicate Equality is, when each of two Quantities to be equated to ſquare numbers conſiſts of an unknown Root or number a, with ſome abſolute or known number, and the numbers prefixt to the Root a are equal to one another; as in the five following Examples.

1.	If . . . . $a+192 = \square$ And . . . . $a+128 = \square$ What is the number a?	Reſolved in *Queſt.* 8. of this *Book* 3.
2.	If . . . . $10a+54 = \square$ And . . . . $10a+6 = \square$ What is the number a?	Reſolved in *Queſt.* 8.
3.	If . . . . $192-a = \square$ And . . . . $64-a = \square$ What is the number a?	Reſolved in *Queſt.* 14. See the like in *Queſt.* 50.
4.	If . . . . $a-27 = \square$ And . . . . $a-15 = \square$ What is the number a?	Reſolved in *Queſt.* 15.
5.	If . . . . $a+12 = \square$ And . . . . $a-8 = \square$ What is the number a?	Reſolved in *Queſt.* 16.

II. The ſecond kind of Duplicate Equality is, when each of two Quantities to be equated to Squares conſiſts of an unknown Root a, and of one and the ſame known ſquare number. Alſo, when the numbers prefixt to a in both Quantities are unequal, and the abſolute numbers are unequal Squares; as in the ſix following Examples.

1.	If . . . . $4+a = \square$ And . . . . $4-a = \square$ What is the number a?	Reſolved in *Queſt.* 17.
2.	If . . . . $9+a = \square$ And . . . . $4-a = \square$ What is the number a?	Reſolved in *Queſt.* 18.
3.	If . . . . $8a+4 = \square$ And . . . . $6a+4 = \square$ What is the number a?	Reſolved in *Queſt.* 33.
4.	If . . . . $4-2a = \square$ And . . . . $4-3a = \square$ What is the number a?	Reſolved in *Obſervat.* 1. *Queſt.* 33.
5.	If . . . . $4+2a = \square$ And . . . . $4-3a = \square$ What is the number a?	Reſolved in *Obſervat.* 1. *Queſt.* 33.
6.	If . . . . $10a+9 = \square$ And . . . . $5a+4 = \square$ What is the number a?	Reſolved in *Obſervat.* 2. *Queſt.* 33.

III. The third kind of Duplicate Equality is, when each of two Quantities to be equated to Squares conſiſts of ſome number of a, and an abſolute number not a Square, but the numbers prefixt to a have ſuch proportion to one another as a ſquare number to a ſquare number; as in the three following Examples.

1.	If . . . . $A + 12 = \square$ And . . . $4A + 12 = \square$ What is the number A ?	Resolved in *Quest.* 29.
2.	If . . . . $\frac{12}{10}A + \frac{48}{10} = \square$ And . . . $\frac{1}{10}A + \frac{1}{10} = \square$ What is the number A ?	Resolved in *Quest.* 35.
3.	If . . . . $\frac{48}{9} - \frac{2}{3}A = \square$ And . . . $\frac{48}{9} - \frac{1}{3}A = \square$ What is the number A ?	Resolved in *Quest.* 50.

IV. The fourth kind of Duplicate Equality is, when two Quantities to be equated to Squares are diverſly compos'd of AA and A, or of AA, A and abſolute numbers; in which caſes, to the end the Duplicate Equality propos'd may be explicable by rational numbers, theſe two things are requiſite to be found therein; *viz.* Firſt, either the numbers prefixt to AA, or the abſolute numbers muſt be rational Squares: Secondly, the difference of the Quantities propoſed muſt be either ſome ſole number of A, or compos'd of ſome number of A and an abſolute number, or of ſome numbers of AA and A; as in the following Examples.

1.	If . . . . $AA + A = \square$ And . . . . $AA - A = \square$ What is the number A ?	Resolved in *Quest.* 19.
2.	If . . . . $9AA + 2A = \square$ And . . . $9AA + A = \square$ What is the number A ?	Resolved in *Quest.* 20.
3.	If . . . . $4AA + 5A = \square$ And . . . $9AA + 5A = \square$ What is the number A ?	Resolved in *Quest.* 21. See the like in *Quest.* 22.
4.	If . . . $4AA + 3A - 1 = \square$ And . . . $4AA - A - 1 = \square$ What is the number A ?	Resolved in *Quest.* 20.
5.	If . . . $4AA + 15A = \square$ And . . . $4AA - A - 4 = \square$ What is the number A ?	Resolved in *Quest.* 31.
6.	If . . . $4AA + A + 1 = \square$ And . . . $3A + 1 = \square$ What is the number A ?	Resolved in *Quest.* 32.
7.	If . . . . $AA + D = \square$ And . . . . $BA + D = \square$ What is the number A ?	Resolved in *Quest.* 55.
8.	If . . $AA + 2PA + BB = \square$ And . . . $\frac{A}{P} + 1 = \square$ What is the number A ?	Resolved in *Quest.* 59.
9.	If . . $3AA - 6A + 4 = \square$ And . . . $4AA + 4 = \square$ What is the number A ?	Resolved in *Quest.* 94. See the like in *Quest.* 93.
10.	If . . . . $EE + 1 = \square$ And . . . . $E + 1 = \square$ What is the number E ?	Resolved in *Quest.* 99.
11.	If $AA - 6144A + 1048576 = \square$ And . . . $A + 64 = \square$ What is the number A ?	Resolved in *Quest.* 111.

A brief Expoſition upon Monſieur Fermat's *Analytical Invention, inſerted in the late Edition of Monſieur* Bachet's *Comment upon* Diophantus, *printed at* Toloze, *Anno* 1670.

JAC. *de* BILLY, who collected the ſaid Invention out of *Fermat*'s Letters, divides it into three Parts. The firſt is an Improvement of one of *Diophantus*'s kinds of Duplicate Equality, (to wit, the fourth and laſt kind in my preceding Proſpect, but the ſixth with *Bachet* in his Comment upon *Queſt.* 24. *Book* VI. *Diophant.*) for whereas *Diophantus*'s method of reſolving that kind of Duplicate Equality, finds out but one value or two at the moſt of the Root ſought, and ſometimes the value found out is Negative, *viz.* leſs than nothing, *Fermat*'s method finds out innumerable affirmative values.

The ſecond part ſhews how to reſolve two kinds of Triplicate Equality, by reducing them to a Duplicate Equality of the ſame kind with that above mentioned.

The third part ſhews how to equate a Quantity compos'd of five Terms to a Square; likewiſe a Quantity of four Terms to a Square or Cube, and for the moſt part to find out innumerable affirmative values of the Root ſought.

Theſe three parts I ſhall explain in order, and according to my uſual method put A; (inſtead of N by *Diophantus*) for a Root or number unknown, AA for the Square of that Root, AAA for the Cube, *&c.*

Concerning the Firſt part of Fermat's *Analytical Invention.*

Fermat's Rule to find out innumerable affirmative values of the Root ſought, in a Duplicate equality of the kind before mentioned, is this,

Firſt, by the vulgar method of *Diophantus*, (explain'd in divers preceding Queſtions of this Book,) find out one value of the ſought Root, (A,) it matters not whether it be affirmative or negative; then to A joyn that value with its own ſign, whether it be $+$ or $-$, and take the ſumm for a new Root inſtead of A; then according to the ſaid new Root let a new Duplicate equality be deduced from the firſt, and find out the value of A in the new Duplicate equality by the vulgar method. That done, connect this latter value to the firſt before found, with their own ſigns, and it will give a ſecond value of the Root ſought in the firſt Duplicate equality. In like manner by the help of this ſecond value you may find out a third, and from the third a fourth, and ſo infinitely.

Note. After one value of the Root ſought is found out in the vulgar way, there will always certainly ariſe a known Square in each of the two new Algebraick quantities to be equated to Squares, in the ſecond, third, or any following Duplicate equality deduced from the firſt as the Rule doth direct, the reaſon whereof will be evident to him that diligently examines the Operation. But when thoſe two known Squares are unequal, the leſſer muſt be reduced to the greater, (in ſuch manner as before hath been ſhewn in *Queſt* 21, and 111. of this Book,) to the end there may be one and the ſame known ſquare number in each of the two quantities to be equated, and then the difference of the ſaid quantities will be compoſed of ſome number of AA and ſome number of A; which kind of difference is abſolutely neceſſary in the uſe of *Fermat*'s Rule before expreſt. All which will be made manifeſt by the following Queſtions 113, 114, 115, 116.

QUEST. 113.

To find a number, that if to the Product of its Square multiplied by 32 you add unity, and from the ſumm ſubtract eight times the number ſought, the remainder may be a Square. Alſo, that eight times the number ſought, together with unity, may make a Square.

Let a be put for the number ſought, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $32aa - 8a + 1 = \square$
2. And . . . . . $8a + 1 = \square$

} what is the number a ?

RESOLUTION.

3. To reſolve that Duplicate equality, firſt, (according to the vulgar method before explained in *Queſt.* 32. of this Book,) I take the difference of the two quantities to be equated, and becauſe in this Example, either of them may be taken for the greater, I ſuppoſe the firſt to be the greater, ſo their difference is $32aa - 16a$; then I ſeek two ſuch quantities that the Product of their multiplication may make $32aa - 16a$, and that as well in half the ſumm as in half the difference of the ſame two quantities, there may be found the abſolute number 2, (to wit, the double of the ſquare Root of 1, the known Square in each of the two quantities to be equated;) ſo I find $4a - 2$ and $8a$, whoſe Product is $32aa - 16a$, and their difference is $4a + 2$; the half of this difference is $2a + 1$, whoſe Square $4aa + 4a + 1$ being equated to $8a + 1$, (which was ſuppoſed to be the leſſer of the two quantities in the Duplicate equality,) will give, after due Reduction, $a = 1$, according to which value, the firſt of the two quantities propos'd to be equated, to wit, $32aa - 8a + 1$, makes the Square 25, and the other quantity $8a + 1$ makes the Square 9; wherefore the Queſtion is ſolved, and the ſaid value of a, to wit, 1, is the only value either affirmative or negative that can be found out by the common method. But *Fermat*'s Rule, by the help of the ſaid firſt value finds out a ſecond, and from the ſecond a third, *&c.* As, for example, to find out another number, or value of a beſides 1 to ſolve the Queſtion, let $a + 1$ be taken for a new Root inſtead of a, ($+1$, becauſe the firſt Root was found $+1$, but if it had been found -1, then $a - 1$ ought to be put for the new Root;) then let a new Duplicate equality be deduced from that before reſolved in this manner, *viz.*

4. Inſtead of $32aa$ take 32 times the Square of the new Root $a + 1$, that is, ... $32aa + 64a + 32$
5. To that Product add unity, and it makes this ſumm, . $32aa + 64a + 33$
6. From that ſumm ſubtract eight times $a + 1$, (inſtead of $8a$ in the firſt Duplicate equality,) that is, . . . $\ldots + 8a + 8$
7. And the remainder muſt be equal to a Square, *viz.* . $32aa + 56a + 25 = \square$
8. Again, inſtead of $+8a$ the later quantity of the firſt Duplicate equality, take eight times $a + 1$, that is, . $\ldots + 8a + 8$
9. And to that Product add unity, ſo this ſumm muſt be equal to a Square, *viz.* . . . . . $\ldots + 8a + 9 = \square$
10. the ſeventh and ninth ſteps give a new Duplicate equality, *viz.* . . . . . $32aa + 56a + 25 = \square$ $+ 8a + 9 = \square$
11. But becauſe in this Duplicate equality the Squares 25 and 9 are unequal, and *Fermat*'s Rule takes place only in a Duplicate equality which hath one and the ſame known ſquare number in each quantity to be equated, let 25 be divided by 9, and by the Quotient $2\frac{7}{9}$ multiply $8a + 9$; ſo this quantity comes forth (inſtead of $8a + 9$) to be equated to a Square, *viz.* . . . . . $\ldots + \frac{22}{9}a + 25 = \square$
12. Thus at length, from the Duplicate equality before expreſt in the firſt and ſecond ſteps, another is deduced, and qualified as *Fermat*'s Rule requires, *viz.* . . . . $32aa + 56a + 25 = \square$ $+ \frac{200}{9}a + 25 = \square$
13. Then by reſolving this laſt Duplicate equality in like manner as the firſt, you will find . . . . . $a = -\frac{2102222}{1697809}$
14. To which negative value of a if you add 1, becauſe $a + 1$ was taken for the new Root, you will find $-\frac{1882714}{1697809}$ for a ſecond Root or value of a in the Duplicate equality in the firſt and ſecond ſteps, *viz.* . . $a + 1 = -\frac{1882714}{1697809}$

I ſay $-\frac{1882714}{1697809}$, beſides 1 firſt found, will ſolve the Queſtion; for, if you multiply the Square of $-\frac{1882714}{1697809}$ by 32, and add 1 to the Product, and from this ſumm ſubtract the Product of the ſaid $-\frac{1882714}{1697809}$ into 8, there will remain a Square whoſe ſide is $\frac{2802842}{1697809}$. Moreover, if the Product of the ſaid $-\frac{1882714}{1697809}$ into 8, be increaſed with 1, it makes a Square, whoſe ſide is $\frac{1221}{1303}$.

In

In like manner, by taking or ſubſtituting $a + \frac{1882714}{1697809}$ for a new Root inſtead of a, and proceeding as before, you may find out a third number, and from the third a fourth, *&c.* to ſolve the Queſtion propoſed; but the operation in finding out Anſwers by this method is ſo exceſſively laborious, that for the moſt part he that hath found out two Anſwers, will hardly have patience to find out a third.

QUEST. 114.

To find a number, that if to the double of its Square you add unity, and from that ſumm ſubtract four times the number ſought, the remainder may be a Square. Alſo, that if from unity you ſubtract the double of the number ſought, the remainder may be a Square.

Let a be put for the number ſought, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . . $2aa - 4a + 1 = \square$
2. And . . . . . $-2a + 1 = \square$

} what is the number a ?

RESOLUTION.

3. If that Duplicate equality be reſolved by the vulgar method, the only value of a will be found -4; but this being leſs than nothing, I ſearch out an affirmative value of a by *Fermat*'s Rule, thus,
4. Inſtead of a I take for a new Root . . . . . $a - 4$
5. Then inſtead of $2aa$ I take the double of the Square of the new Root $a - 4$, *viz.* . . . . . $2aa - 16a + 32$
6. To which double Square I add unity, (as the Queſtion requires,) and it makes this ſumm, ... $2aa - 16a + 33$
7. Then from that ſumm I ſubtract four times the new Root $a - 4$, (inſtead of $4a$ in the Duplicate equality in the firſt ſtep,) *viz.* I ſubtract . . . . . $\ldots + 4a - 16$
8. So this remainder muſt be equal to a Square, *viz.* . . $2aa - 20a + 49 = \square$
9. Again, if from unity, the double of the new Root $a - 4$ be ſubtracted, the remainder muſt be equal to a Square, *viz.* $\ldots - 2a + 9 = \square$
10. So in the eighth and ninth ſteps there is a new Duplicate equality deduced from that in the firſt and ſecond ſteps; but becauſe in the new Duplicate equality the known ſquare numbers 49 and 9 are unequal, I divide 49 by 9, and by the Quotient $5\frac{4}{9}$ I multiply $-2a + 9$ in the ninth ſtep, and it gives to be equated to a Square . . . $\ldots - \frac{22}{9}a + 49 = \square$
11. The eighth and tenth ſteps give this new Duplicate equality to be reſolved, in which there is the ſame known ſquare number, (to wit, 49, as *Fermat*'s Rule requires,) *viz.* $2aa - 20a + 49 = \square$ $- \frac{22}{9}a + 49 = \square$
12. Now to reſolve the laſt preceding Duplicate equality, according to the vulgar method, I take the difference of the two quantities to be equated, which, by ſuppoſing the firſt quantity to be the greater, is . . . . . $2aa - \frac{22}{9}a$
13. Then I ſearch out two ſuch quantities, that being mutually multiplied may make the ſaid difference, and that as well in half the ſumm as in half the difference of the ſaid two quantities there may be found 7, (the ſquare Root of 49,) ſo I find the two quantities to be . . $\frac{112}{41}a - 14$ and $\frac{44}{63}a$
14. Half the difference of thoſe two quantities is . . . $\frac{68842}{11106}a - 7$
15. Then the Square of the ſaid half-difference being equated to $-\frac{22}{9}a + 49$ (in the eleventh ſtep) will give . . . $a = \frac{1412.1214}{391300049}$
16. From which value of a I ſubtract 4, becauſe the new Root was aſſumed to be $a - 4$, and the remainder is . $-\frac{111122442}{391300049}$

I ſay $-\frac{111122442}{391300049}$ ſhall be a value of the Root a in the Duplicate equality in the firſt and ſecond ſteps, and therefore will ſolve the Queſtion propoſed; for if $a = -\frac{111122442}{391300049}$, then $2aa - 4a + 1$ makes a Square whoſe ſide is $\frac{174668814}{391300049}$; alſo, $1 - 2a$ makes a Square whoſe ſide is $\frac{11117}{62557}$. Thus although a number leſs than nothing, to wit, -4 was found out for the firſt value of the Root a, yet by the help thereof an affirmative Root or number is found out to ſolve the Queſtion propoſed, and from that ſecond Root you may find out a third.

QUEST. 115.

QUEST. 115.

To find a right-angled Triangle in rational numbers, that the ſumm of the ſides about the right-angle may make a Square. Alſo, that either of the ſame ſides being added to the Square of the other of them may make a Square: And that one of the ſaid ſides about the right-angle, together with a given multiple (ſuppoſe the triple) of the other of them may make a Square.

RESOLUTION.

1. It is manifeſt by the Theorem in *Queſt.* 24. of this *Book*, that if the Fraction $\frac{1}{4}$ be divided into any two parts, then either part added to the Square of the other makes a Square; therefore, to ſolve the firſt part of the Queſtion infinitely, put for the ſides about the right-angle . . . . — a and $\frac{1}{4} - a$
2. The ſumm of the Squares of thoſe ſides ſhall be equal to the Square of the Hypothenuſal, therefore . . . . $2aa - \frac{1}{2}a + \frac{1}{16} = \square$
3. And becauſe the Queſtion requires that one of the ſides about the right-angle, together with the triple of the other, may make a Square, let $\frac{1}{4} - a$ be added to $3a$, and then the ſumm muſt be equal to a Square, *viz.* . . . . . $+2a + \frac{1}{4} = \square$
4. So in the ſecond and third ſteps there is a Duplicate equality, which, (by the method in divers preceding Queſtions) may be reduced to this having equal abſolute Squares, *viz.* $32aa - 8a + 1 = \square$ $+8a + 1 = \square$
5. Which Duplicate equality laſt expreſt being reſolved by the vulgar method gives but one ſingle value of a, to wit, unity, (as hath been ſhewn in the preceding *Queſt.* 113.) But according to the Poſitions in the firſt ſtep of the Reſolution of this Queſtion, the number ſignified by a ought to be leſs than $\frac{1}{4}$, and conſequently $1 = a$ will not ſolve this Queſtion by affirmative numbers; for if a be 1, then $\frac{1}{4} - a$, (which was put for one of the ſides about the right-angle) will be leſs than nothing. Therefore to ſearch out another value of a, I proceed according to *Fermat*'s Rule thus, *viz.* Inſtead of a, I take for a new Root $a + 1$, by which the Duplicate equality in the fourth ſtep will be reduced to this that follows, (as before hath been ſhewn in *Queſt.* 113.)

viz. $\begin{cases} 32aa + 56a + 25 = \square \\ + \frac{222}{9}a + 25 = \square. \end{cases}$

6. Now if this Duplicate equality be reſolved in the vulgar way, it will give $a = -\frac{1120}{16978}$, therefore $-\frac{1120}{16978}$ $(= a + 1)$ ſhall be a value of a, (beſides 1,) in the Duplicate equality in the fourth ſtep; and becauſe the ſaid $-\frac{1120}{16978}$ is leſs than $\frac{1}{4}$, it ſhall be the firſt of the ſides about the right-angle of the Triangle ſought; then ſince $\frac{1}{4} - a$ was put for the other ſide, this latter, by ſubtracting the former from $\frac{1}{4}$, ſhall be $\frac{2242}{67912}$, and by reducing the firſt ſide to the ſame Denominator with the latter, the two deſired ſides about the right-angle are $\frac{2142}{67912}$ and $\frac{2242}{67912}$, therefore the Hypothenuſal is $\frac{120241}{67912}$; which right-angled Triangle will ſolve the Queſtion: For, firſt, the ſumm of the ſides about the right-angle makes a Square, to wit, $\frac{1}{4}$; ſecondly, if $\frac{2142}{67912}$ the firſt ſide about the right-angle, be added to the Square of the latter ſide $\frac{2242}{67912}$, it makes a Square whoſe ſide is $\frac{241}{67912}$; thirdly, if $\frac{2242}{67912}$ the latter ſide about the right-angle, be added to the Square of the firſt ſide $\frac{2142}{67912}$, it makes a Square, whoſe ſide is $\frac{2402}{67912}$; and laſtly, if $\frac{2242}{67912}$ the latter ſide about the right-angle, be added to the triple of the firſt ſide $\frac{2142}{67912}$, it makes a Square, whoſe ſide is $\frac{1201}{2606}$.

QUEST. 116.

To find a right-angled Triangle, that the Square of the Area being added to the ſumm of the ſides about the right-angle may make a Square.

RESOLUTION.

1. For the ſides about the right-angle put . . . . . — a and 1
2. Then the ſumm of the Squares of thoſe ſides muſt be equal to a Square, to wit, the Square of the Hypothenuſal, therefore . . $aa + 1 = \square$
3. And the Square of the Area being added to the ſumm of the ſides about the right-angle muſt make a Square, *viz.* . . . $\frac{1}{4}aa + a + 1 = \square$

4. So

4. So in the two laſt ſteps there is a Duplicate equality qualified as *Fermat*'s Rule preſuppoſeth, but if it be reſolved in the ordinary way, it gives no value of a either affirmative or negative, I therefore reduce that Duplicate equality to another that ſhall have equal numbers of aa, by multiplying $\frac{1}{4}aa + a + 1$ in the third ſtep by the Square 4, ſo it produceth $aa + 4a + 4$, and now this following Duplicate equality comes to be reſolved,

viz. $\begin{cases} aa + 1 = \square. \\ aa + 4a + 4 = \square. \end{cases}$

5. Which Duplicate equality laſt expreſt being reſolved in the vulgar way, (explain'd in *Queſt* 20, 31. of this *Book*,) will give this negative value of a, viz. $a = -\frac{1}{8}$
6. But that value of a being leſs than nothing, I renew the work according to *Fermat*'s method, *viz.* inſtead of a I take for a new Root . . . . . $a - \frac{1}{8}$
7. By which new Root $a - \frac{1}{8}$ I form this Duplicate equality out of that in the ſecond and third ſteps, (after the manner before explain'd in *Queſt.* 113.) *viz.* . . . . . $aa - \frac{1}{4}a + \frac{1}{64} = \square$ $aa - \frac{1}{4}a + \frac{255}{64} = \square$
8. Then I reduce that new Duplicate equality to another that ſhall have in each quantity one and the ſame abſolute ſquare number, (as *Fermat*'s Rule requires;) ſo this ariſeth, *viz.* . . . . . $64aa - 240a + 289 = \square$ $18496aa - 69360a + 289 = \square$
9. Now if the Duplicate equality laſt expreſt be reſolved in the vulgar way, the value of a will again be found negative; and if I ſhould aſſume a third Root and proceed as before, the work would be intollerably tedious, and perhaps produce another negative value. This difficulty leads me to another way by which *Fermat* hath ſolved divers knotty Problems, but it ſeems to me to depend more upon chance than Art; however, I ſhall try whether it will find out an affirmative value of a to ſolve the Duplicative equality in the eighth ſtep or not.

First then, ſuppoſing the latter of the two quantities in the eighth ſtep to be the greater, their difference is $18432aa - 69120a$, then I ſeek two ſuch numbers that being mutually multiplied may make 18432, and that their ſumm may make 272; (to wit, the double of 136 the ſide of the Square 18496 which is prefixt to aa in the eighth ſtep;) ſo I find 144 and 128, which are very luckily rational, and therefore fit for the preſent purpoſe; then theſe two quantities $144a - 540$ and $128a$ being mutually multiplied will produce the above-mentioned difference, and in half the ſumm of thoſe Factors there will be found $136a$ the ſide of the Square $18496aa$ in the eighth ſtep: Then by equating the Square of half the ſumm of the ſaid Factors $144a - 540$ and $128a$, *viz.* the Square of $136a - 270$, to $18496aa - 69360a + 289$ (in the eighth ſtep,) there will thence ariſe $\frac{289}{408}$ for the value of a; wherefore $a - \frac{1}{8}$, (which was aſſumed for the new Root in the ſixth ſtep inſtead of a,) gives $\frac{241}{4080}$ for the value of a in the firſt Duplicate equality in the ſecond and third ſteps, that is, one of the ſides about the right-angle of the Triangle ſought; but the other of the ſaid ſides was aſſumed to be 1; therefore the Hypothenuſal ſhall be $\frac{12961}{4080}$; and the three ſides of the right-angled Triangle ſought are $\frac{2337}{4080}$, 1, $\frac{1712}{4080}$; for if the ſumm of the ſides about the right-angle, that is, of $\frac{4049}{4080}$ and 1 be added to the Square of the Area it makes a Square, whoſe ſide is $\frac{2111}{8160}$. And if you deſire a ſecond right-angled Triangle to ſolve the Queſtion, you may put $a + \frac{241}{4080}$ for a new Root, and proceed as before.

Concerning the ſecond Part of Fermat's *Analytical Invention.*

Diophantus doth not ſo much as mention a Triplicate Equality, but *Fermat* ſhews how to ſolve two kinds thereof. The firſt is, when three quantities to be equated to Squares are ſuch, that every one of them is compos'd of ſome number of a, either affirmative or negative, and one and the ſame known affirmative ſquare number. The ſecond kind of Triplicate equality is, when three quantities to be equated to Squares are ſuch, that every one of them is compos'd of the ſame affirmative number of aa, and ſome number of a either affirmative or negative, without any abſolute number. How each of theſe kinds of Triplicate equality may be reſolved the following Queſtions will make manifeſt.

QUEST. 117.

QUEST. 117.

To find a number, that if it be multiplied by three given numbers ſeverally, ſuppoſe by 1, 2, 5, and to every one of the Products one and the ſame given Square, ſuppoſe 1, be added, the three ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . } a
2. Then the Queſtion requires that every one of theſe three ſumms may make a Square, *viz.* . . . . . } $a+1 = \square$; $2a+1 = \square$; $5a+1 = \square$

3. Now to reſolve that Triplicate equality, firſt form a Square from any number of $a+$ the ſide of the given Square 1, as from $a+1$, whoſe Square is $aa+2a+1$; then divide $aa+2a$ (the two firſt terms of that Square) by any one of the three numbers prefixt to a in the ſaid Triplicate equality, as by 1, which is tacitly prefixt to a in the firſt quantity $a+1$, and take the Quotient $aa+2a$ for a new Root inſtead of a, (which was put for the number ſought,) whereby the firſt part of the Queſtion is ſolved indefinitely; for if $aa+2a$ be put for the number ſought, then unity added to it makes a Square, to wit, $aa+2a+1$. Then multiply the ſaid $aa+2a$ by 2 which is prefixt to a in the ſecond quantity $2a+1$; and it produceth $2aa+4a$, to which add the given Square 1, and it makes $2aa+4a+1$ to be equated to a Square. Again, multiply $aa+2a$ by 5 which is prefixt to a in the third quantity $5a+4$, and it makes $5aa+10a$; to this add unity and it makes $5aa+10a+1$, to be equated to a Square; hence the following Duplicate equality ariſeth,

$$\textit{viz.}\ \begin{cases} 2aa+4a+1 = \square \\ 5aa+10a+1 = \square. \end{cases}$$

4. This Duplicate equality being reſolved by the vulgar way, will give $a = -6$, by which value of a if $aa+2a$ be expounded, it makes 24; (for the Square of -6 is $+36$, to which if you add -12 the double of -6, it makes $+24$;) wherefore 24 is the number ſought, and will ſolve the Queſtion: For if unity be added firſt to 24, then to 48 the double of 24, and laſtly, to 120 the quintuple of 24, the three ſumms are Squares, to wit, 25, 49, 121.

If you deſire another number beſides 24 to ſolve the Queſtion propoſed, you may aſſume $a-6$ for a new Root of the Duplicate equality laſt reſolved, and thence (by the method before explained in the firſt Part) find out a ſecond number to ſolve that Duplicate equality, and conſequently the Queſtion.

Note. When in a Triplicate equality of the firſt kind before defined, the greateſt of the three numbers of a is equal to the ſumm of the other two, then in ſuch caſe that Triplicate equality, although it may be poſſible in it ſelf, is inexplicable by the method of reſolving the preceding *Queſt.* 117. As, for example, if theſe three quantities be propoſed to be equated to as many Squares, *viz.* $5a+1$; $16a+1$; $21a+1$; where the greateſt number of a is equal to the ſumm of the other two, (for $21 = 16+5$,) and the value of the Root a is 3, according to which, thoſe three quantities being expounded will give theſe three Squares, 16, 49, 64; it will be in vain to ſeek out any Anſwer to that Triplicate equality by *Fermat*'s Rule, for it will produce an abſurd or fruitleſs Equation, as you will find upon tryal.

QUEST. 118.

To find a number, that if it be multiplied by three given numbers, ſuppoſe by 3, 4, 5, and the Products be ſeverally ſubtracted from 1 a given Square, the three remainders may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . } a
2. Then the Queſtion requires that theſe three Remainders may be Squares, *viz.* . . . . . } $1-3a = \square$; $1-4a = \square$; $1-5a = \square$

3. This Triplicate equality may be reſolved like that in the foregoing *Queſt.* 117. For firſt, I form a Square from $1-$ any number of a, as from $1-3a$, whoſe Square is $9aa-6a+1$; then I divide $9aa-6a$ (the two firſt terms of that Square,) by 3 which

which is prefixt to a in the firſt quantity $1-3a$, and the Quotient is $3aa-2a$, this with its contrary ſigns is $-3aa+2a$, which I put for a new Root inſtead of a (the number ſought,) whereby the firſt part of the Queſtion is ſatisfied infinitely; for if the triple of $-3aa+2a$ be ſubtracted from the given Square 1, the remainder is a Square, to wit, $9aa-6a+1$. Then I multiply the ſaid new Root $-3aa+2a$ by 4 and 5 ſeverally, (which are prefixt to a in the ſecond and third quantities of the Triplicate equality in the ſecond ſtep,) and ſubtracting the Products $-12aa+8a$ and $-15aa+10a$ ſeverally from the given Square 1, the remainders $12aa-8a+1$ and $15aa-10a+1$ are to be equated to Squares, ſo it remains only to ſolve this following Duplicate equality,

$$\textit{viz.}\ \begin{cases} 12aa-8a+1 = \square \\ 15aa-10a+1 = \square. \end{cases}$$

4. This Duplicate equality being ſolved in the ordinary way, gives $a = \frac{6}{11}$, by which value of a, the new Root $-3aa+2a$ being expounded will give $\frac{24}{121}$ for the value of a in the Triplicate equality in the ſecond ſtep. Wherefore $\frac{24}{121}$ will ſolve the Queſtion; for if its triple, quadruple and quintuple be ſeverally ſubtracted from unity, the three remainders are Squares, to wit, $\frac{49}{121}$, $\frac{25}{121}$, $\frac{1}{121}$.

QUEST. 119.

To find a number, as alſo four other numbers in Geometrical proportion continued, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . . } $a-1$
2. Then multiply a into any four known numbers continually proportional, as into 1, 2, 4, 8, and aſſume the Products to be the four Proportionals ſought, *viz.* . . . . . } a , $2a$, $4a$, $8a$
3. Then ſubtract the number in the firſt ſtep from thoſe four Proportionals ſeverally, and every one of the remainders muſt make a Square; but the firſt remainder is manifeſtly the Square 1, it remains therefore to reſolve this Triplicate equality, *viz.* . . } $a+1 = \square$; $3a+1 = \square$; $7a+1 = \square$

Now to reſolve that Triplicate equality you may take $aa+2a$ for a new Root inſtead of a, whereby the firſt part of the Triplicate equality will be ſolved indefinitely; for if $aa+2a$ be increaſed with 1 it makes a Square, to wit, $aa+2a+1$; then the two other parts of the ſaid Triplicate equality (by the like Operation as was uſed in *Queſt.* 117.) will be converted into this following Duplicate equality,

$$\textit{viz.}\ \begin{cases} 3aa+6a+1 = \square \\ 7aa+14a+1 = \square, \end{cases}$$

4. Which Duplicate equality being reſolved in the vulgar way gives $a = -12$, whence $aa+2a$ (the new Root) will be found 120; (for the Square of -12 is $+144$, to which if you add -24, that is, $2a$, it makes 120.) Therefore the firſt number ſought by the Queſtion is 119, (that is, $a-1$;) and the four numbers required to be in continual proportion are 120, 240, 480, 960; (which anſwer to a, $2a$, $4a$, $8a$, in the ſecond ſtep:) for if 119 be ſubtracted from thoſe four Proportionals ſeverally, the remainders are the Squares 1, 121, 361, 841.

QUEST. 120.

Three ſquare numbers Geometrically proportional being given, to find a number, that if it be added to thoſe Proportionals ſeverally, the three ſumms may be Squares.

RESOLUTION.

1. Let the three given Squares in continual proportion be . . . } 1 , 4 , 16 $\div\div$
2. For the number ſought put . . . . . } a
3. Then the Queſtion requires . . . . . } $a+1 = \square$; $a+4 = \square$; $a+16 = \square$

4. Now in regard the three known Squares in that Triplicate equality are unequal, they muſt be reduced to the ſame Square, which may be done thus, *viz.* Becauſe a Square multiplied by a Square produceth a Square, multiply the firſt quantity $a + 1$ by 64, (the Product of 4 into 16,) and it makes $64a + 64$ to be equated to a Square. Again, multiply the ſecond quantity $a + 4$ by 16, (the Product of 1 into 16,) and it gives $16a + 64$ to be equated to a Square. Likewiſe, multiply the third quantity $a + 16$ by 4, (the Product of 1 into 4,) and it produceth $4a + 64$ to be equated to a Square. So this Triplicate equality comes forth to be reſolved,

$$viz. \begin{cases} 64a + 64 = \square \\ 16a + 64 = \square \\ 4a + 64 = \square \end{cases}.$$

5. This Triplicate equality having in every one of its three quantities the ſame Square 64 may be reſolved (like that in the foregoing Queſtions 117, 119.) thus, *viz.* Firſt, form a Square from any number of $a +$ the ſide of the known Square 64, as from $2a + 8$, the Square whereof is $4aa + 32a + 64$; then divide $4aa + 32a$ by 4, which is prefixt to a in the third quantity $4a + 64$, and take the Quotient $aa + 8a$ inſtead of the Root a, (which was put for the number ſought;) whence the laſt part of the Triplicate equality in the fourth ſtep is ſolved indefinitely, for four times $aa + 8a$ together with 64 makes a Square, to wit, $4aa + 32a + 64$. Again, by taking the ſaid $aa + 8a$ inſtead of the Root a, the firſt and ſecond parts of the ſaid Triplicate equality will be reduced to this Duplicate equality,

$$viz. \begin{cases} 64aa + 512a + 64 = \square \\ 16aa + 128a + 64 = \square \end{cases}.$$

6. Which Duplicate equality being reſolved in the vulgar way gives $a = \frac{304}{105}$, whence $aa + 8a$ (the new Root) will be found $\frac{471736}{11025}$, which is the number ſought by the Queſtion; for if 1, 4, 16 be ſeverally added to the ſaid $\frac{471736}{11025}$, the three ſumms will be the Squares of theſe ſides $\frac{322}{105}$, $\frac{626}{105}$, $\frac{224}{105}$.

I ſhall now proceed to the ſecond kind of Triplicate equality, and explain it by Queſtions.

QUEST. 121. (Probl. 4. cap. 1. part. 2. *Dioph.* redivivi.)

To find a number, that if it be multiplied by every one of three given numbers Geometrically proportional, ſuppoſe by 1, 2, 4, and the Products be added ſeverally to the Square of the number ſought, the three ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then multiply that number a by the three given Proportionals 1, 2, 4, and to the Products ſeverally add the Square of the ſaid number a, ſo (according to the import of the Queſtion) theſe three ſumms muſt make Squares, *viz.* . . . . . $\begin{cases} aa + a = \square \\ aa + 2a = \square \\ aa + 4a = \square \end{cases}$
3. Now that Triplicate equality being of the ſecond kind before defined, muſt be reduced to a Triplicate equality of the firſt kind thus, *viz.* Take $\frac{1}{a}$ inſtead of the Root a, (the number ſought,) whence the firſt quantity $aa + a$ will be converted into $\frac{1}{aa} + \frac{1}{a}$; the ſecond quantity $aa + 2a$ will be reduced to $\frac{1}{aa} + \frac{2}{a}$; and the third quantity $aa + 4a$ to $\frac{1}{aa} + \frac{4}{a}$. Then becauſe a Square multiplied by a Square produceth a Square, if every one of the three new terms $\frac{1}{aa} + \frac{1}{a}$; $\frac{1}{aa} + \frac{2}{a}$ and $\frac{1}{aa} + \frac{4}{a}$, which are to be equated to Squares, be multiplied by the Square aa, that Triplicate equaquality will be reduced to this,

$$viz. \begin{cases} 1 + a = \square \\ 1 + 2a = \square \\ 1 + 4a = \square \end{cases}.$$

4. This Triplicate equality falling under the firſt form may be reſolved like that in the preceding *Queſt.* 117. thus, *viz.* Inſtead of a I take for a new Root $aa + 2a$, which increaſed with 1 makes a Square, to wit, $aa + 2a + 1$, whereby the firſt part of the Triplicate

Triplicate equality is ſolved indefinitely; then by multiplying $aa + 2a$ by 2 and 4 ſeverally, and adding 1 to each Product, this Duplicate equality ariſeth,

$$viz. \begin{cases} 2aa + 4a + 1 = \square \\ 4aa + 8a + 1 = \square \end{cases}.$$

5. Whence $a = -\frac{20}{7}$, by which if $aa + 2a$ (the new Root) be reſolved it makes $\frac{120}{49}$; this divided by 1, becauſe $\frac{1}{a}$ was taken inſtead of a in the firſt Triplicate equality, gives $\frac{42}{120}$, which will ſolve the Queſtion; for if $\frac{42}{120}$ be multiplied ſeverally by 1, 2, 4, and the Products be added ſeverally to the Square of the ſaid $\frac{42}{120}$, the three ſumms will be Squares, whoſe ſides are $\frac{91}{120}$, $\frac{112}{120}$, $\frac{161}{120}$.

QUEST. 122.

To find three ſquare numbers, whoſe ſumm added to their three ſides ſeverally, may make Squares.

RESOLUTION.

1. Divide ſome ſquare number, ſuppoſe 121, into three ſuch Squares, that the greateſt may exceed the ſumm of the other two, ſuch are 4, 36, 81; then for the three Squares ſought put . . . . . $4aa$; $36aa$; $81aa$
2. The ſides of thoſe Squares are . . . . . $2a$, $6a$, $9a$
3. And the ſumm of the ſame Squares is . . . . . $121aa$
4. But if the ſaid ſumm $121aa$ be added to the three ſides in the ſecond ſtep ſeverally, every one of the three ſumms muſt make a Square; hence this Triplicate equality ariſeth, *viz.* . . . . $\begin{cases} 121aa + 2a = \square \\ 121aa + 6a = \square \\ 121aa + 9a = \square \end{cases}$
5. Which Triplicate equality; (by the method delivered in the preceding *Queſt.* 121.) will be reduced to this Triplicate equality, *viz.* . . . . . $\begin{cases} 121 + 2a = \square \\ 121 + 6a = \square \\ 121 + 9a = \square \end{cases}$
6. Now to reſolve the Triplicate equality laſt expreſt, I take (according to the Rule before given) $2aa + 2a$ for a new Root inſtead of a, whence the firſt part of the ſaid Triplicate equality will be ſolved indefinitely, and the ſecond and third parts will be converted into this Duplicate equality,

$$viz. \begin{cases} 12aa + 132a + 121 = \square \\ 18aa + 198a + 121 = \square \end{cases}.$$

7. Which Duplicate equality being reſolved in the ordinary way will give $a = -\frac{212}{47}$, according to which, the new Root $2aa + 22a$ will be found $\frac{4829440}{2209}$, and this divided by 1, (becauſe $\frac{1}{a}$ was put inſtead of a, in reducing the Triplicate equality in the fourth ſtep to that in the fifth,) gives $\frac{4829440}{629920}$ for the value of a in the Triplicate equality in the fourth ſtep; according to which laſt mentioned value, the three ſides in the ſecond ſtep are $\frac{44118}{629920}$, $\frac{122814}{629920}$, $\frac{228841}{629920}$, whoſe Squares will ſolve the Queſtion; for if thoſe ſides be ſeverally added to the ſumm of their Squares, the three ſumms will be Squares, whoſe ſides are $\frac{224629}{629920}$, $\frac{122841}{629920}$, $\frac{432241}{629920}$.

Again, if this Triplicate equality were propoſed to be reſolved, *viz.* . . . . . $\begin{cases} aa + 2a = \square \\ 4aa + 3a = \square \\ 16aa + 6a = \square \end{cases}$

It may be reduced to this by putting $\frac{1}{a}$ for a, . . . $\begin{cases} 1 + 2a = \square \\ 4 + 3a = \square \\ 16 + 6a = \square \end{cases}$

And that laſt expreſt, (by what hath been ſaid in *Queſt.* 120.) may be reduced to this, ... $\begin{cases} 64 + 128a = \square \\ 64 + 48a = \square \\ 64 + 24a = \square \end{cases}$

And this Triplicate equality being reſolved like thoſe in *Queſt.* 117, 120. will give $\frac{2106216}{529}$ for the value of a in either of the two laſt preceding Triplicate equalities; and laſtly, by dividing 1 by $\frac{2106216}{529}$, it gives $\frac{529}{2106216}$ for the value of a in the Triplicate equality propoſed in the eighth ſtep.

QUEST. 123. (Probl. 9. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, that if it be multiplied by three given numbers in Arithmetical Progreſſion, and the Products be ſubtracted from the Square of the number ſought, the three remainders may be Squares. But the following Reſolution preſuppoſeth, (for the reaſon before given in the *Note* at the end of *Queſt.* 117.) that the greateſt of the three numbers given is not equal to the ſumm of the other two. Let therefore the given numbers be 3, 4, 5.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then (according to the import of the Queſtion) this Triplicate equality muſt be reſolved, *viz.* . . . . . $\begin{cases} aa - 3a = \square \\ aa - 4a = \square \\ aa - 5a = \square \end{cases}$
3. But that (according to the *Rule* in *Queſt.* 121.) will be reduced to this, *viz.* . . . . . $\begin{cases} 1 - 3a = \square \\ 1 - 4a = \square \\ 1 - 5a = \square \end{cases}$

In which Triplicate equality laſt expreſt the value of the Root a was by *Queſt.* 118. found $-\frac{24}{121}$, but becauſe in the reduction of the Triplicate equality in the ſecond ſtep, $\frac{1}{a}$ was taken for a new Root inſtead of a, we muſt divide 1 by the ſaid $-\frac{24}{121}$, ſo there ariſeth $\frac{121}{24}$ for the number ſought by the Queſtion; for if $\frac{121}{24}$ be multiplied by 3, 4, 5 ſeverally, and the Products be ſeverally ſubtracted from the Square of $\frac{121}{24}$, the three remainders will be the Squares of $\frac{77}{24}$, $\frac{55}{24}$ and $\frac{11}{24}$.

Note 1. Sometimes, when four, five or more quantities are to be equated to Squares, they may be reſolved by the method before explain'd: As, for example,

If this Quadruplicate equality be propos'd to be reſolved, $\begin{cases} 20a + 64 = \square \\ 12a + 16 = \square \\ 8a + 4 = \square \\ 2a + 1 = \square \end{cases}$

You may (by the method in the preceding *Queſt.* 120.) reduce that four-fold equality to this, *viz.* . . . . . $\begin{cases} 20a + 64 = \square \\ 48a + 64 = \square \\ 128a + 64 = \square \\ 128a + 64 = \square \end{cases}$

Which laſt Quadruplicate equality is in effect but a Triplicate equality, for there are two Terms which happen to be the ſame, to wit, $128a + 64$; and therefore you may reſolve that Triplicate equality by the method before delivered.

Note 2. Sometimes alſo by the preceding method of reſolving a Triplicate equality of the firſt kind, you may reſolve one of *Diophantus*'s kinds of Duplicate equality, (to wit, that explain'd in the preceding *Queſt.* 33.) more eaſily than by his method, as will appear by the following *Queſt.* 124, 125, 126.

QUEST. 124. (The ſame with the foregoing *Queſt.* 33.)

To find a number leſs than 2, and ſuch, that if it be multiplied ſeverally by two numbers given, ſuppoſe by 6 and 8, and to each of the Products there be added the ſame given Square 4, the two ſumms may be Squares.

RESOLUTION.

1. For the number ſought put . . . . . a
2. Then the Queſtion requires this Duplicate equality to be reſolved, ... $\begin{cases} 6a + 4 = \square \\ 8a + 4 = \square \end{cases}$
3. To which end you may proceed thus; Firſt, (according to the method of reſolving a Triplicate equality before delivered,) form a Square from $a + 2$, (2 being the ſide of the given Square 4,) ſo that Square will be $aa + 4a + 4$; then take $\frac{3}{6}$ part of $aa + 4a$, ($\frac{1}{6}$ part, becauſe 6 is prefixt to a in the firſt part of the given Duplicate equality,) and it is $\frac{1}{6}aa + \frac{2}{3}a$, which is to be aſſumed for a new Root inſtead of a. Whence 'tis evident, that if the given Square 4 be added to ſix times $\frac{1}{6}aa + \frac{2}{3}a$, it makes a Square, to wit, $aa + 4a + 4$, whereby the firſt part of the Queſtion is ſolved indefinitely. Then multiply the new Root $\frac{1}{6}aa + \frac{2}{3}a$ by 8, and to the Product add 4, ſo

ſo there comes forth $\frac{4}{3}aa + \frac{16}{3}a + 4$ to be equated to a Square, the ſide whereof muſt be ſo feigned that $\frac{1}{6}aa + \frac{2}{3}a$ (the new Root) may be leſs than 2; but if $\frac{1}{6}aa + \frac{2}{3}a$ be leſs than 2, it will follow that a is leſs than 2. Wherefore let $\frac{4}{3}aa + \frac{16}{3}a + 4$ be equated to a Square, ſo, as that a may be leſs than 2; to which end the ſide of the ſaid Square may be feigned $-2 +$ any number of a greater than $3\frac{44}{100}a$; let therefore the ſaid ſide be feigned $4a - 2$, and then the Square of $4a - 2$ being equated to $\frac{4}{3}aa + \frac{16}{3}a + 4$, will give $a = \frac{16}{11}$, by which, if the new Root $\frac{1}{6}aa + \frac{2}{3}a$ be reſolved, it makes $\frac{160}{121}$ for the number ſought by the Queſtion propoſed: For firſt, it is leſs than 2; ſecondly, ſix times $\frac{160}{121}$ together with 4 makes a Square, to wit, $\frac{1444}{121}$, whoſe ſide is $\frac{38}{11}$; and laſtly, eight times $\frac{160}{121}$ together with 4 makes the Square $\frac{1764}{121}$, whoſe ſide is $\frac{42}{11}$.

Again, if this Duplicate equality were propoſed, ... $\begin{cases} 4 - 6a = \square \\ 4 - 8a = \square \end{cases}$

You may put $\frac{2}{3}a - \frac{1}{6}aa$ for a new Root inſtead of a, and then proceed as before.

Again, if this Duplicate equality were propoſed, ... $\begin{cases} 4 + 6a = \square \\ 4 - 8a = \square \end{cases}$

You may take $\frac{1}{6}aa + \frac{2}{3}a$ for a new Root inſtead of a, and then proceed as before.

QUEST. 125. (Probl. 10. in cap. 1. part. 2. *Dioph. redivivi.*)

To find a number, as alſo three other numbers in Geometrical proportion, that if from theſe Proportionals ſeverally the firſt number be ſubtracted, the three remainders may be Squares.

RESOLUTION.

1. For the firſt number ſought put . . . . . $2a - 1$
2. And for the three continual Proportionals ſought put . . . $a, 2a, 4a$
3. From which if you ſubtract the firſt number ſought, the remainders are . . . . . $1 - a, 1, 1 + 2a$
4. Among which remainders the mean is a Square, wherefore it remains to equate each of the two extremes to a Square, *viz.* $\begin{cases} 1 - a = \square \\ 1 + 2a = \square \end{cases}$

Now to reſolve that Duplicate equality, you may take $2aa + 2a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $4aa + 4a + 1$, and $1 - a$ into $1 - 2a - 2aa$, which muſt be equated to a Square, but with this Caution, That the new Root $2aa + 2a$ may be leſs than 1, and that $4aa + 4a$ may exceed 1, and conſequently that the value of a may fall between $-\frac{20}{100}$ and $-\frac{16}{100}$; to which end, the ſide of the ſaid Square may be feigned $1 - 2a$, whoſe Square $4aa - 4a + 1$ being equated to $1 - 2a - 2aa$, gives $a = \frac{1}{3}$; therefore the new Root $2aa + 2a$ is $\frac{8}{9}$, and the firſt number ſought, which was repreſented by $2a - 1$, is conſequently $\frac{7}{9}$, and the three deſired Proportionals are $\frac{8}{9}$, $\frac{12}{9}$, $\frac{18}{9}$; which will ſolve the Queſtion, as may eaſily be proved.

QUEST. 126.

To find two numbers, ſuch, that if their ſumm be increaſed and leſſened, as well by their difference as the difference of their Squares, the ſumms and remainders may be Squares.

RESOLUTION.

1. If unity be divided into any two parts their difference is equal to the difference of their Squares, (as hath been ſhewn in *Queſt.* 53. of this Book,) therefore for the two numbers ſought put . . . . . $\frac{1}{2} + a$ and $\frac{1}{2} - a$
2. Whence their difference, as alſo the difference of their Squares is . . . $2a$
3. It remains therefore that the ſumm of the two numbers in the firſt ſtep, to wit, unity, being increaſed and leſſened by $2a$, the ſumm and remainder may be Squares; hence this Duplicate equality ariſeth, *viz.* . . . . . $\begin{cases} 1 + 2a = \square \\ 1 - 2a = \square \end{cases}$
4. Now to reſolve that Duplicate equality, you may take $\frac{1}{2}aa + a$ for a new Root inſtead of a, whence $1 + 2a$ will be converted into the Square $aa + 2a + 1$, and $1 - 2a$ into $1 - 2a - aa$, which muſt be equated to a Square, but with this Caution, That the new Root $\frac{1}{2}aa + a$ may be leſs than $\frac{1}{2}$, and conſequently a leſs than $\sqrt{2} - 1$, that

that is, leſs than $\frac{414}{1000}$; but to cauſe that effect, the ſide of the ſaid Square may be feigned to be $1 -$ any number of a greater than a, as $1 - 3a$, the Square whereof being equated to the ſaid $1 - 2a - aa$ will give $a = \frac{2}{5}$, therefore the new Root $\frac{1}{2}aa + a$ is $\frac{12}{25}$, according to which, the Poſitions in the firſt ſtep being expounded, will give $\frac{42}{50}$ and $\frac{1}{50}$ for the numbers ſought: For if to their ſumm, which is 1, you add and ſubtract their difference $\frac{48}{50}$, the ſumm and remainder are Squares, to wit, $\frac{42}{25}$ and $\frac{1}{25}$; and theſe will alſo come forth when the difference of the Squares of the two numbers $\frac{42}{50}$ and $\frac{1}{50}$ is added to and ſubtracted from their ſumm 1, becauſe the difference of the two numbers is equal to the difference of their Squares.

Concerning the third Part of Fermat's *Analytical Invention.*

I. The Scope of this third Part is chiefly to ſhew how to equate a quantity compos'd of five Terms, *viz.* of ſome numbers of *aaaa, aaa, aa, a* with an abſolute (or known) number, to a Square; as alſo to equate a quantity compos'd of four Terms to a Square or Cube; and that in ſuch manner, that for the moſt part innumerable values of the unknown Root a may be found out.

II. In equating a quantity compos'd of five Terms to a Square, one of theſe two things is abſolutely neceſſary, *viz.* either the firſt Term muſt be a Biquadrate, or elſe the laſt Term, to wit, the abſolute number, muſt be a rational Square. Likewiſe, in equating a quantity conſiſting of four Terms to a Square, one of the extremes muſt be a Square. And laſtly, in equating a quantity of four Terms to a Cube, one of the extremes muſt be a Cube.

III. When a quantity compos'd of five Terms is given to be equated to a Square, and the firſt Term is a Biquadrate, but the abſolute number, that is, the laſt Term, hath not a rational ſquare Root, then the ſide of the Square muſt be feigned ſo, as that in the Square it ſelf there may be found the ſame numbers of *aaaa, aaa* and *aa* as are in the quantity given to be equated, to the end that thoſe three firſt Terms may by Reduction deſtroy one another, and conſequently an Equation remain between ſome number of a and an abſolute number, whence the value of the Root a, if it hath any poſſible value, may be expreſſible by ſome rational number either affirmative or negative; but how to feign the ſaid ſide ſo as to cauſe that effect, the following Propoſition and Canon will ſhew; where to evidence the certainty thereof, I ſhall aſſume b, c, d to ſtand for Coefficients or known numbers prefixt to *aaa, aa* and a, and n for the abſolute number, (or laſt Term,) which in this caſe is ſuppoſed to have no rational ſquare Root.

P R O P.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . . $aaaa + baaa + caa + da + n = \square$

C A N O N.

2. When $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, then let the ſide of the Square ſought be feigned . . . . . $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$
3. But if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, then let the ſide of the Square be feigned . . . . . $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers.

4. Let this quantity be given to be equated to a Square, *viz.* . . . . . $aaaa + 4aaa + 6aa + 2a + 7 = \square$
5. Here, becauſe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, *viz.* half 6 exceeds $\frac{1}{8}$ of the Square of 4, the firſt part of the Canon gives this feigned ſide of a Square, *viz.* . $aa + 2a + 1$
6. Then by equating the Square of the ſaid ſide $aa + 2a + 1$ to the given quantity $aaaa + 4aaa + 6aa + 2a + 7$, the value of a, after due Reduction, will be found 3, by which if the given quantity be reſolved, it makes the Square 256, whoſe ſide is 16.
7. Again, let this quantity be given to be equated to a Square, ... $aaaa + 4aaa + 2aa - 6a + 11 = \square$
8. Here, becauſe $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, *viz.* $\frac{1}{8}$ of the Square of 4 exceeds the half of 2, the latter part of the Canon gives this feigned ſide of a Square, *viz.* $1 - 2a - aa$

9. Then

9. Then by equating the Square of the ſaid feigned ſide $1 - 2a - aa$ to the given quantity $aaaa + 4aaa + 2aa - 6a + 11$, the value of the Root a will be found 5, by which if the given quantity be reſolved it makes the Square 1156, whoſe ſide is 34.

IV. When a quantity compos'd of five Terms is to be equated to a Square, and the the firſt Term is not a perfect Biquadrate, but the laſt Term, that is, the abſolute number is a Square, then the ſide of a Square muſt be feigned ſo, as that in the Square it ſelf there may be found the ſame numbers of *aa*, a and abſolute number, as are in the quantity given to be equated; to the end that thoſe three laſt Terms by due Reduction may vaniſh out of each part, and an Equation remain between ſome numbers of *aaaa* and *aaa*, whence the value of the Root a, if it hath any poſſible value, may be expreſſible by ſome rational number either affirmative or negative. But how to feign the ſaid ſide ſo as to cauſe ſuch an effect, the following Propoſition and Canon will ſhew; where to evidence the certainty thereof, I put b, c, d to ſtand for the Coefficients or known numbers prefixt to *aaa, aa, a*; alſo, rr (whoſe ſide is r) for the laſt Term, which in this Caſe is a rational ſquare number, and f, which is prefixt to *aaaa*, ſtands for a number not a Square.

P R O P.

1. Let this quantity be given to be equated to a Square, *viz.* . . . . . $faaaa + baaa + caa + da + rr = \square$

C A N O N.

2. When $4crr$ exceeds dd, then let the ſide of the Square ſought be feigned . . . . . $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$
But if dd exceeds $4crr$, then let the ſide of the Square be feigned . . . . . $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

Examples in Numbers.

3. Let this quantity be given to be equated to a Square, *viz.* . . . . . $10aaaa + 4aaa + 19aa + 6a + 9 = \square$
4. Here, becauſe $4crr$ exceeds dd, *viz.* four times $19 \times 9 \times 9$ exceeds the Square of 6, the firſt part of the Canon gives this feigned ſide of a Square, *viz.* . . . . . $3aa + 3a + 3$
5. Then by equating the Square of the ſaid ſide $3aa + 3a + 3$ to the given quantity $10aaaa + 4aaa + 19aa + 6a + 9$, the value of the Root a will be found 2, according to which the ſaid given quantity being expounded makes the Square 289, whoſe ſide is 17.
6. Again, let this quantity be given to be equated to a Square, *viz.* . . . . . $2aaaa + 3aaa + 3aa + 6a + 1 = \square$
7. Here, becauſe dd exceeds $4crr$, *viz.* the Square of 6 exceeds four times $3 \times 1 \times 1$, the latter part of the Canon gives this feigned ſide of a Square, *viz.* . . . . $1 + 3a - 3aa$
8. Then by equating the Square of the ſaid ſide $1 + 3a - 3aa$, to the given quantity $2aaaa + 3aaa + 3aa + 6a + 1$, the value of the Root a will be found 3; according to which, the ſaid given quantity being expounded makes the Square 289.

V. When a Quantity compos'd of five Terms given to be equated to a Square is ſuch, that the firſt Term is a Biquadrate, and the laſt Term (that is, the abſolute number) hath a rational ſquare Root, then the ſide of a Square to be equated to ſuch Quantity may be varied ſix ſeveral ways, (including into this number the two laſt preceding Canons,) by every one of which the value of the Root a may oftentimes be found out, and expreſt by a rational number either affirmative or negative. To evidence this, I ſhall (as before) put b, c, d to ſtand for the Coefficients or known numbers prefixt to *aaa, aa* and a; and rr (whoſe ſide is r,) for the rational ſquare number which is the laſt Term. As, for example,

Let this Quantity be propos'd to be equated to a Square, *viz.* . . . . . $aaaa + baaa + caa + da + rr$

1. Then to the end that $aaaa + da + rr$ may be found in a Square to be equated to the quantity propoſed, the ſide of that Square may be feigned $aa + \frac{d}{2r}a + r$

2. Or,

2. Or, to cauſe the ſame effect, the ſide of the ſaid Square may be feigned . . . . . } $r + \frac{d}{2r}a - aa$

3. Again, that $caa + da + rr$ may be found in a Square to be equated to the quantity propoſed, the ſide of ſuch Square, when $4crr$ exceeds dd, may (agreeable to the Canon in the preceding *Sect.* 4.) be feigned . . } $\frac{4crr - dd}{8rrr}aa + \frac{d}{2r}a + r$

4. But to cauſe the ſame effect, if dd exceeds $4crr$, then let the ſide of the Square be feigned . . . . } $r + \frac{d}{2r}a - \frac{dd - 4crr}{8rrr}aa$

5. Again, that $aaaa + baaa + rr$ may be found in a Square to be equated to the quantity propoſed, let the ſide of the Square be feigned . . . . . } $aa + \frac{1}{2}ba + r$

6. Or, to cauſe the ſame effect, the ſide of the ſaid Square may be feigned . . . . . } $aa + \frac{1}{2}ba - r$

7. Again, that $aaaa + baaa + caa$ may be found in a Square to be equated to the quantity propoſed, the ſide of that Square, when $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, may (agreeable to the Canon in *Sect.* 3.) be feigned . . . . } $aa + \frac{1}{2}ba + \frac{1}{2}c - \frac{1}{8}bb$

8. But, to cauſe the ſame effect, if $\frac{1}{8}bb$ exceeds $\frac{1}{2}c$, let the ſide of the ſaid Square be feigned . . . . . } $\frac{1}{8}bb - \frac{1}{2}c - \frac{1}{2}ba - aa$

Examples in Numbers of the preceding ſides of Squares expreſt by Letters.

Let this quantity be propoſed to be equated to a Square, *viz.* . . . . . } $aaaa + 4aaa + 10aa + 20a + 1 = \square$

1. Then ſuppoſing b, c, d, r, to ſtand for 4, 10, 20, 1, the firſt of the preceding literal ſides will give . . . . . } $aa + 10a + 1$

The Square of which ſide $aa + 10a + 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, there will ariſe, after due Reduction, $16aaa = -92aa$; whence by dividing each part by $16aa$, there comes forth $-5\frac{3}{4}$ for the value of the Root a, according to which, the propoſed quantity being expounded will give the Square $\frac{1400225}{15625}$, whoſe ſide is $\frac{375}{125}$.

2. Again, the third literal ſide gives . . . } $1 + 10a - aa$

The Square of which ſide $1 + 10a - aa$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -1\frac{1}{3}$, according to which, the ſame quantity being reſolved, makes the Square of $2\frac{7}{9}$.

3. Again, becauſe in the quantity propoſed, dd exceeds $4crr$, the fourth literal ſide gives . } $1 + 10a - 45aa$

The Square of which ſide $1 + 10a - 45aa$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = \frac{252}{253}$, according to which, the ſame quantity being expounded makes the Square of the ſide $\frac{254106}{64009}$.

4. Again, the fifth literal ſide gives . . . } $aa + 2a + 1$

The Square of which ſide $aa + 2a + 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -4$, according to which, the ſame quantity being expounded makes the Square 81.

5. Again, the ſixth literal ſide gives . . . } $aa + 2a - 1$

The Square of which ſide $aa + 2a - 1$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = -3$, according to which the ſaid quantity being reſolved makes the Square 4.

6. Laſtly, becauſe $\frac{1}{2}c$ exceeds $\frac{1}{8}bb$, the ſeventh literal ſide gives . . . . . } $aa + 2a + 3$

The Square of which ſide $aa + 2a + 3$ being equated to the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$, will give $a = 1$, according to which the ſame quantity being reſolved makes the Square 36.

VI. Some-

VI. Sometimes, when a quantity compos'd of five Terms is equated to a Square formed from one of the eight literal ſides before expreſt in *Sect.* V. no value of the Root a either affirmative or negative can thence be diſcovered. As, for example,

If this quantity be propos'd to be equated to a Square, *viz.* . . . . } $aaaa + 18aaa + 120aa + 351a + 1519 = \square$

The Canon in *Sect.* III. (or the ſeventh literal ſide in *Sect.* VI.) will give this feigned ſide of a Square, *viz.* . } $aa + 9a + 19\frac{1}{2}$

The Square of which ſide $aa + 9a + 19\frac{1}{2}$ is . . . . . } $aaaa + 18aaa + 120aa + 351a + 380\frac{1}{4}$

Which Square being equated to the quantity propoſed will give this fruitleſs and abſurd Equation, *viz.* . . . . } $1138\frac{1}{4} = 0$

VII. When negative Terms are intermingled with affirmative, in a quantity compos'd of five Terms given to be equated to a Square, the ſide of the Square, (when ſuch an Equation is poſſible,) ſhall be one of the eight literal ſides before expreſt in *Sect.* V. ſaving that one, and ſometimes two of its ſigns $+$ muſt be changed into $-$. As, for example,

If $aaaa - 8aaa + 28aa - 40a + 4$ be given to be equated to a Square, it may be variouſly done, in regard the extremes $aaaa$ and 4 are Squares. Firſt then, I imagine all the Terms of the propoſed quantity to be affirmative, ſo it will be $aaaa + 8aaa + 28aa + 40a + 4$; now to feign the ſide of a Square, that $aaaa + 8aaa + 4$ may by due Reduction vaniſh out of each part, the fifth literal ſide in *Sect.* V. being reſolved into numbers will give $aa + 4a + 2$ for the feigned ſide; but here two of its ſigns $+$ muſt be changed into $-$, that in its Square there may be found $aaaa - 8aaa + 4$ to deſtroy $aaaa - 8aaa + 4$ in the quantity given to be equated, to which end, the ſaid ſide $aa + 4a + 2$ muſt be converted into $aa - 4a - 2$, and then the Square of this ſide being equated to the propoſed quantity $aaaa - 8aaa + 28aa - 40a + 4$ will give $a = \frac{2}{2}$.

Again, to feign the ſide of a Square to be equated to the ſame given quantity $aaaa - 8aaa + 28aa - 40a + 4$, ſo, as to deſtroy the firſt, ſecond and laſt Terms in each part, the firſt of the literal ſides in *Sect.* V. being reſolved into numbers, gives $aa + 10a + 2$ for the feigned ſide, if all the Terms of the given quantity were affirmative; but that $aaaa - 8aaa + 4$ may vaniſh out of each part, the ſaid ſide $aa + 10a + 2$ muſt be changed into $aa - 10a + 2$, and then the Square of this ſide being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = \frac{2}{3}$.

Again, to feign the ſide of a Square to be equated to the ſame given quantity $aaaa - 8aaa + 28aa - 40a + 4$, in ſuch manner that $aaaa - 40a + 4$ may vaniſh out of each part, the firſt of the eight literal ſides in *Sect.* V. being reſolved into numbers, gives $aa + 10a + 2$ for the feigned ſide, if all the Terms of the given quantity were affirmative; but that $aaaa - 40a + 4$ may be expunged out of each part, the ſaid ſide $aa + 10a + 2$ muſt be changed into $2 - 10a - aa$, and then the Square of this ſide being equated to the given quantity $aaaa - 8aaa + 28aa - 40a + 4$, will give $a = -\frac{1}{7}$. Other ſides might likewiſe be feigned, as is evident by the foregoing *Sect.* V. and to him that is a little exercis'd in this Method it will not be difficult to change the Signs in any poſſible caſe.

VIII. When a quantity conſiſting of five Terms is to be equated to a Square, and one or more values of the Root a are found out, either affirmative or negative, by the Rules before given, you may from every one of thoſe Primitive Roots or values, find out other values of the Root a, even as many as you pleaſe, which latter, *Fermat* calls Derivative Roots of the firſt, ſecond, third, *&c.* degree. As, for example,

To find a Derivative Root of the firſt degree out of the quantity $aaaa + 4aaa + 10aa + 20a + 1$, before propoſed in *Sect.* V. to be equated to a Square, take one of its Primitive Roots there found out, to wit, -3, and connect it to a, ſo it makes $a - 3$, then inſtead of a take $a - 3$ for a new Root, according to which the propoſed quantity $aaaa + 4aaa + 10aa + 20a + 1$ will be converted into $aaaa - 8aaa + 28aa - 40a + 4$, as here you ſee;

aaaa	*aaaa* — 12*aaa* -|- 54*aa* — 108*a* -|- 81
4*aaa*	-|- 4*aaa* — 36*aa* -|- 108*a* — 108
10*aa*	-|- 10*aa* — 60*a* -|- 90
20*a*	-|- 20*a* — 60
1	-|- 1
Summ,	*aaaa* — 8*aaa* -|- 28*aa* — 40*a* -|- 4

This ſumm muſt be equated to a Square, whoſe ſide (as before hath been ſhewn in *Sect.* 7.) may be feigned $aa - 4a - 2$, the Square whereof being equated to the ſaid ſumm will give $a = \frac{7}{2}$; but becauſe the new Root was put $a - 3$, out of $\frac{7}{2}$ ſubtract 3, and there will remain $\frac{1}{2}$ for a ſecond value of the Root a in the propoſed quantity $aaaa$ -\|- $4aaa$ -\|- $10aa$ -\|- $20a$ -\|- 1, which ſecond value may be called a Derivative Root of the firſt degree.

In like manner by the help of the ſaid ſecond value $\frac{1}{2}$ you may find out a third, by joyning -\|- $\frac{1}{2}$ to a, and taking a -\|- $\frac{1}{2}$ for a new Root, according to which, the given quantity $aaaa$ -\|- $4aaa$ -\|- $10aa$ -\|- $20a$ -\|- 1 will be converted into $aaaa$ -\|- $6aaa$ -\|- $8\frac{1}{2}aa$ -\|- $\frac{67}{2}a$ + $\frac{225}{16}$ to be equated to a Square, the ſide whereof may be feigned aa + $3a$ + $\frac{15}{4}$, (agreeable to the fifth literal ſide in *Sect.* 5.) the Square whereof being equated to the given quantity, there will thence ariſe $a = -11$, therefore the new Root a + $\frac{1}{2}$ gives $a = -\frac{21}{2}$ for a third value of the Root a in the given quantity, that is, a Derivative Root of the ſecond degree.

Nor will the Operation be otherwiſe to find out a fourth value, or derivative Root of the third degree, by putting for a new Root $a - \frac{21}{2}$, for according to this, every member of the propoſed quantity $aaaa$ -\|- $4aaa$ -\|- $10aa$ -\|- $20a$ -\|- 1 being reſolved, there will come forth $aaaa$ — $38aaa$ -\|- $\frac{1021}{2}aa$ — $\frac{6221}{2}a$ -\|- $\frac{132481}{16}$, this ſumm is to be equated to a Square, whoſe ſide may be feigned $aa - 19a - \frac{361}{4}$, and the Square of this ſide being equated to the ſaid ſumm will give $a = \frac{8121}{46}$, from which if you ſubtract $\frac{21}{2}$, (becauſe the new Root was put $a - \frac{21}{2}$,) there will remain $\frac{1221}{23}$ for a fourth value of the Root a, that is, a derivative Root of the third degree, out of the quantity firſt propoſed to be equated to a Square.

Laſtly, as by the help of one of the primitive Roots of the propoſed quantity $aaaa$ -\|- $4aaa$ -\|- $10aa$ -\|- $20a$ -\|- 1 other Roots have been derived, ſo by the help of any one of the reſt of the primitive Roots of the ſame quantity, found out in the Examples of *Sect.* 5. you may proceed to find out other derivative Roots, but ſometimes you will meet with fruitleſs Equations.

IX. A quantity compos'd of four Terms may be equated to a Square, when either the abſolute number, that is, the laſt Term is a Square, or the firſt Term a Biquadrate.

Firſt, let $20aaaa$ -\|- $5aa$ -\|- $40a$ -\|- 16 be given to be equated to a Square. Feign the ſide ſo, that $40a$ -\|- 16 may vaniſh out of each part, to which purpoſe, let the ſide be $5a$ + 4, (5 being the Quotient that ariſeth by dividing 40 in the $4a$, by 8 the double of the ſide of the given Square 16;) then by equating the Square of $5a$ -\|- 4 to the given quantity $20aaaa$ + $5aa$ + $40a$ + 16 you will find $a = 1$, according to which, that quantity being reſolved makes the Square 81. Now to find a ſecond value of the Root a, you may put for a new Root a + 1, according to which, the given quantity $20aaaa$ + $5aa$ + $40a$ + 16 will be converted into $20aaaa$ + $65aa$ + $110a$ + 81 to be equated to a Square, the ſide whereof, (that $110a$ + 81 may vaniſh out of each part,) may be feigned $\frac{11}{9}a$ + 9, whence after due Reduction, there will ariſe $a = -\frac{112}{81}$; therefore a + 1 (which was put for the new Root) gives $1 - \frac{112}{81}$, that is, $\frac{31}{81}$ for a ſecond value of the Root a, (or a derivative Root of the firſt degree,) and by putting a + $\frac{31}{81}$ for a new Root you may find out a third value, and ſo infinitely.

Secondly, an Example where the firſt Term is $aaaa$ may be this, *viz.* Let $aaaa$ + $4aaa$ — $3aa$ + $2a$ be given to be equated to a Square. That $aaaa$ + $4aaa$ may vaniſh out of each part, feign the ſide of a Square to be aa + $2a$, (2 in the $2a$ being the half of 4 prefixt to aaa in the given quantity;) then the Square of aa + $2a$ being equated to $aaaa$ + $4aaa$ — $3aa$ + $2a$ will give $a = \frac{2}{7}$; and to find out a ſecond value of a you may put a + $\frac{2}{7}$ for a new Root, and proceed as in former Examples.

Thirdly,

Thirdly, although ſome intermediate Term be omitted in a quantity compos'd of four Terms, ſuch quantity may be equated to a Square: As, to equate $5aaaa$ -\|- $16aaa$ -\|- $24aa$ + 16 to a Square, you may feign its ſide to be $3aa$ + 4, (3 in the $3aa$ being the Quotient that ariſeth by dividing 24 which is prefixt to aa in the given quantity, by 8 the double of the ſide of the given Square 16,) and thence the value of a will be found 4; then you may put a + 4 for a new Root to find out a ſecond value of a.

In like manner, if $aaaa$ + $60aa$ + $80a$ + 500 be propoſed to be equated to a Square, you may feign its ſide to be aa + 30, (30 being the half of 60 which is prefixt to aa in the propoſed quantity,) whence you will find $a = 5$, and for a derivative Root you may put a + 5.

X. A Quantity compos'd of four Terms may be equated to a Cube, when either the abſolute number, (that is, the laſt Term,) or the firſt Term is a perfect Cube.

Firſt, let $2aaa$ -\|- aa -\|- $3a$ -\|- 1 be given to be equated to a Cube. That the two laſt Terms may vaniſh out of each part, feign the ſide of a Cube to be a -\|- 1, (1 being the ſide of the given Cube 1, and a being the Quotient that ariſeth by dividing $3a$ in the given quantity by 3 the triple Square of 1, the cubick Root of the given Cube 1,) then the Cube of a -\|- 1, that is, aaa -\|- $3aa$ -\|- $3a$ -\|- 1 being equated to the given quantity $2aaa$ -\|- aa -\|- $3a$ -\|- 1, will give $a = 2$; then to find out a ſecond value of a you may put a -\|- 2 for a new Root.

Secondly, but if the firſt Term be a rational Cube; as, if $8aaa$ -\|- $24aa$ -\|- $2a$ -\|- 48 be given to be equated to a Cube; that the firſt and ſecond Terms may vaniſh out of each part, feign the ſide of a Cube to be $2a$ -\|- 2, ($2a$ being the ſide of the Cube $8aaa$; and 2 being the Quotient that ariſeth by dividing 24 which is prefixt to aa, by 12 the triple Square of 2 the cubick Root of 8 in $8aaa$;) then the Cube of $2a$ -\|- 2 being equated to the given quantity $8aaa$ -\|- $24aa$ -\|- $2a$ -\|- 48, will give $a = \frac{22}{11}$; whence you may find out derivative Roots as before.

XI. If the firſt Term of a Quantity compos'd of four Terms given to be equated to a Cube be a rational Cube, and the laſt Term, to wit, the abſolute number be alſo a Cube, then that given Quantity may be equated to a Cube in a threefold manner.

As, for exampie, if aaa -\|- $2aa$ -\|- $4a$ -\|- 1 be propoſed to be equated to a Cube; firſt, that the firſt and laſt Terms may vaniſh out of each part, feign the ſide of the Cube to be a + 1, (which is compos'd of the cubick Roots of aaa and 1; then the Cube of a -\|- 1 being equated to the quantity propoſed will give $a = 1$. Secondly, that the firſt and ſecond Terms may vaniſh out of each part, you may feign the ſide to be a -\|- $\frac{2}{3}$, (a being the ſide of the Cube aaa, and $\frac{2}{3}$ being the Quotient that ariſeth by dividing 2 which is prefixt to aa, by 3 the triple Square of the cubick Root of 1 which is prefixt to aaa,) whence $a = -\frac{22}{7^2}$. And laſtly, that the third and fourth Terms may vaniſh, you may feign the ſide to be $\frac{2}{3}a$ -\|- 1, (1 being the ſide of the given Cube 1, and $\frac{2}{3}a$ being the Quotient that ariſeth by dividing $4a$ by 3 the triple Square of the cubick Root of the given Cube 1,) whence $a = \frac{22}{37}$; and by the help of thoſe three primitive Roots you may find out derivatives, in like manner as before.

XII. Sometimes when a Quantity compos'd of four Terms, whereof one or both the extremes are Cubes, is to be equated to a Cube, no value of the Root a either affirmative or negative can be found out by any of the Rules before delivered.

As, if $4aaaa$ -\|- $3aa$ -\|- $3a$ -\|- 1 be given to be equated to a Cube, its ſide can only be feigned a -\|- 1, the Cube whereof being equated to the given quantity will give $3aaa = 0$; and therefore the given quantity cannot be equated to a Cube.

Alſo, if aaa -\|- $2aa$ -\|- $3a$ -\|- 1 be to be equated to a Cube, there can but one primitive Root be found out, although there be a threefold way of feigning the ſide of the Cube according to *Sect.* XI. which primitive Root will be diſcovered from the feigned ſide a -\|- $\frac{2}{3}$, but neither of the other two ways will prove effectual.

I ſhall now add a few Queſtions to illuſtrate the foregoing Third Part of *Fermat's* Invention, and ſo conclude this Book.

QUEST. 127.

(The ſame with the foregoing Queſt. 105. *but reſolved after another manner.)*

To find a right-angled Triangle, that the Area being ſubtracted as well from the Hypothenuſal as from one of the ſides about the right-angle, each remainder may be a Square.

RESOLUTION.

1. Let h, b, p repreſent the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle in numbers. Divide thoſe three ſides ſeverally by a, and put the Quotients for the three ſides of the Triangle ſought, *viz.* . . . . . $\frac{h}{a}, \frac{b}{a}, \frac{p}{a}$

2. Then by ſubtracting the Area $\frac{\frac{1}{2}bp}{aa}$, as well from $\frac{b}{a}$, (one of the ſides about the right-angle, as from the Hypothenuſal $\frac{h}{a}$, each remainder muſt be equal to a Square, and by multiplying each remainder by the Denominator aa, this Duplicate equality ariſeth, *viz.* . . . . . $ba - \frac{1}{2}bp = \square$; $ha - \frac{1}{2}bp = \square$

3. Let the firſt of thoſe two quantities be equated to ſome Square, *viz.* ſuppoſe . . . . . $ba - \frac{1}{2}bp = bb$

4. Whence, after due Reduction, ... $a = b + \frac{1}{2}p$

5. Therefore by multiplying h into $b + \frac{1}{2}p$, (inſtead of a,) the latter of the two quantities in the ſecond ſtep, will be converted into this quantity, which muſt be equated to a Square, *viz.* . . . . . $hb + \frac{1}{2}hp - \frac{1}{2}bp = \square$

6. Now ſince h, b, p, were put for the Hypothenuſal, Baſe and Perpendicular of a right-angled Triangle, the quantity in the fifth ſtep ſhews that a right-angled Triangle muſt be found out ſuch, that if the Hypothenuſal be multiplied into the ſumm of one of the ſides about the right-angle and half the other ſide, and the Product be leſſened by the Area, the remainder muſt be a Square: But ſuch a right-angled Triangle, by *Fermat*'s method, (before explained,) may be found out thus, *viz.*

7. Form a right-angled Triangle from two numbers taken at pleaſaure, as from $a + 1$ and 1, ſo the three ſides will be theſe, *viz.* . . . . $aa + 2a + 2$ = Hypoth. ; $aa + 2a$ = Baſe, ; $+ 2a + 2$ = Perpend.

8. The Product of the Hypothenuſal into the ſumm of the Baſe and half the Perpendicular is . $aaaa + 5aaa + 9aa + 8a + 2$

9. The Area is . . . . . $. . + aaa + 3aa + 2a$

10. Which ſubtracted from the ſaid Product leaves this quantity to be equated to a Square, *viz.* $aaaa + 4aaa + 6aa + 6a + 2 = \square$

11. Feign the ſide of that Square, (according to the 7th literal ſide in the foregoing *Sect.* 5. *Part* 3.) $aa + 2a + 1$

12. Then the Square of the ſaid ſide $aa + 2a + 1$ being equated to the quantity in the tenth ſtep will give $a = -\frac{1}{2}$, therefore $a + 1$ and 1 the numbers forming the Triangle ſhall be $\frac{1}{2}$ and 1, or (in Integers in the ſame Reaſon) 1 and 2; by which, if a right-angled Triangle be formed, one of the ſides about the right-angle will be leſs than nothing, to wit, -3; (for the Square of 2 is to be ſubtracted from the Square of 1, becauſe 1 and 2 anſwer to $a + 1$ and 1 the numbers that formed the Triangle in the ſeventh ſtep;) to cauſe therefore all the ſides to be affirmative, the work muſt be renewed in manner following, *viz.*

13. Let a right-angled Triangle be formed from $a + 1$ and 2, ſo the three ſides will be theſe, *viz.* . . . . . $aa + 2a + 5$ = Hypoth. ; $aa + 2a - 3$ = Baſe, ; $+ 4a + 4$ = Perpend.

14. The Product of the Hypothenuſal into the ſumm of the Baſe and half the Perpendicular is $aaaa + 6aaa + 12aa + 18a - 5$

15. The Area is . . . . . $. . + 2aaa + 6aa - 2a - 6$

16. Which being ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz.* $aaaa + 4aaa + 6aa + 20a + 1 = \square$

17. The ſide of that Square may be variouſly feigned, (according to the preceding *Sect.* 5.) let it be the ſecond literal ſide in that *Sect.* viz. $1 + 10a - aa$

18. Then the Square of the ſaid ſide $1 + 10a - aa$ being equated to the quantity in the ſixteenth ſtep will give $a = \frac{22}{6}$; therefore $a + 1$ and 2 ſhall be $\frac{28}{6}$ and 2, that is, in Integers in the ſame Reaſon, 29 and 12, by which if you form a right-angled Triangle, the three ſides will be $985, 697, 696$, that is, h, b, p; then according to the Poſitions in the firſt ſtep, divide every one of thoſe ſides by 1045, that is, by $b + \frac{1}{2}p = a$, (as appears by the fourth ſtep,) ſo the Quotients $\frac{985}{1045}, \frac{697}{1045}, \frac{696}{1045}$ ſhall be the ſides of a right-angled Triangle to ſolve the Queſtion: For if the Area be ſubtracted from the Hypothenuſal $\frac{985}{1045}$ and the Baſe $\frac{697}{1045}$ ſeverally, the remainders will be the Squares of theſe ſides $\frac{288}{1045}$ and $\frac{627}{1045}$.

And becauſe the quantity in the ſixteenth ſtep is capable of being equated to innumerable Squares, (according to the Method before explained,) the Queſtion is alſo capable of innumerable Anſwers; but in larger numbers than thoſe, that may be found out by the foregoing *Queſt.* 105. which is the ſame with this.

QUEST. 128. (Probl. 1. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that if the double of its Area be ſubtracted from every one of the three ſides, the remainders may be Squares.

RESOLUTION.

1. Let b, p, h repreſent the Baſe, Perpendicular, and Hyothenuſe of a right-angled Triangle. Divide thoſe ſides ſeverally by a, and aſſume the Quotients to be the three ſides of the Triangle ſought, *viz.* . . . . . $\frac{b}{a}, \frac{p}{a}, \frac{h}{a}$

2. Then by ſubtracting the double Area $\frac{bp}{aa}$ from every one of thoſe three ſides, the remainders muſt be Squares, and multiplying the remainders ſeverally by the Denominator aa, this Triplicate equality ariſeth to be reſolved, *viz.* . . . $ba - bp = \square$; $pa - bp = \square$; $ha - bp = \square$

3. Now in order to reſolve that Duplicate equality, let the firſt of its three quantities be equated to ſome Square, *viz.* ſuppoſe $ba - bp = bb$

4. Whence, after due Reduction to find out the value of a, you will diſcover . . . . . $a = b + p$

5. Then by multiplying $b + p$, inſtead of a, into p, the ſecond of the three quantities in the ſecond ſtep (to wit, $pa - bp$,) will be converted into this quantity, which is manifeſtly a Square, *viz.* . . . . . $bp + pp - bp = \square = pp$

6. And by multiplying $b + p$, inſtead of a, into h, the third quantity in the ſecond ſtep will be converted into this quantity to be equated to a Square, *viz.* . . . . . $hb + hp - bp = \square$

7. Thus the Triplicate equality in the ſecond ſtep is reduced to a Duplicate equality in the fifth and ſixth ſteps, and becauſe the firſt of the two quantities in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate $hb + hp - bp$ (in the ſixth ſtep,) to a Square, which diſcovers the Scope of our ſearch muſt be this, *viz.* to find a right-angled Triangle, ſuch, that if the Hypothenuſal be multiplied by the ſumm of the ſides about the right-angle, and the Product be leſſened by the double of the Area, the remainder muſt be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a + 1$ and 2, ſo the three ſides will be theſe, *viz.* . . . . $aa + 2a + 5$ = Hyp. ; $aa + 2a - 3$ = Baſe, ; $+ 4a + 4$ = Perp.

9. The Product of the Hypothenuſal into the ſumm of the Baſe and Perpendicular is . . $aaaa + 8aaa + 18aa + 32a + 5$

10. The double Area is . . . . . $. . + 4aaa + 12aa - 4a$

11. Which ſubtracted from the ſaid Product, leaves this quantity to be equated to a Square, *viz.* . . . . . $aaaa + 4aaa + 6aa + 36a + 17 = \square$

12. Feign

12. Feign the side of that Square according to the Canon in the preceding *Sect.* 3. *Part* 3. and it will be . . . . . $aa + 2a + 1$

13. Then the Square of the said side $aa + 2a + 1$ being equated to the quantity in the eleventh step, will give $a = -\frac{1}{2}$; therefore $a + 1$ and 2, the numbers forming the Triangle in the eighth step, shall be $\frac{1}{2}$ and 2, or, (in Integers in the same Reason,) 1 and 4, by which if a right-angled Triangle be formed, one of the sides about the right-angle will be less than nothing, to wit, -15; (for the Square of 4 is to be subtracted from the Square of 1, because 1 and 4 answer to $a+1$ and 2, the numbers that formed the Triangle in the eighth step, where $a+1$ was supposed to exceed 2.) To cause therefore all the sides to be affirmative, the work must be renewed thus, *viz.*

14. Form a right-angled Triangle from $a + 1$ and 4, so the three sides will be these, *viz.*
$aa + 2a + 17 =$ Hyp.
$aa + 2a - 15 =$ Base,
$+ 8a + 8 =$ Perp.

15. The Product of the Hypothenusal into the summ of the Base and Perpendicular is . . $aaaa + 12aaa + 30aa + 156a - 119$

16. The double Area is . . . . . $+ 8aaa + 24aa - 104a - 120$

17. Which subtracted from the said Product, leaves this quantity to be equated to a Square, *viz* . . . . . $aaaa + 4aaa + 6aa + 260a + 1 = \square$

18. The side of that Square may be variously feigned, (according to the preceding *Sect.* V.) let it be the second literal side in that *Sect.* *viz.* . . . . . $1 + 130a - aa$

19. The Square of the said side $1 + 130a - aa$ being equated to the quantity in the seventeenth step, will give $a = \frac{4223}{66}$, therefore $a + 1$ and 4 shall be $\frac{4289}{66}$ and 4, or, (in Integers in the same Reason,) 4289 and 4223; from which, a right-angled Triangle being formed, the three sides will be 18465217, 18325825, 2264592, that is, h, b, p: Then according to the Positions in the first step, divide those three sides severally by 20590417, that is, by $b + p = a$, (as is evident by the fourth step;) so the Quotients $\frac{18465217}{20590417}$, $\frac{18325825}{20590417}$, $\frac{2264592}{20590417}$ shall be the sides of a right-angled Triangle to solve the Question, as may easily be proved.

Note 1. Although the Question be truly solved, yet 'tis evident that it was by chance that the Triplicate equality in the second step came to be reduced to a single equality; for if the quantity to be equated to a Square in the fifth step, had not happened to have been a Square, there would have been an inexplicable Duplicate equality.

Note 2. It is easie to perceive by the second, third and fourth steps, that instead of bp the double Area, the Product of bp multiplied by any square number may be given in the Question: As, if it were required to find out a right-angled Triangle, that $4bp$, that is, eight times the Area being subtracted from every one of the three sides may leave Squares, you need only to multiply the Denominator 20590417 of the three sides before found by 4, without altering the Numerators; or, if $9bp$, that is, eighteen times the Area, were prescribed, then to multiply the Denominator by 9.

QUEST. 129. (Probl. 1. in cap. 2. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, that the Product of the Hypothenusal into one of the sides about the right-angle, being subtracted from every one of the three sides, may leave Squares.

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, and for the three sides of the Triangle sought put . . . . . $\frac{h}{a}$; $\frac{b}{a}$, $\frac{p}{a}$

2. Then from $\frac{hp}{aa}$, (the Product of the Hypothenusal into the Perpendicular,) subtract every one of the three sides, and the remainders must be Squares; therefore also those remainders multiplied into the Denominator aa must make Squares; hence this Triplicate equality ariseth, *viz.* . . . . . $ha - hp = \square$
$ba - bp = \square$
$pa - bp = \square$

3. Now

3. Now in order to resolve that Triplicate equality, let the first of its three quantities be equated to some Square, *viz.* suppose $ha - hp = hh$

4. Whence, after due Reduction to find out the value of a, you will discover . . . . . $a = h + p$

5. Then by multiplying $h + p$, instead of a, into b, the second of the three quantities in the second step will be reduced to this to be equated to a Square, *viz.* . . . . . $hb + bp - hp = \square$

6. Likewise by multiplying $h + p$, instead of a, into p, the third quantity in the second step will be reduced to this quantity, which is manifestly a Square, *viz.* . . . . . $hp + pp - hp = \square = pp$

7. Thus the Triplicate equality in the second step is reduced to a Duplicate equality in the fifth and sixth steps; and because the latter quantity in that Duplicate equality happens to be a Square, to wit, pp, it remains only to equate the former, that is, $hb + bp - hp$, to a Square; which shews that a right-angled Triangle must be found, such, that if the summ of the Hypothenusal and Perpendicular be multiplied by the Base, and from the Product you subtract the Product of the Hypothenusal into the Perpendicular, the remainder may be a Square: But such a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from $a + 2$ and 1, (2 and 1 being numbers taken at pleasure,) so the three sides will be these, *viz.*
$aa + 4a + 5 =$ Hyp.
$aa + 4a + 3 =$ Base,
$+ 2a + 4 =$ Perp.

9. The summ of the Hypothenusal and Perpendicular being multiplied by the Base, produceth . . . . . $aaaa + 10aaa + 36aa + 54a + 27$

10. The Product of the Hypothenusal and Perpendicular is . . . . . $2aaa + 12aa + 26a + 20$

11. Which latter Product being subtracted from the former, leaves to be equated to a Square, ... $aaaa + 8aaa + 24aa + 28a + 7 = \square$

12. Feign the side of that Square according to the Canon in the preceding *Sect.* III. *Part* 3. and it will be . . . . . $aa + 4a + 4$

13. Then the Square of the said side being equated to the quantity in the eleventh step will give $a = -\frac{2}{4}$, and therefore $a + 2$ and 1, which were the numbers forming the Triangle in the eighth step, shall be -1 and 4, but one of these being negative, the work must be renewed; and now a right-angled Triangle may be confidently formed from $a - 1$ and 4, which Triangle being used like the former in the ninth, tenth and eleventh steps, at length there will remain $aaaa - 4aaa + 6aa - 260a + 1$ to be equated to a Square, the side whereof may be variously feigned, let it be $1 - 130a + aa$, then the Square of this side being equated to the said $aaaa - 4aaa + 6aa - 260a + 1$, will give $a = 66$; therefore $a - 1$ and 4 the numbers forming the Triangle shall be 65 and 4, by which if you form a right-angled Triangle, the three sides will be found 4241, 4209, 520, that is, h, b, p. Then according to the Positions in the first step, divide those three numbers severally by 4761, that is, by $b + p = a$, as appears by the fourth step, and the Quotients $\frac{4241}{4761}$, $\frac{4209}{4761}$, $\frac{520}{4761}$ shall be the sides of a right-angled Triangle to solve the Question, as may easily be proved.

QUEST. 130. (Probl. 39. in cap. 1. part. 1. *Dioph.* redivivi.)

To find a right-angled Triangle, whose Area subtracted from one of the sides about the right-angle, may leave a given number, suppose 2, (or n.)

RESOLUTION.

1. Let h, b, p represent the Hypothenusal, Base and Perpendicular of a right-angled Triangle, then multiply those sides severally by a, and put the Products for the sides of the Triangle sought, *viz.* . . . . . ha ; ba ; pa

2. The Area of which Triangle being subtracted from one of its sides about the right-angle, suppose from ba, must leave a remainder equal to the given number 2, (or n,) therefore . . . . . $ba - \frac{1}{2}bpaa = n = 2$

3. Which Equation divided by $\frac{1}{2}bp$, gives . . . . . $\frac{b}{\frac{1}{2}bp}a - aa = \frac{n}{\frac{1}{2}bp}$

4. Now

4. Now that the value of a in the laſt Equation may be expreſſible by a rational number, it is evident by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. that if the abſolute quantity in the latter part of that Equation be ſubtracted from the Square of half the Coefficient which is drawn into a, the remainder muſt have a rational ſquare Root, therefore . . . . . $\frac{\frac{1}{4}bb - \frac{1}{2}bpn}{\frac{1}{4}bbpp} = \square$

5. And becauſe the Denominator $\frac{1}{4}bbpp$ is a Square, it remains only to equate the Numerator to a Square, therefore . . . . $\frac{1}{4}bb - \frac{1}{2}bpn = \square$

6. Or, to avoid Fractions, let the ſaid $\frac{1}{4}bb - \frac{1}{2}bpn$ be multiplied by 4, and then it remains to equate . . . . . $bb - 2bpn = \square$

7. Whence it appears, that in order to ſolve the Queſtion propoſed, a right-angled Triangle muſt firſt be found, ſuch, that if from the Square of one of its ſides about the right-angle, the Product of the quadruple of the Area multiplied by the given number n be ſubtracted, the remainder may be a Square: But ſuch a right-angled Triangle may be found out thus, *viz.*

8. Form a right-angled Triangle from two numbers taken at pleaſure, as from $a - 1$ and 4, ſo the three ſides will be theſe, *viz.* . . . . . $aa - 2a + 17 = \text{Hyp.}$; $aa - 2a - 15 = \text{Baſe}$; $+ 8a - 8 = \text{Perp.}$

9. The Square of the Baſe is . . . . $aaaa - 4aaa - 26aa + 60a + 225$

10. The Product of the quadruple of the Area into the given number 2 is . . . $+ 32aaa - 96aa - 416a + 480$

11. Which Product being ſubtracted from the ſaid Square of the Baſe, leaves this quantity to be equated to a Square, *viz.* $aaaa - 36aaa + 70aa + 476a - 255$

12. Feign the ſide of the deſired Square to be $aa - 18a - 127$

13. Then the Square of the ſaid ſide being equated to the quantity in the eleventh ſtep, will give $a = -4$. But becauſe this value is negative, let $a - 4$ be put for a new Root, and according to that let all the members of the quantity in the eleventh ſtep be reſolved; ſo this new quantity $aaaa - 52aaa + 598aa - 2068a + 1521$ comes forth to be equated to a Square, the ſide whereof may be feigned $39 - \frac{1214}{39}a + aa$, whence $a = \frac{272226}{1560}$; therefore $a - 4$ (the new Root) ſhall be $\frac{212222}{390}$, and therefore $a - 1$ and 4 the numbers that formed the Triangle in the eighth ſtep ſhall be $\frac{212222}{390}$ and 4; or in Integers in the ſame proportion, 67609 and 1560. Wherefore the preparatory Triangle formed from thoſe numbers, and agreeable to the Scope mentioned in the ſeventh ſtep, ſhall be 4573410481, 4568543281, 210940080; For if its quadruple Area be multiplied by the given number 2, and the Product be ſubtracted from the Square of the ſecond ſide, there will remain a Square, whoſe ſide is 412514632 1.

14. Now let thoſe three ſides of the preparatory Triangle be taken for the values of b, b, p, and the given number 2 for n; in the Equation in the third ſtep, then that Equation being reſolved will give $a = \frac{1144246}{120461110647100321}$; and if this number be multiplied into the three ſides of the preparatory Triangle it will give the Triangle ſought, whoſe three ſides conſiſt of theſe three Numerators, 25347953801344222, 25320977530297822, 11691273776769 60, having 120461110647200 31 for a common Denominator.

Note. This Queſtion is very eaſily ſolved when the given number is leſs than unity, by the Canon of the preceding *Queſt.* 100. of this *Book*.

The End of the Third BOOK.

THE ELEMENTS OF THE ALGEBRAICAL ART.

BOOK IV.

CHAP. I.

Concerning the Scope of this fourth Book, and the Signification of Characters, Abbreviations and Citations uſed therein

THE Deſign of this Fourth Book is, to ſhew the excellent Uſe of the *Algebraical* Art in the Reſolution and Compoſition of Plane Problems, to wit, ſuch as may be ſolved or effected by drawing only Right (or ſtraight) and Circular Lines. In purſuance of that Deſign I have divided this Book into Ten Chapters, whereof the firſt Six are Preparatory to the reſt, which contain Four *Claſſes* or Forms of Examples, ſhewing how to find out as well Theorems, as Geometrical Effections of Plane Problems, with their Demonſtrations, by the Steps of Algebraical Reſolution. All which I have endeavour'd to render clear and intelligible to ſuch Readers as are competently exerci'd in the firſt Six Books of *Euclid*'s Elements, and in the Firſt and Second Books of theſe *Algebraical Elements*.

The Explication of the Signs *or* Characters.

$+$	Signifies	*More.*
$-$		*Leſs.*
$\times$		*Into, or By.*
$::$		*Proportionals.*
$\div$		*Continual Proportionals.*
$\surd$		*The Square Root, or Side of a Square.*
$=$		*Equal*
$\sqsubset$		*Greater.*
$\sqsupset$		*Leſſer.*
$\parallel$		*Parallel.*
$\angle$		*A plain Angle in general.*
$\sqsupset$		*A Right-angle.*
$\perp$		*Perpendicular.*
$\odot$		*A Circle.*
$\square$		*A Square.*
$\square$		*A long Square.*
$\triangle$		*A plain Triangle in general.*

Examples, ſhewing more at large the Signification of the foregoing Characters.

$a + b$	Signifies the ſumm of the right lines or numbers repreſented by *a* and *b*.
$a - b$	The exceſs by which the right line or number *a* exceeds the right line or number *b*; or, it imports that the latter quantity *b* is ſubtracted, or to be ſubtracted from the former Quantity *a*.
$a \times b$, or ab	The Rectangle or Product made by the multiplication of the right line or number *a*, by the right line or number *b*.
aa	The Square of the right line or number ſignified by *a*.
$\frac{aa}{c}$	The right line ariſing by the Application of the Square of the right line *a*, to the right line *c*; or, the Quotient ariſing by the Diviſion of the Square of the number *a*, by the number *c*.
$\frac{ab}{c}$	The right line or number ariſing by the Application or Diviſion of the Rectangle or Product of *a* into *b*, by *c*.
$a . b :: c . d$,	*viz.* As *a* is to *b*; ſo *c* to *d*.
$a , b , c \div\div$,	*viz.* As *a* is to *b*; ſo *b* to *c*.
$\sqrt{ab}$	The ſquare Root of the Product of *a* into *b*; or, the ſide of a Square equal to the Rectangle *ab*.
$\sqrt{:aa + bb:}$	The ſquare Root univerſal of $aa + bb$; or, the ſide of a Square equal to the ſumm of the Squares *aa* and *bb*.
$\sqrt{:aa - bb:}$	The ſquare Root univerſal of $aa - bb$; or, the ſide of a Square equal to the exceſs of the Square *aa* above the Square *bb*.
$a = b$	The line or number *a* is equal to the line or number *b*.
$a = 5c$	The line or number *a* is equal to five times the line or number *c*.
$a = \frac{1}{2}d$	The line or number *a* is equal to half the line or number *d*.
$a \sqsubset f$	The line or number *a* is greater than the line or number *f*.
$a \sqsupset g$	The line or number *a* is leſs than the line or number *g*.
AB ‖ CD	The line AB is parallel to the line CD.
∠ABC	The angle ABC. Obſerve here, that when an angle is expreſt by three letters, the middle letter ſtands at the angular point.
∠A	The angle A.
∠ABC is ⌟	The angle ABC is a right-angle.
AB ⊥ BC	The right-line AB is perpendicular to the right-line BC.
ABCD is ⊙	ABCD is a Circle. Obſerve here, that the firſt letter towards the left hand is uſually ſet at the Center.
□AD	Signifies either the Square AD when the letters A and D ſtand at the oppoſite angles of the Square; or elſe, the Square of the right-line AD, when A and D ſtand at the ends of the ſide of the Square.
□A	The Square of the right-line A.
□AB, BC	The long Square, or Rectangle, made of the right-lines AB and BC.
□AB, C	The Rectangle of the right-lines AB and C.
□A, B	The Rectangle of the right-lines A and B.
△ABC	The Triangle ABC.

Explication of Abbreviations *and* Citations.

Probl.	Problem.
Suppoſ.	Suppoſitions.
Req.	It is (or, let it be) required.
Prepar.	Preparation.
Conſtr.	Conſtruction.
Req. demonſtr.	It is (or, let it be) required to Demonſtrate.
Concluſ.	Concluſion.
Coroll.	Corollary.
Annot.	Annotation.
Explicat.	Explication.
Per Prop. 11. *Elem.* 5.	By the 11th Propoſition of the fifth Book of *Euclid*'s Elements.

Per

Per Defin. 29. *Elem.* 1.	By the 29th Definition of the firſt Book of *Euclid*'s Elements.
Per Ax. 1. *Chap.* 2.	By the firſt Axiom of the ſecond Chapter of this fourth Book.

Note. In the handling of every Propoſition, whether it be a Theorem or Problem, I proceed from the beginning to the end by Steps numbred in the Margin by 1, 2, 3, 4, 5, *&c.* that by referring to preceding Steps, the riſe of the following may be apparent: So when 'tis ſaid, [*Therefore ont of*, or, *From* 3°. *and* 4°.] it imports, that the thing aſſerted or inferr'd is manifeſt by the third and fourth Steps of the Propoſition in hand. If any other Abbreviations occurr, their meaning will be obvious to every intelligent Reader.

CHAP. II.

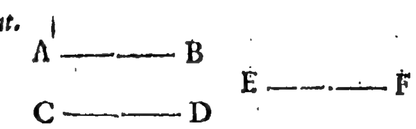
The explication of Axioms, or common notions, upon which the force of Inferences or Concluſions, about the Equality, Majority and Minority of Quantities compared to one another, doth chiefly depend.

Axiom 1.

1. IF each of two Quantities be equal to a third, thoſe two are equal between themſelves.

Explicat.

If AB = EF,
And CD = EF,
Then AB = CD; *per Ax.* 1.



That is to ſay,

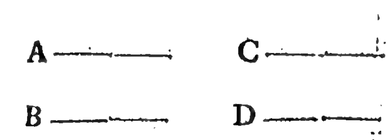
If AB be equal to EF, and CD be equal to EF, then AB is equal to CD, by the firſt *Axiom* of *Chap.* 2.

Axiom 2.

2. Quantities which are equal to equal Quantities, are alſo equal between themſelves.

Explicat.

If C = D,
A = C,
B = D,
Then A = B; *per Ax.* 2.

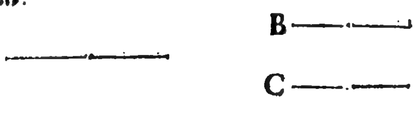


Axiom 3.

3. That which is greater or leſs than one of two equal Quantities, is alſo greater or leſs than the other.

Explicat.

If B = C,
And A ⊏ B,
Then A ⊏ C; *per Ax.* 3.



That is to ſay,

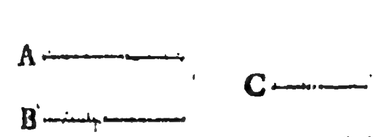
If B be equal to C, and A be greater than B, then A is greater than C, by *Ax.* 3.

Axiom 4.

4. If one of two equal Quantities be greater or leſs than a third, the other of thoſe two ſhall be alſo greater or leſs than the ſame third.

Explicat.

If A = B,
And A ⊏ C,
Then B ⊏ C; *per Ax.* 4.

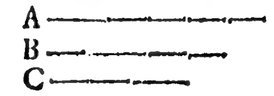


Axiom 5.

5. That which is greater than the greater of two Quantities, is alſo greater than the leſſer; and that which is leſs than the leſſer of two Quantities, is alſo leſs than the greater.

Explicat.

If $B \sqsubset C$,
And $A \sqsubset B$,
Then $A \sqsubset C$; *per Ax.* 5.

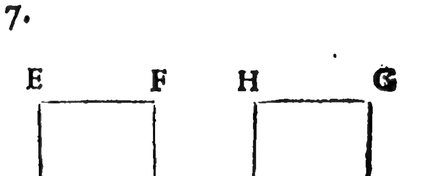


Axiom 6.

6. The exchanging of equal Quantities doth not alter equality.

Explicat.

If $A+B = C+D$,
And $E = B$,
Then $A+E = C+D$; *per Ax.* 6.



Axiom 7.

7. Interpretation doth not change equality.

Explicat.

Suppoſ. $\begin{cases} \square AF = \square CG, \\ AF \text{ is } \square AB, \\ CG \text{ is } \square CD, \end{cases}$

$\square AB = \square CD$; *per Ax.* 7.

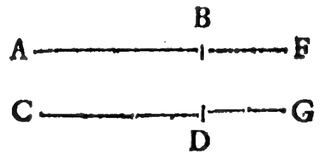
That is to ſay,
The Square AF is equal to the Square CG, by ſuppoſition.
AF is the Square of the ſide AB, by ſuppoſition.
CG is the Square of the ſide CD, by ſuppoſition.
The Square of the ſide AB is equal to the Square of the ſide CD, by the ſeventh *Axiom* of *Chap.* 2.

Axiom 8.

8. If to equal Quantities you add equal Quantities, or one and the ſame Quantity, the wholes ſhall be equal.

Explicat.

If $AB = CD$,
And $BF = DG$,
Then $AB+BF = CD+DG$,
That is, $AF = CG$; *per Ax.* 8.

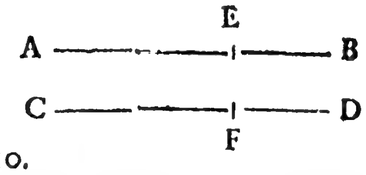


Axiom 9.

9. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining ſhall be equal to one another.

Explicat.

If $AB = CD$,
And $AE = CF$,
Then $AB - AE = CD - CF$,
That is, $EB = FD$; *per Ax.* 9.

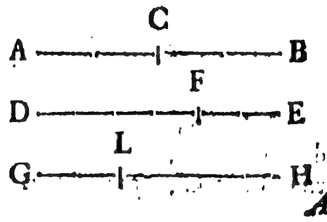


Axiom 10.

10. If from a whole the half be taken away, half will remain; and if more than half be taken away, leſs than half will remain; but if one third part be taken away, two thirds will remain, *&c.*

Explicat.

If $AC = \frac{1}{2}AB$,
Then $CB = \frac{1}{2}AB$, *per Ax.* 10.
If $DF \sqsubset \frac{1}{2}DE$,
Then $FE \sqsupset \frac{1}{2}DE$, *per Ax.* 10.
If $GL = \frac{1}{3}GH$,
Then $LH = \frac{2}{3}GH$, *per Ax.* 10.



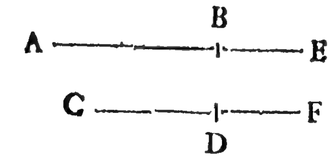
Axiom 11.

Axiom 11.

11. If to unequal quantities equal quantities be added, the wholes are unequal.

Explicat.

If $AB \sqsubset CD$,
And $BE = DF$,
Then $AE \sqsubset CF$; *per Ax.* 11.

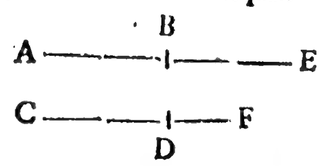


Axiom 12.

12. If to equal quantities you add unequal quantities, the wholes are unequal.

Explicat.

If $AB = CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; *per Ax.* 12.

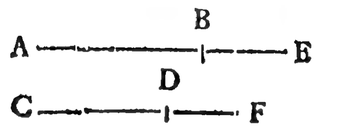


Axiom 13.

13. If to unequal quantities unequal quantities be added, the greater to the greater, and the leſs to the leſs, the wholes are unequal, to wit, the former the greater, and the latter the leſſer.

Explicat.

If $AB \sqsubset CD$,
And $BE \sqsubset DF$,
Then $AE \sqsubset CF$; *per Ax.* 13.

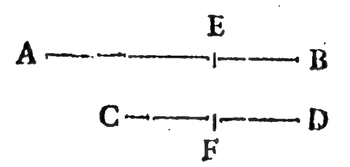


Axiom 14.

14. If from unequal quantities equal quantities or one and the ſame quantity be taken away, the remainders will be unequal.

Explicat.

If $AB \sqsubset CD$,
And $EB = FD$,
Then $AE \sqsubset CF$; *per Ax.* 14.

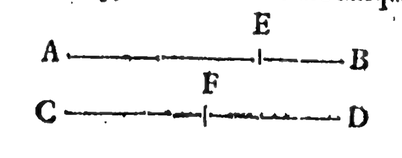


Axiom 15.

15. If from equal quantities unequal quantities be taken away, the remainders are unequal.

Explicat.

If $AB = CD$,
And $AE \sqsubset CF$,
Then $EB \sqsupset FD$; *per Ax.* 15.



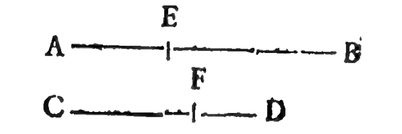
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Axiom 16.

16. If from unequal quantities unequal quantities be taken away, from the greater the leſſer, and from the leſſer the greater, the remainders are unequal; to wit, the former the greater, and the latter the leſſer.

Explicat.

If $AB \sqsubset CD$,
And $CF \sqsubset AE$,
Then $EB \sqsubset FD$; *per Ax.* 16.



Axiom 17.

17. Quantities which are the doubles of one and the ſame quantity, or of equal quantities, are equal between themſelves. Conceive the ſame of triples, quadruples, *&c.*

Explicat.

If $A = 2C$,
And $B = 2C$,
Then $A = B$; *per Ax.* 17.

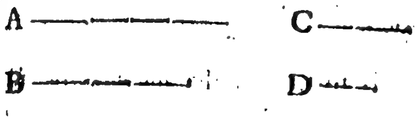


Axiom 18.

18. The double of the greater of two quantities is greater than the double of the leſſer.

Explicat.

If $\begin{cases} C \sqsubset D, \\ A = 2C, \\ B = 2D, \end{cases}$
Then $A \sqsubset B$; *per Ax.* 18.



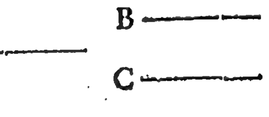
Axiom 19.

Axiom 19.

19. That which is the double of one of two equal quantities, is alſo the double of the other.

Explicat.

If $B = C$,
And $A = 2B$,
Then $A = 2C$; *per Ax.* 19.

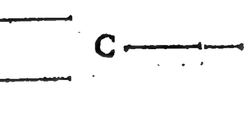


Axiom 20.

20. If one of two equal quantities be the double of a third, the other of thoſe two is alſo the double of the ſame third.

Explicat.

If $A = B$,
And $A = 2C$,
Then $B = 2C$; *per Ax.* 20.



Axiom 21.

21. Quantities which are the halves of one and the ſame quantity, or of equal quantities, are equal between themſelves. Underſtand the ſame of thirds, fourths, *&c.*

Explicat.

If $A = \frac{1}{2}C$,
And $B = \frac{1}{2}C$,
Then $A = B$; *per Ax.* 21.

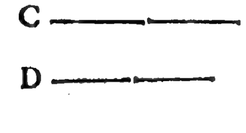


Axiom 22.

22. The half of the greater of two quantities is greater than the half of the leſſer.

Explicat.

If $\begin{cases} C \sqsubset D, \\ A = \frac{1}{2}C, \\ B = \frac{1}{2}D, \end{cases}$
Then $A \sqsubset B$; *per Ax.* 22.

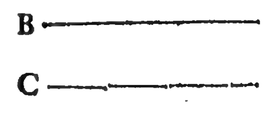


Axiom 23.

23. That which is the half of one of two equal quantities is alſo the half of the other.

Explicat.

If $B = C$,
And $A = \frac{1}{2}B$,
Then $A = \frac{1}{2}C$; *per Ax.* 23.

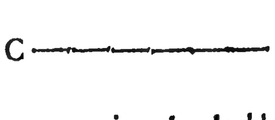


Axiom 24.

24. If one of two equal quantities be the half of a third, the other of thoſe two ſhall be the half of the ſame third.

Explicat.

If $A = B$,
And $A = \frac{1}{2}C$,
Then $B = \frac{1}{2}C$; *per Ax.* 24.



What hath been ſaid in the eight laſt preceding *Axioms* concerning the double and the half, may be alſo underſtood of the triple, quadruple, quintuple, *&c.* and of thirds, fourths, fifths, *&c.*

Axiom 25.

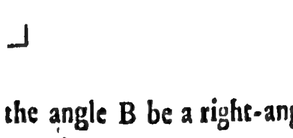
25. Every whole is greater than its part.

Axiom 26.

26. All right angles are equal between themſelves.

Explicat.

If $\lessdot A$ be $\lrcorner$,
And $\lessdot B$ be $\lrcorner$,
Then $\lessdot A = \lessdot B$, *per Ax.* 26.



That is to ſay, if the angle A be a right-angle, and the angle B be a right-angle; then the angle A is equal to the angle B.

Axiom 27.

Axiom 27.

27. If one of two or more equal angles be a right-angle, every one of the reſt of thoſe equal angles is alſo a right-angle.

Explicat.

If $\lessdot A = \lessdot B = \lessdot C$,
And $\lessdot A$ be $\lrcorner$,
Then $\lessdot B$ and $\lessdot C$ are $\lrcorner$; *per Ax.* 27.

A ⌟, B ⌟, C ⌟.

That is to ſay, If the angles A, B and C be equal to one another, and the angle A be a right-angle, then the angles B and C are alſo right-angles. *Per Axiom.* 24. *Chap.* 2.

Axiom 28.

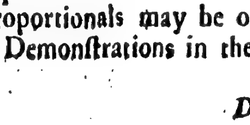
28. Every whole is equal to all its parts taken together.

Axiom 29.

29. If a quantity be neither greater nor leſs than another quantity, thoſe quantities are equal between themſelves.

Explicat.

If A be neither $\sqsubset$ nor $\sqsupset$ than B,
Then $A = B$; *per Ax.* 29.



CHAP. III.

The explication of Definitions, concerning the ways of arguing uſed by Mathematicians, to inferr one Analogie from another.

THE ways of arguing about *Reaſons*, or *Proportions*, are principally ſix, which are explain'd in this Chapter in ſuch order as they are expreſt by the 12th, 13th, 14th, 15th, 16th and 17th Definitions at the beginning of the fifth Book of *Euclid*'s Elements; to which ſix ways of reaſoning, ſix others are alſo here inſerted as Annotations, being the *Scholies* of *Clavius* and *Herigonius* upon ſuch Propoſitions of *Euclid*'s Elements as are hereafter cited. All which are very uſeful in Mathematical Reſolution and Compoſition, as will appear in the following 7th, 8th, 9th and 10th Chapters.

Definition I.

1. *Alternate Reaſon* is the comparing of the Antecedent to the Antecedent, and the Conſequent to the Conſequent.

Explicat.

If this Analogy be propoſed, ... $\begin{cases} a : b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Then alternately, or by permutation, . . $\begin{cases} a . c :: b . d \\ 6 . 12 :: 4 . 8 \end{cases}$ *Per Prop. 16. Elem. 5.*

That is to ſay, If *a* hath ſuch Reaſon (or Proportion) to *b*, as *c* to *d*; then alternately, or by permutation of Reaſon, as *a* is to *c*, ſo ſhall *b* be to *d*.

But note diligently, that in this firſt way of arguing, all the four Proportionals in the Analogy propounded muſt neceſſarily be Quantities of one and the ſame kind, that is, either all Lines, or all Planes, *&c.* For although it may properly be ſaid, as the line *a* is to the line *b*, ſo is the Plane *c* to the Plane *d*; yet it cannot be thence inferr'd by Alternate Reaſon, that the line *a* is to the Plane *c*, as the line *b* to the Plane *d*, becauſe there is no Proportion between a Line and a Plane, which are quantities of different kinds: But in all the following ways of arguing, the two firſt Proportionals may be of one kind, and the two latter of another, as is manifeſt by the Demonſtrations in the fifth Book of *Euclid*'s Elements.

Defini-

Definit. II.

2. *Inverſe Reaſon* is the taking of the Conſequent as the Antecedent, to compare it to the Antecedent as if it were the Conſequent.

Explicat

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Coroll. prop. 4. Elem. 5. Then inverſly, ... $\begin{cases} b . a :: d . c \\ 4 . 6 :: 8 . 12 \end{cases}$

Definit. III.

3. *Compoſition of Reaſon* is the taking of the Antecedent and Conſequent both as one, to compare it to the ſame Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 12 :: 4 . 8 \end{cases}$

Per prop. 18. Elem. 5. Then by Compoſition, ... $\begin{cases} a+b . b :: c+d . d \\ 18 . 12 :: 12 . 8 \end{cases}$

Annot. 1.

4. *Compoſition of Reaſon converſe* is the taking of the Antecedent and Conſequent both as one, to compare it to the ſame Antecedent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 1. Clavii in prop. 18. Elem. 5. Then by Compoſ. converſe, ... $\begin{cases} a+b . a :: c+d . c \\ 10 . 6 :: 20 . 12 \end{cases}$

Annot. 2.

5. *Compoſition of Reaſon contrary* is the comparing of the Antecedent to the Antecedent and Conſequent taken both as one.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 2. Clavii in prop. 18. Elem. 5. Then by Compoſ. contrary, ... $\begin{cases} a . a+b :: c . c+d \\ 6 . 10 :: 12 . 20 \end{cases}$

Annot. 3.

6. *Compoſition of Reaſon inverſly contrary* is the comparing of the Conſequent to the Antecedent and Conſequent taken both as one.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Schol. 2. Herigon. in prop. 18. Elem. 5. Then by Compoſition inverſly contrary, . $\begin{cases} b . a+b :: d . c+d \\ 4 . 10 :: 8 . 20 \end{cases}$

Definit. IV.

7. *Diviſion of Reaſon* is the comparing of the exceſs whereby the Antecedent exceeds the Conſequent, to the ſame Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per prop. 17. Elem. 5. Then by Diviſion, ... $\begin{cases} a-b . b :: c-d . d \\ 2 . 4 :: 4 . 8 \end{cases}$

But in this way of arguing by Diviſion of Reaſon, 'tis manifeſt that the Antecedent muſt neceſſarily be greater than the Conſequent.

Annot. 1.

Annot. 1.

8. *Diviſion of Reaſon converſe* is the comparing of the Conſequent to the exceſs whereby the Antecedent exceeds the Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 9 . 4 :: 18 . 8 \end{cases}$

Per Schol. 1. Clavii in prop. 17. Elem. 5. Then by Diviſion converſe, ... $\begin{cases} b . a-b :: d . c-d \\ 4 . 5 :: 8 . 10 \end{cases}$

Annot. 2.

9. *Diviſion of Reaſon contrary* is the comparing of the Antecedent, to the exceſs whereby the Conſequent exceeds the Antecedent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 4 . 6 :: 8 . 12 \end{cases}$

Per Schol. 2. Clavii in prop. 17. Elem. 5. Then by Diviſion contrary, ... $\begin{cases} a . b-a :: c . d-c \\ 4 . 2 :: 8 . 4 \end{cases}$

But here 'tis manifeſt that the Conſequent muſt be greater than the Antecedent.

Annot. 3.

10. *Diviſion of Reaſon inverſly contrary* is the comparing of the exceſs whereby the Conſequent exceeds the Antecedent, to the ſame Antecedent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 4 . 6 :: 8 . 12 \end{cases}$

Per Schol. 2. Herigon. in prop. 17. Elem. 5. Then by Diviſion inverſly contrary, . . $\begin{cases} b-a . a :: d-c . c \\ 2 . 4 :: 4 . 8 \end{cases}$

Definit. V.

11. *Converſion of Reaſon* is the comparing of the Antecedent to the exceſs by which the Antecedent exceeds the Conſequent.

Explicat.

If . . . . . $\begin{cases} a . b :: c . d \\ 6 . 4 :: 12 . 8 \end{cases}$

Per Coroll. prop. 19. Elem. 5. Then by converſe Reaſon, ... $\begin{cases} a . a-b :: c . c-d \\ 6 . 2 :: 12 . 4 \end{cases}$

Definit. VI.

12. *Reaſon of equality* is, when more than two quantities in one Rank, and as many in another are ſuch, that if two to two be compared, they are in the ſame Reaſon; and it alſo happens, that as the firſt is to the laſt in the firſt rank of Quantities, ſo is the firſt to the laſt in the latter rank. Or otherwiſe, 'tis a compariſon of the extremes to one another, the mean quantities being taken away.

But there are two ways of arguing by Reaſon of equality, to wit, one when the Proportion is Ordinate, the other when the Proportion is Inordinate or Diſturbed; both which are explain'd in the two following Definitions.

Definit. VII.

13. *Ordinate proportion* is, when in the firſt rank of quantities, as the Antecedent is to the Conſequent, ſo in the latter rank is the Antecedent to the Conſequent: and when in the firſt rank as the Conſequent is to ſome other, ſo in the latter rank is the Conſequent to ſome other.

Explicat.

If to these quantities propounded, { A,4 . B,6 . C,12 . D,8 / E,10 . F,15 . G,30 . H,20

These Analogies do happen, . . { A,4 . B,6 :: E,10 . F,15 / B,6 . C,12 :: F,15 . G,30 / C,12 . D,8 :: G,30 . H,20

Per prop. 22. Elem. 5. Then by Reason of equality, . . ∢ A,4 . D,8 :: E,10 . H,20

That is to say, when the proportion in both ranks of quantities propounded is ordinate, (according to *Defin.* 7.) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C, D; so shall the first be to the last in the second rank of quantities E, F, G, H.

Definit. VIII.

14. *Inordinate proportion* is, when three quantities standing in one rank and three in another do afford these Analogies, *viz.* as the first quantity in the first rank is to the second in the same rank; so is the second quantity in the second rank to the third in the same rank: and as the second quantity in the first rank, is to the third in the same rank; so is the first quantity in the second rank to the second in the same rank.

Explicat.

If to these quantities propounded, { A,4 . B,6 . C,3 / D,20 . E,10 . F,15

These Analogies do happen; . . { A,4 . B,6 :: E,10 . F,15 / B,6 . C,3 :: D,20 . E,10

Per prop. 23. Elem. 5. Then by reason of Equality, . . ∢ A,4 . C,3 :: D,20 . F,15

That is to say, when the proportion in both ranks of quantities propounded is inordinate, (according to *Defin.* 8.) then by Reason of equality, as the first is to the last in the first rank of quantities, A, B, C; so shall the first be to the last in the latter rank of quantities, D, E, F.

CHAP. IV.

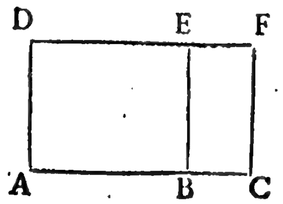
Various fundamental Theorems frequently used in Mathematical Resolution and Composition.

Theorem I.

A Rectangle (or right-angled Parallelogram) comprehended under any right-line and the difference of any two right-lines, is equal to the difference of two Rectangles comprehended under the first line and each of the two latter.

Suppos.

1. A D is a right-line,
2. A C and B C are right-lines,
3. A B C is a right-line,
4. A B = A C — B C.



5. . . *Req. demonstr.* . . . . □ A D,A B = □ A D,A C — □ A D,B C.

Preparat.

6. Make □ A F to be contain'd under A D and A C,
7. Make B E ⊥ A C.

Demon-

Demonstration.

8. By Constr. in 6°, and 7°. (and *per prop.* 1. *Elem.* 2.) . . . . } □ A E + □ B F = □ A F
9. Therefore, by subtracting □ B F from each part of that Equation, } □ A E = □ A F — □ B F
10. That is, (*per Ax.* 7. *Chap.* 2.) Which was to be demonstrated. } □ A D,A B = □ A D,A C — □ A D,B C / B E

Illustration Algebraical.

Let three right-lines be represented by . . . . { *a* . . . | 3 / *b* . . . | 6 / *c* . . . | 2

Suppose also . . . . . ∢ *b* ⊏ *c*

Then if the first line be multiplied by the difference of the second and third, that is, $a \times \overline{b - c}$, the Product will be { *ab* — *ac* | 18 — 6 = 12

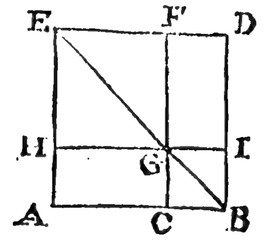
Which Product is manifestly the difference between the Product of the first line *a* into the second *b*, and the Product of the first line *a* into the third *c*, according to the tenour of *Theorem* 1.

Theorem II.

If a right-line be cut into any two parts, the Square described upon the whole line is equal to the Squares described upon the parts, and to twice the Rectangle comprehended under the parts.

Suppos.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C + C B = A B.



4. . . *Req. demonstr.* . . . . □ A B = □ A C + □ C B + 2 □ A C,C B.

Prepar.

5. Upon A B describe the □ A D, (*per prop.* 46. *Elem.* 1.)
6. Draw the Diameter E B.
7. Draw C F ∥ A E (or B D,) and cutting E B in G, (*per prop.* 31. *Elem.* 1.)
8. By the point G draw H G I ∥ A B, (or E D.)

Demonstration.

9. By Constr. in 5°, ... } A D is □ A B.
10. Therefore, (*per* 29. *Defin.* 1. *Elem.*) } ∢A, ∢A E D, ∢D, ∢D B A are ⌟.
11. And out of 7°, 8°, and 10°; (*per* 29. *prop.* 1. *Elem.*) . . . . . } ∢E H G, ∢E F G, ∢H G F are ⌟.
12. And out of 5°, (*per* 29. *defin* 1. *Elem.*) } A E = A B = B D = D E.
13. Therefore out of 10°, and 12°; (*per prop.* 5, & 32. *Elem.* 1.) . . . . } ∢A E B is ½ ⌟.
14. Likewise, out of 10°, and 12°, . . } ∢D E B is ½ ⌟.
15. Likewise out of 11°, and 13°; . . } ∢H G E is ½ ⌟.
16. Likewise out of 11°, and 14°; . . } ∢F G E is ½ ⌟.
17. Therefore out of 13°, and 15°; (*per prop.* 6. *Elem.* 1.) . . . . . } H E = H G.
18. Likewise out of 14°, and 16°; . . } E F = F G.
19. And from 7°, and 8°; (*per prop.* 34. *Elem.* 1.) . . . . . } E F = H G.
20. Wherefore out of 11°, 17°, 18°, 19°; (*per* 29. *defin.* 1. *Elem.*) . . . . . } H F is □ H G, or □ A C.
21. And in the same respect, ... } C I is □ C B.
22. Therefore from 21°, ... } C G = C B.
23. And from 22°, (*per* 36. *prop.* 1. *Elem.*) } □ A G, (or □ A C,C G,) = □ A C,C B.

24. But (*per* 43. *prop.* 1. *Elem.*) . ⟩ □AG = □GD.
25. Therefore out of 23°, and 24°; (*per Ax.* 1. *Chap.* 2.) . . . ⟩ □GD = □AC, CB.
26. But (*per Ax.* 28. *Chap.* 2.) . ⟩ □AD = □HF + □CI + □AG + □GD.
27. Wherefore, out of 5°, 20°, 21°, 24°, 25°, 26°; (*per Ax.* 7. *Chap.* 2.) ⟩ □AB = □AC + □CB + 2□AC, CB.

Which was to be dem.

Coroll. 1.

Hence it is manifeſt that the Parallelograms which are about the Diameter of a Square, are alſo Squares themſelves.

Coroll. 2.

It alſo appears that the Diameter of any Square divides its angles into two equal parts.

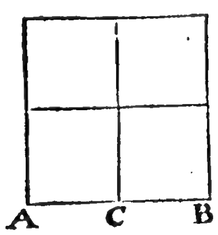
Illuſtration Algebraical.

Suppoſe a right-line to be cut into two parts, to wit,	*a* and *b* . .	6 and 2
Then the ſumm of the parts is	*a* + *b* . . .	8
Which ſumm multiplied by it ſelf produceth the Square of the whole line, to wit, ...	*aa* + 2*ba* + *bb*	64

Which Product or Square doth manifeſtly conſiſt of the Squares of the parts *a* and *b*, and twice the Product (or Rectangle) of the ſame parts; according to the tenour of the preceding *Theor.* 2.

Theorem III.

A Square deſcribed upon any right-line is equal to four times the Square of the half of the ſame line; and conſequently, a quarter of the former Square is equal to the latter.



Suppoſ.

1. AB is a right-line;
2. AC = CB, therefore
3. AB = AC + CB = 2AC or 2CB.
4. . . . *Req. demonſtr.* . . . . ⟩ □AB = 4□AC, (or 4□CB;) *Alſo,* ¼□AB = □AC, or □CB.

Demonſtration.

5. By ſuppoſition, ... ⟩ . . AC = CB.
6. Therefore, (*per Sch.* of *prop.* 46. *El.* 1.) ⟩ □AC = □CB.
7. And out of 5°, (*per prop.* 36. *Elem.* 1.) ⟩ □AC = □AC, CB.
8. And out of 7°, (*per Ax.* 17. *Chap.* 2.) ⟩ 2□AC = 2□AC, CB.
9. And out of 6°, (*per Ax.* 8. *Chap* 2.) ⟩ 2□AC = □AC + □CB.
10. And out of 8°, and 9°; (*per Ax.* 8.) ⟩ 4□AC = □AC + □CB + 2□AC, CB.
11. But *per Theor.* 2. of this *Chap.* . ⟩ □AB = □AC + □CB + 2□AC, CB.
12. Wherefore out of 10°, and 11°; (*per Ax.* 1.) . . . . . ⟩ □AB = 4□AC, (or 4□CB.)

Conclu.
13. And out of 12°, (*per Ax.* 21. *Chap.* 2.) . . . . . ⟩ ¼□AB = □AC, (or □CB.)

Which was to be dem.

Illuſtration Algebraical.

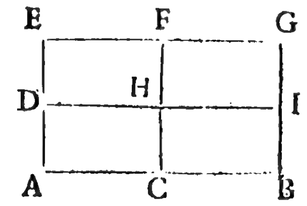
Let a right-line be repreſented by	2*a*	10
The half thereof is	*a*	5
The Square of the whole line 2*a* is	4*aa*	100
The Square of the half, to wit, of *a* is	*aa*	25

The firſt of thoſe Squares is evidently equal to four times the latter, and conſequently a quarter of the former is equal to the latter; as is affirmed by *Theor.* 3.

Theor. IV.

Theorem IV.

A Rectangle (or long Square) comprehended under any two unequal right-lines is equal to four times the Rectangle comprehended under the half of each of thoſe lines; and conſequently a quarter of the firſt Rectangle is equal to the latter.



Suppoſ.

1. AB and AE are right-lines,
2. AC = CB = ½AB,
3. AD = DE = ½AE.
4. . . *Req. demonſtr.* . . ⟩ □AB, AE = 4□AC, AD. Alſo, ¼□AB, AE = □AC, AD.

Prepar.

5. Make □AG to be contain'd under AB and AE,
6. Draw CF ∥ AE, (or BG.) Likewiſe, DI ∥ EG, (or AB.)

Demonſtration.

7. By Conſtr. in 5°, . . ⟩ AG is □AB, AE.
8. Therefore (*per* 30. *defin.* 1. *Elem.*) . . . . . ⟩ ∠A, ∠E, ∠G, ∠B are ⌟
9. And becauſe by Conſtr. in 6°, ... ⟩ CF ∥ AE ∥ BG.
10. Likewiſe by Conſtr. in 6°, ⟩ DI ∥ EG ∥ AB.
11. Therefore out of 8°, 9°, and 10°; (*per prop.* 29. *Elem.* 1.) . . . . . ⟩ AH, DF, HG, CI are □.
12. And becauſe by Suppoſ. in 2° and 3°, ... ⟩ AC = CB. Alſo AD = DE.
13. Therefore out of 11° and 12°, (*per prop.* 36. *Elem.* 1.) . . . . . ⟩ □AH = □DF = □HG = □CI.
14. And from 13°, (*per Ax.* 8. *Chap.* 2.) . . . . . ⟩ □AH + □DF + □HG + □CI = 4□AH.
15. But (*per Ax.* 28°. *Ch.* 2.) ⟩ □AH + □DF + □HG + □CI = □AG.
16. Therefore out of 14° and 15°, (*per Ax.* 1.) . . ⟩ □AG = 4□AH.
17. That is, (*per Ax.* 7. *Chap.* 2.) . . . . . ⟩ □AB, AE = 4□AC, AD.
18. And conſequently from 17°, (*per Ax.* 21.) . ⟩ ¼□AB, AE = □AG, AD.

Conclu.

Which was to be dem.

Illuſtration Alegebraical.

Let a right-line be repreſented by	2*a*	6
And another right-line by	2*b*	4
The half of the former line is	*a*	3
And the half of the latter is	*b*	2
The Product or Rectangle of the two whole lines is	4*ab*	24
The Product of the half of each line is	*ab*	6

The firſt of thoſe Products is evidently equal to four times the latter, and conſequently a quarter of the former is equal to the latter, according to the tenour of *Theor.* 4.

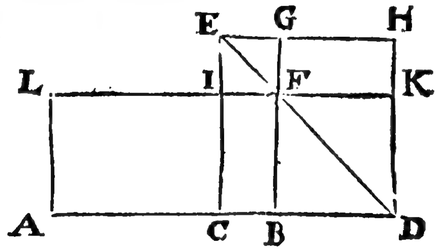
Theor. V.

Theorem V.

If a right-line be cut into any two unequal parts, the Square of the difference of the parts is equal to the Squares of the parts, leſs by twice the Rectangle (or long Square) comprehended under the parts: Alſo, the Square of half the difference of the ſaid parts is equal to a quarter of each of the Squares of the parts, leſs by half the Rectangle of the parts.

Suppoſ.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C ⊏ C B.



Prepar.

4. Produce A B to D, ſo, that A C = C D, then 'tis manifeſt that B D is the difference of the parts A C and C B; for C D (or A C) — C B = B D.
5. Upon C D deſcribe the □ C H; (*per prop.* 46. *Elem* 1.)
6. Draw the Diameter E D.
7. By the point B draw B G ‖ C E (or D H,) and cutting E D in F.
8. By the point F draw L F K ‖ A D, or E H.
9. By the point A draw A L ‖ C E.
10. . . *Req. demonſtr.* . . { □ B D = □ A C + □ C B − 2 □ A C, C B. Alſo, □ ½ B D = ¼ □ A C + ¼ □ C B − ½ □ A C, C B.

Demonſtration.

11. By Conſtr. in 4° and 5°, C H is □ C D or □ A C.
12. Therefore (*per Coroll.* 1. *Theor.* 2. of this *Chapt.*) B K is □ B D.
13. Likewiſe . . . . IG is □ E G or □ C B.
14. And (*per prop.* 43. *Elem.* 1.) . . . . □ C F = □ F H.
15. Therefore (*per Ax.* 8. *Chap.* 2.) by adding □ IG to each part of the Equation in 14°, ... □ C G = □ I H.
16. Again, (by Conſtr. in 4° and 5°, ... C E = A C.
17. And 'tis evident that . C B = C B.
18. Therefore from 16° and 17°, (*per prop.* 35. *Elem.* 1.) □ C E, C B (or □ C G) = □ A C, C B.
19. Therefore out of 15° and 18°, (*per Ax.* 6, & 8 *Ch* 2.) □ C G + □ I H = 2 □ A C, C B.
20. And becauſe (*per Ax.* 28, & 8. *Chap.* 2.) . □ B K + □ C G + □ I H = □ C H + □ I G.
21. Therefore out of 19° and 20°, (*per Ax.* 6 *Ch.* 2.) □ B K + 2 □ A C, C B = □ C H + □ I G.
22. Therefore from 21°, (*per Ax.* 9. *Chap.* 2.) . □ B K = □ C H + □ I G − 2 □ A C, C B.
23. Therefore out of 22°, 12°, 11°, 13°; (*per Ax.* 7. *Chap.* 2. . . . . □ B D = □ A C + □ C B − 2 □ A C, C B.

Concluſ. 1. Which was to be dem.

24. Moreover, out of 23°, (*per Ax.* 21. *Chap.* 2.) . ¼ □ B D = ¼ □ A C + ¼ □ C B − ½ □ A C, C B.
25. And *per The.* 3. of this *Ch.* ¼ □ B D = □ ½ B D
26. Therefore out of 24° and 25°, (*per Ax.* 1.) □ ½ B D = ¼ □ A C + ¼ □ C B − ½ □ A C, C B.

Concluſ. 2. Which was alſo to be dem.

Illuſtra-

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	*a* and *b* . . .	16 and 10
Suppoſe alſo	*a* ⊏ *b* . . .	
Then the difference of the parts is	*a* − *b* . . .	6
And half the difference of the parts is . . .	½*a* − ½*b* . . .	3
The Square of the whole difference is . . .	*aa* + *bb* − 2*ba*	36
The Square of the half difference is . . .	¼*aa* + ¼*bb* − ½*ba*	9

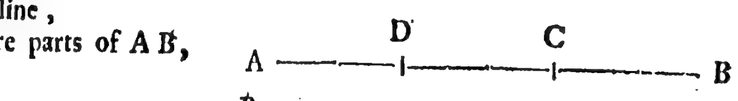
Which two Squares do manifeſtly prove the certainty of what is affirmed in the foregoing *Theor.* 5.

Theorem VI.

If a right-line be cut into any two unequal parts, the Square of the whole line together with the Square of the difference of the parts is equal to twice the Squares of the parts; and conſequently half the Square of the whole line together with half the Square of the difference of the parts is equal to the ſumm of the Squares of the parts.

Suppoſ.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C ⊏ C B.



Prepar.

4. From C A cut off C D = C B, thence it follows that A D (= A C − C B) is the difference of the parts A C and C B.
5. . . *Req. demonſtr.* . . { □ A B + □ A D = 2 □ A C + 2 □ C B. *Alſo,* ½ □ A B + ½ □ A D = □ A C + □ C B.

Demonſtration.

6. By *Theor.* 2. of this *Chapt.* □ A B = □ A C + □ C B + 2 □ A C, C B.
7. And by *Theor.* 5. of this *Chapt.* □ A D = □ A C + □ C B − 2 □ A C, C B.
8. Therefore out of 6° and 7°, (*per Ax.* 8. *Chap.* 2.) . . . . □ A B + □ A D = 2 □ A C + 2 □ C B.
9. And conſequently, (*per Ax.* 21. *Chap.* 2.) . . . . . ½ □ A B + ½ □ A D = □ A C + □ C B. } *Concluſ.*

Which was to be dem.

Illuſtration Alegebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	*a* and *b* . . .	6 & 4
Suppoſe alſo, ...	*a* ⊏ *b*	
Then the ſumm of the parts is	*a* + *b* . . .	10
And the difference of the parts is	*a* − *b* . . .	2
The Square of the whole line, that is, the Square of the ſumm of the parts, is	*aa* + *bb* + 2*ba*	100
The Square of the difference of the parts is . . .	*aa* + *bb* − 2*ba*	4
The ſumm of thoſe Squares is	2*aa* + 2*bb* . .	104

Which ſumm, (according to the tenour of the preceding *Theor.* 6.) is manifeſtly equal to twice the ſumm of the Squares of the parts.

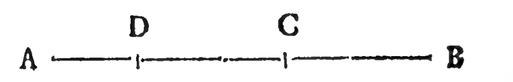
Theorem VII.

If a right-line be cut into any two unequal parts, the Square of the whole line is equal to four times the Rectangle (or long Square) comprehended under the parts, together with the Square of the difference of the parts: Alſo, the Square of half the ſaid right-line, (or of half the ſumm of the parts,) is equal to the Rectangle of the parts together with a quarter of the Square of the difference of the parts.

Suppoſ.

Suppoſ.

1. A B is a right-line,
2. A C and C B are parts of A B,
3. A C ⊏ C B.



Prepar.

4. From C A cut off C D = C B, whence A D (= A C — C B) is the difference of the parts A C and C B.

5. . . . *Req. demonſtr.* . . { □ A B = 4 □ A C, C B + □ A D. Alſo, □ ½ A B = □ A C, C B + ¼ □ A D.

Demonſtration.

6. By *Theor.* 5. of this *Chapt.* . . □ A D = □ A C + □ C B — 2 □ A C, C B.
7. Therefore by adding 4 □ A C, C B to each part of that Equation this ariſeth, (*per ax.* 8. *ch.* 2.) 4 □ A C, C B + □ A D = □ A C + □ C B + 2 □ A C, C B.
8. But *per Theor.* 2. of this *Chap.* . . □ A B = □ A C + □ C B + 2 □ A C, C B.
9. Therefore from 7° and 8°, (*per Ax.* 1. *Chap.* 2.) . . . □ A B = 4 □ A C, C B + □ A D.

Which was to be dem.

10. Moreover, from 9°, (*per Ax.* 21. *Chap.* 2.) . . . ¼ □ A B = □ A C, C B + ¼ □ A D.
11. And by *Theor.* 3. of this *Cha.* . . ¼ □ A B = □ ½ A B.
12. Therefore from 10° and 11°, (*per Ax.* 1.) . . . . . □ ½ A B = □ A C, C B + ¼ □ A D.

Which was alſo to be dem.

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two parts, to wit,	*a* and *b* . . .	6 and 4
Suppoſe alſo	*a* ⊏ *b*	
Then the ſumm of the parts is	*a* + *b* . . .	10
And the difference of the parts is	*a* — *b* . . .	2
The Square of the ſumm of the parts, that is, the Square of the whole line is	*aa* + *bb* + 2*ba*	100
And the Square of the difference of the parts is .	*aa* + *bb* — 2*ba*	4
To which Square of the difference if you add the quadruple Product of the parts, to wit, . . .	. . . + 4*ba*	96

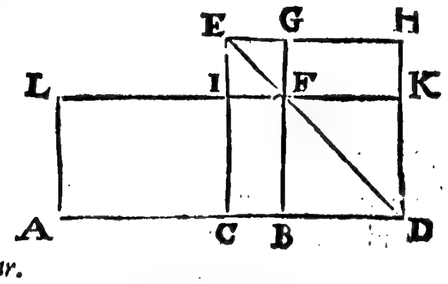
The ſumm (according to the tenour of *Theor.* 7.) makes the Square of the whole line, to wit, *aa* + *bb* + 2*ba*.

Theorem VIII.

If a right-line be cut into any two unequal parts, the Rectangle (or long Square) comprehended under the whole line and the difference of the parts is equal to the difference of the Squares of the parts. Alſo, the Rectangle under half the ſaid right-line, (that is, half the ſumm of the parts) and half the difference of the parts, is equal to a quarter of the difference of the Squares of the parts.

Suppoſ.

1. A B is a right-line.
2. A C and C B are parts of A B.
3. A C ⊏ C B.



Prepar.

4. Produce A B to D, ſo, that A C = C D, thence it follows that B D is the difference of the parts A C and C B; for C D (or A C) — C B = B D.

5. Upon

5. Upon C D deſcribe the □ C H.
6. Draw the Diameter E D.
7. By the point B draw B G ‖ C E (or D H,) and cutting E D in F.
8. By the point F draw L F K ‖ A D, or E H.
9. By the point A draw A L ‖ C E.

16. . . *Req. demonſtr.* . . { □ A B, B D = □ A C — □ C B. Alſo, □ ½ A B, ½ B D = ¼ □ A C — ¼ □ C B.

Demonſtration.

11. By Conſtr. in 5°, ... C H is □ C D.
12. Therefore out of 6° and 11°, (*per Cor.* 1. *The.* 2. of this *Cha.*) I G = □ I F, and B K = □ B D.
13. And from 12°, (*per* 29. *defin.* 1. *Elem.*) . . . . . B F = B D.
14. By Conſtr. in 7° and 8°, . . . C F is □.
15. Therefore from 14°, (*per* 34. *prop.* 1. *Elem.*) . . . . . B F = C I, and I F = C B.
16. And from 13° and 15°, (*per Ax.* 1.) . . . . . B D = C I.
17. By Conſtr. in 4° and 5°, . . . D H = C A.
18. Therefore out of 16° and 17°, (*per* 36. *prop.* 1. *Elem.*) . . . □ B D, D H = □ C I, C A.
19. That is, (*per Ax.* 7. *Chap.* 2.) □ B H = □ A I.
20. Therefore by adding □ C F to each part of the Equation in 19°, *Gnomon*, I C D H G = A F = □ A B, B D (B F.)
21. But 'tis manifeſt (*per Ax.* 9. *Chap.* 2.) that . . . . . *Gnomon*, I C D H G = □ C H — □ I G.
22. Therefore from 20° and 21°, (*per Ax.* 1.) . . . . . □ A B, B D = □ C H — □ I G.
23. And becauſe from 4°, 5°, 11°, and 15°, ... □ C D (or □ A C) — □ C B (□ I G) = □ C H — □ I G.
24. Therefore from 22° and 23°, (*per Ax.* 1.) . . . . . □ A B, B D = □ A C — □ C B. *Concluſ.* 1.

Which was to be dem.

25. Moreover, from 24°, (*per Ax.* 21. *Chap.* 2.) . . . . ¼ □ A B, B D = ¼ □ A C — ¼ □ C B.
26. And by *Theor.* 4. of this *Chapt.* ¼ □ A B, B D = □ ½ A B, ½ B D.
27. Therefore out of 25° and 26°, (*per Ax.* 1.) . . . . . □ ½ A B, ½ B D = ¼ □ A C — ¼ □ C B. *Concluſ.* 2.

Which was alſo to be dem.

Illuſtration Algebraical.

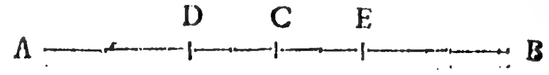
Suppoſe a right-line to be cut into two parts, to wit,	*a* and *b*	6 and 4
Suppoſe alſo	*a* ⊏ *b*	
Then the ſumm of the parts is	*a* + *b*	10
And the difference of the parts is	*a* — *b*	2
Therefore the Rectangle (or Product) of the ſumm and difference of the parts is	*aa* — *bb*	20

Which Rectangle (according to the tenour of *Theor.* 8.) is manifeſtly equal to the difference of the Squares of the parts *a* and *b*. And by multiplying ½*a* + ½*b* into ½*a* — ½*b*, the latter part of the ſaid Theorem will be alſo manifeſt.

Theorem IX.

If a right-line be cut into any two unequal parts, the greater part ſhall be equal to half the whole line, together with half the difference of the parts: And, the leſſer part ſhall be equal to half the whole line leſs by half the difference of the parts.

Suppoſ.



1. AB is a right-line,
2. AE and EB are parts of AB,
3. AE ⊏ EB.

Prepar.

4. From AB cut off AD = EB, thence it follows that DE is the difference of the parts AE and EB; for DE = AE — AD (EB.)
5. Divide DE into two equal parts in C; therefore DC = CE = ½DE.
6. . . . *Req. demonſtr.* . . { AE = ½AB + ½DE. EB = ½AB — ½DE.

Demonſtration.

7. Becauſe by Conſtr. in 4°, . . } AD = EB.
8. And by Conſtr. in 5°, . . . } DC = CE = ½DE.
9. Therefore the ſumm of the Equations in 7° and 8°, gives (*per Ax.* 8. *Chap.* 2.) . . . . } AC = CB = ½AB.
10. And the ſumm of the Equations in 8° and 9° gives . . . . } AE = ½AB + ½DE.
11. And the Equation in 8° ſubtracted from the Equation in 9°, gives } EB = ½AB — ½DE. } Which was to be Dem.

Illuſtration Algebraical.

Suppoſe a right-line to be cut into two unequal parts, to wit, ...	*a* and *b*	6 and 4
Suppoſe alſo	*a* ⊏ *b*	
Then half the whole line, that is, half the ſumm of the parts is	½*a* + ½*b*	5
And half the difference of the parts is	½*a* — ½*b*	1

Now (according to the import of *Theor.* 9.) the ſumm of the ſaid half ſumm and half difference doth manifeſtly make *a* the greater part: And the exceſs of the ſaid half ſumm above the ſaid difference is manifeſtly equal to *b* the leſſer part.

CHAP. V.

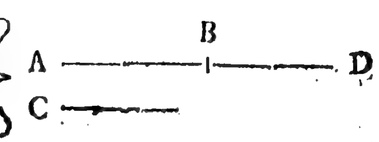
A Collection of Canonical Geometrical Effections, frequently uſed in the Conſtruction of Plane Problems; more eſpecially of thoſe whoſe Solutions are found out by the Algebraical Art.

AS all Arithmetical Operations are compris'd under five kinds, to wit, *Addition*, *Subtraction*, *Multiplication*, *Diviſion* and the *Extraction of Roots*, ſo all thoſe Geometrical Conſtructions which are formed according to Canons deduced from the Algebraical Reſolutions of Problems, do principally depend upon the like kinds of Operations, or Effections; but how theſe Geometrical Effections, (or the Arithmetick of Geometry) may be perform'd, ſo far as is neceſſary to the Conſtruction of Plane Problems, to wit, ſuch as may be ſolved by drawing only right-lines and deſcribing the Circumferences of Circles, I ſhall ſhew in this Chapter, the Contents whereof are extracted out of the firſt ſix Books of *Euclid*'s Elements, wherein I preſuppoſe the Reader to be competently verſed.

Problem I.

To add a given right-line to a right-line given.

Let AB and C be two right-lines given to be added together, *viz.* let it be required to find out a right-line which ſhall be equal to both the given right-lines taken together as one right-line, ... }



Con-

Conſtruction.

By *prop.* 2. *Elem.* 1. produce (or continue) the given line AB to D, that BD may be equal to C, ſo is AD the right-line ſought; for by Conſtruction AD = AB + BD, and BD = C, therefore (*per Ax.* 6. *Chap.* 2.) AD = AB + C, as was required.

Probl. II.

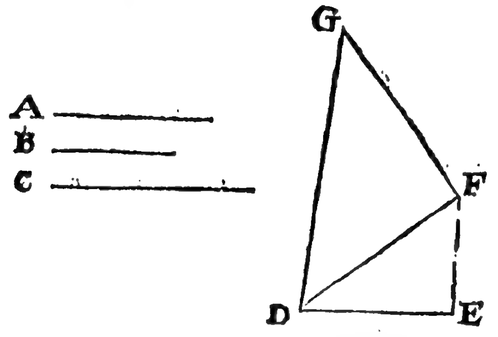
Two or more Squares being given, to find a Square equal to them all.

Suppoſ.

1. A, B, C are the ſides of three Squares given.

Req. to find

2. DG a right-line, ſuch, that □DG = □A + □B + □C.



DE = 16 = A
EF = 12 = B
DF = 20
FG = 21 = C
DG = 29

Conſtr.

3. Make DE = A.
4. Make EF ⊥ DE.
5. Make EF = B.
6. Draw DF.
7. Make FG ⊥ DF.
8. Make FG = C.
9. Draw DG, which ſhall be the ſide of the Square required.
10. . . . *Req. demonſtr.* . □GD = □A + □B + □C.

Demonſtration.

11. Becauſe by Conſtr. in 7° and 4°, } ∠DFG = ⌟ = ∠DEF.
12. Therefore, (*per prop.* 47. *Elem.* 1.) } □DG = □FG + □FD.
13. Likewiſe . . . . . } □FD = □DE + □EF.
14. Therefore from 12° and 13°, (*per Ax.* 6. *Chap.* 2.) . . . } □DG = □DE + □EF + □FG = □A + □B + □C.

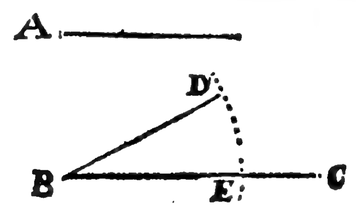
Which was to be done.

After the ſame manner of Conſtruction, as many Squares as one will may be added into one. But if Planes of any other kind, as Long-Squares, Rhombs, Rhomboids, Triangles, *&c.* be given to be added, they muſt firſt be transformed into Squares, which may be done by *Prop.* 14. *Elem.* 2. or by various ways delivered in the practical Geometry of divers Mathematicians, and then they may be added together as before.

Probl. III.

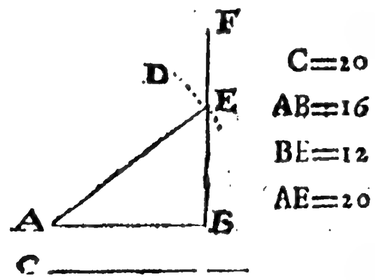
To ſubtract or cut off a right-line given from a greater right-line given.

The ſubtracting or cutting off one right-line from another, to wit, a leſſer from a greater, is perform'd by *Prop.* 3. *Elem.* 1. For, if two unequal right-lines be given, ſuppoſe BC the greater, and A the leſſer, then by deſcribing a Circle from B as a Center, with the diſtance or Semidiameter BD equal to the leſſer line A, the right-line BE = BD or A will be cut off from the greater line BC, and conſequently, EC is the exceſs whereby BC exceeds A or BE.



Probl. IV.

Two unequal Squares being given, to find a Square equal to the exceſs whereby the greater exceeds the leſs.



C = 20
AB = 16
BE = 12
AE = 20

Suppoſ.

1. C and AB are the ſides of two Squares given.
2. C ⊏ AB.

Req. to find

3. BE a right-line, ſuch, that □BE = □C — □AB.

Conſtruction.

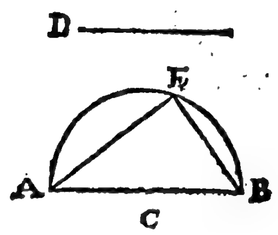
4. Upon the point B, (one of the ends of the given line AB,) erect a Perpendicular, and draw it forth at length, as BF.
5. From A as a Center, at the diſtance of the given line C, deſcribe the arch DE, to cut the Perpendicular BF, ſuppoſe in E; for by ſuppoſition the line C is greater than AB, and therefore a Circle deſcribed upon the Center A, at the diſtance of C ſhall neceſſarily cut the Perpendicular BF produced infinitely.
6. I ſay BE ſhall be the ſide of the Square required.
7. . . . *Req. demonſt.* . . . . . □BE = □C — □AB.

Demonſtration.

8. By Conſtr. in 4° and 5°, ... ∴ ∠ABE is ⌟, and C = AE.
9. Therefore (*per prop.* 47. *Elem.* 1.) . ∴ □AB + □BE = □AE = □C.
10. Therefore (*per Ax.* 9. *Chap.* 2.) . . ∴ . . . □BE = □C — □AB.

Which was to be done.

Another Conſtruction of Probl. 4.



Suppoſ.

11. AB and D are the ſides of two Squares given.
12. AB ⊏ D.

Req. to find

13. EB a right-line, ſuch, that □EB = □AB — □D.

Conſtruction.

14. Upon the given line AB as a Diameter, deſcribe the Semicircle CAEB, and inſcribe AE = D, which is poſſible to be done, for by ſuppoſition AB ⊏ D. Laſtly, draw EB which ſhall be the ſide of the Square required.
15. . . *Req. demonſtr.* . . . . □EB = □AB — □D.

Demonſtration.

16. By Conſtr. in 14°, and *per prop.* 31. *Elem.* 3. . . . . ∠AEB is ⌟, and AE = D.
17. Therefore, *per prop.* 47. *Elem.* 1. . ∴ □AE (or □D) + □EB = □AB.
18. Therefore *per Ax.* 9. *Chap.* 2. . . ∴ □EB = □AB — □D.

Which was to be done.

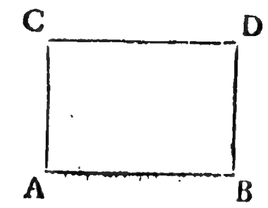
Note. If many Squares be given to be ſubtracted from a Square given, thoſe to be ſubtracted muſt firſt be added together, by the preceding *Probl.* 2. of this *Chapt.* and then ſubtraction may be made by either of the two foregoing Conſtructions of *Probl.* 4. But if Planes which are not Squares be to be ſubtracted, they muſt firſt be reduced to Squares, by *Prop.* 14. *Elem.* 2.

Probl. V.

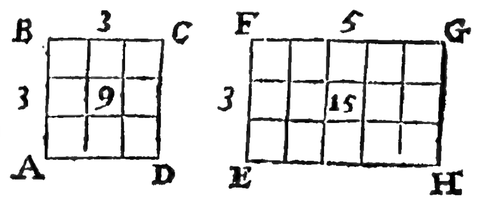
Probl. V.

Concerning Geometrical Multiplication.

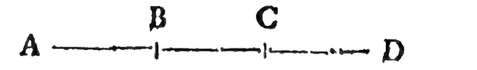
1. A right-line is ſaid to be multiplied by a right-line, when a right-angled Parallelogram, whether it be a Square or a Long-ſquare, is comprehended under one of the ſaid right-lines as a length, and the other as a breadth. As, if the right-line AC be conceived to be moved along the line AB, ſo, as that AC always makes a right-angle with the line AB, until the point C be come to the point D, and the point A to the point B; then the right-angled Parallelogram ACDB is deſcribed by ſuch moving of the line AC, and imports the ſame thing with the Product of the multiplication of the line AB by the line AC. Which Product, or right-angled Parallelogram, is alſo uſually called a Rectangle.



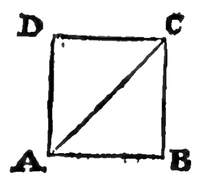
2. A Rectangle is alſo implyed by the Product of the multiplication of any two numbers; for the Area of a Rectangle is equal to the Product made by the multiplication of the number expreſſing the meaſure of one of the ſides about the right-angle, by the number expreſſing the meaſure of the other ſide about the ſame angle. As in the Rectangle or long-Square EG, if its length EH or FG be 5 feet, and the breadth EF or HG, 3 feet, then the Product of the multiplication of 5 by 3, to wit, 15, imports the Area, or number of ſquare feet contain'd in the ſaid Rectangle. Likewiſe, if AE or AD the ſide of the Square AC be 3 feet, the Area is 9 ſquare feet. All which evidently appears by the Diagrams here in view.



3. If a right-line be to be multiplied by a whole number, it may be done by Addition, (by *Probl.* 1. of this *Chapt.*) As, if the line AB be to be multiplied by 3, it implies only the producing or continuing of the ſaid line to ſuch a point D, that the whole line AD may be equal to the triple of the given line AB.

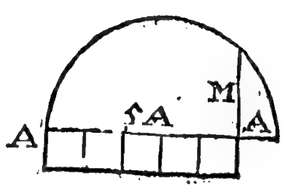


4. But if a right-line AD be to be multiplied by a Fraction, or (which is of the ſame import,) if it be required to cut off ſome ſegment, as ⅔ parts from AD; firſt, (*per Schol.* of *Prop.* 10. *Elem.* 6.) divide the line AD into three equal parts, which ſuppoſe to be AB, BC, CD, then the ſegment AC which is compoſ'd of two of thoſe three parts is manifeſtly ⅔ of the line AD.
5. If a Square be to be multiplied by a whole number, the ſide of the Square ſought may be found out by addition of Squares, as before in *Probl.* 2. So if the Square of the right-line AB, or BC be to be doubled, or multiplied by 2, it implies only the finding of the right-line AC, (by *Probl.* 2. of this *Chapt.*) whoſe Square is equal to the double of the Square of AB or BC. For if AB = BC, and BC ⊥ AB, then (*per prop.* 47. *Elem.* 1.) □AC = □AB + □BC = 2□AB or 2□BC. The ſame thing alſo may be done by the way delivered in the following *Sect.* 6.



AB = 3
BC = 3
AC = √18

6. If a Square be to be multiplied by a whole number, or by any Fraction whatever, whether it be a proper or improper Fraction, that is, if it be required to find the ſide of a Square that ſhall be equal to any preſcribed Multiple, or to any part or parts of a given Square, it may be done thus: Let the right-line A be the ſide of a Square given, and let it be required to find a Square which ſhall contain five times the given Square whoſe



A = 2
5A = 10
5□A = 20
M = √10

ſide

ſide is A. To effect this, find (*per Probl.* 9. of this *Chapt.*) a mean Proportional between the given ſide A, and a right-line equal to five times A; which mean ſuppoſe to be the right-line M: I ſay the line M is the Square required; which I prove thus,

Req. demonſtr. . . . . . □M = 5□A.

Demonſtration.

By Conſtr. theſe are Proportionals, *viz.* . . . } A . M :: M . 5A
Therefore, (*per prop.* 17. *Elem.* 6.) . . . . } □A, 5A = □M
But (*per prop.* 1. *Elem.* 2. reſpect being had to the laſt preceding Diagram,) . . . . . } □A, 5A = 5□A
Therefore from the two preceding Equations, (*per Ax.* 1.) . . . . . } □M = 5□A.

Which was to be done.

In like manner a mean Proportional between A and $\frac{2}{3}$A ſhall be the ſide of a Square equal to $\frac{2}{3}$□A. Alſo a mean Proportional between $3\frac{1}{2}$A (or $\frac{7}{2}$A) and A ſhall be the ſide of a Square equal to $\frac{7}{2}$□A, or $3\frac{1}{2}$□A. The ſame thing may be effected by *Probl.* 11. of this *Chapt.* the proportion of the Square given to the Square ſought being firſt expreſt by two right-lines, by the help of the foregoing *Sect.* 3, or 4. of this *Probl.* 5.

Probl. VI.

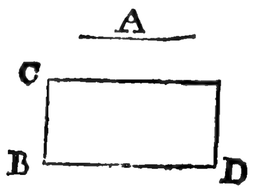
Concerning Geometrical Diviſion, or Application.

That Geometrical Effection which anſwers to *Diviſion* in Arithmetick is called *Application*, the Scope whereof when 'tis exercis'd about the Conſtruction of Plane Problems is only this, *viz.* A Rectangle, (or right-angled Parallelogram) being given, as alſo a right-line, to find out another right-line, ſuch, that a Rectangle contain'd under the line found out, and the line given ſhall be equal to the Rectangle firſt given; which Effection (or Conſtruction) is called the Application of a given Rectangle to a right-line given, and the right-line ariſing out of the Application is called the *Parabola*, or the Geometrical Quotient, which is found out by the *Rule of Three* in right-lines by the following 7th or 8th *Problems* of this *Chapter:* For as the line given is to either of the ſides about the right-angle of the given Rectangle, ſo is the other ſide about the ſame angle, to the line ſought, to wit, the Geometrical Quotient.

This will be made manifeſt by the two following Examples, in the firſt whereof the Rectangle given is a Square, in the latter a long-Square.

Suppoſ. 1.

1. A is the ſide of a Square given,
2. BC is a right-line given.



A = 6
BC = 4
BD = 9

Req. to find

3. BD, a right-line, ſuch, that □BD,BC = □A.

Conſtruction.

4. By *Probl.* 7. of this *Chap.* let it be made as BC to A, ſo A to a third Proportional, which ſuppoſe to be the line BD, therefore, ... } BC . A :: A . BD.
5. I ſay BD is the right-line ſought by the *Probl.* propounded; it remains then to prove that a Rectangle contain'd under the right lines BD and BC is equal to the Square of line A.

Prepar.

6. Let a Rectangle be made of the lines BD and BC, as □CD, (*per prop* 46. *Elem.* 1.) Then
7. . . . *Req. demonſtr.* . . . . . □CD = □A.

Demonſtration.

8. By Conſtr. in 4°, ... } BC . A :: A . BD.
9. Therefore, (*per prop.* 17. *Elem.* 6.) . . . } □A = □BC,BD.
10. But by Conſtr. in 6°, ... } □CD = □BC,BD.
11. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . } □CD = □A.

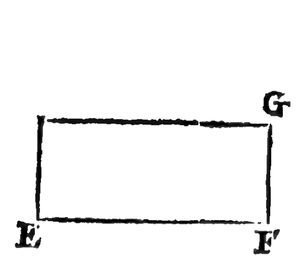
Which was to be done.

Suppoſ. 2.

12. EG is a □ whoſe ſides EF, FG are given ſeverally.
13. BC is a right-line given.

Req. to find

14. BD a right line, ſuch, that . . . . □BC,BD = □EG.



EF = 12
FG = 5
BC = 6
BD = 10

Conſtruction.

15. By *Probl.* 8. of this *Chapt.* let it be made as BC to EF, ſo FG to a fourth Proportional, which ſuppoſe to be the line BD, therefore, } BC . EF :: FG . BD.
16. I ſay BD is the right-line ſought; it remains therefore to prove that a Rectangle contain'd under BC and BD is equal to the given Rectangle EG.

Prepar.

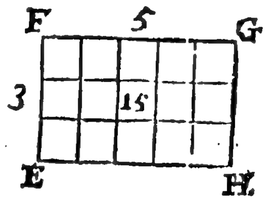
17. Let a Rectangle be made of the lines BC and BD, as □CD, (*per prop.* 46. *Elem.* 1.) Then
18. . . . *Req. demonſtr.* . . . . . □CD = □EG.

Demonſtration.

19. By *Conſtr.* in 15°, ... } BC . EF :: FG . BD.
20. Therefore, (*per prop.* 16. *Elem.* 6.) . . } □EF,FG = □BC,BD.
21. But by *Conſtr.* in 17°, ... } □CD = □BC,BD.
22. Therefore from the two laſt preceding Equations, (*per Ax.* 1. *Chap.* 2.) . . . . . } □CD = □EF,FG, or □EG.

Which was to be done.

23. From the premiſſes 'tis evident that Geometrical Application anſwers to Diviſion in Arithmetick, for the Rectangle applied is correſpondent to the Dividend, and the right-line to which the Rectangle is applied anſwers to the Diviſor, and the right-line ariſing out of the Application, the Quotient: Therefore, if the Area of a Rectangle and either of its ſides be given, the other ſide is alſo given; for if the Area be divided by the given ſide, the Quotient is the other ſide. So if FG, or EH, one of the ſides of the Rectangle EG, be 5, and FE, or GH, the other ſide, 3, the Area 15 divided by 5, (to wit, FG,) gives 3 for FE. Likewiſe the Area 15 divided by 3, (to wit, FE,) gives 5 for FG, or EH.



Probl. VII.

Unto two right-lines given to find a third Proportional.

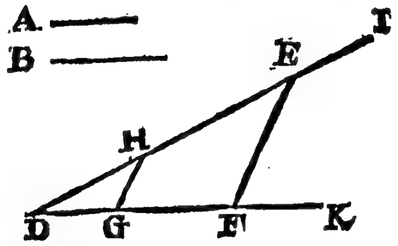
Suppoſ.

1. A and B are two right-lines given.

Req. to find

2. HE a right-line, ſuch, that . . . . . A . B :: B . HE.

A ——



A = 4
B = 6
HE = 9

Construction.

3. Make an angle at pleasure, as $\angle$IDK.
4. Make DG = A. Also GF = B, and DH = B.
5. Draw the right-line HG.
6. By the point F draw FE ‖ GH.
7. I say, HE is the third Proportional sought.
8. . . . *Req. demonstr.* . . . . . A : B :: B . HE.

Demonstration.

9. Because by *Constr.* in 6°, ... } HG ‖ EF (in △DEF.)
10. Therefore, *(per prop. 2. Elem. 6.)* . . . . } DG . GF :: DH . HE.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... } A . B :: B . HE.

Which was to be done.

Probl. VIII.

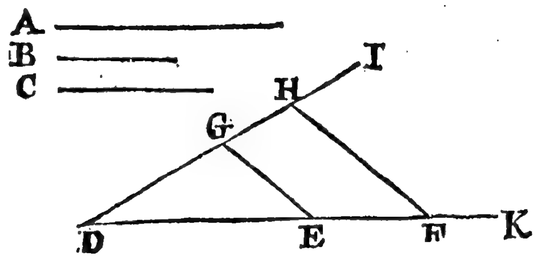
Unto three right-lines given to find a fourth Proportional.

Suppos.

1. A, B, C are three right-lines given.

Req. to find

2. GH a right-line, such, that . . . . . A . B :: C . GH.



A = 27
B = 18
C = 24
GH = 16

Construction.

3. Make an angle at pleasure, as $\angle$IDK.
4. Make DE = A, also EF = B, and DG = C.
5. Draw the right-line EG.
6. By the point F draw FH ‖ EG.
7. I say, GH is the fourth Proportional required.
8. . . . *Req. demonstr.* . . . . . A . B :: C . GH.

Demonstration.

9. Because by *Constr.* in 6°, ... } EG ‖ FH (in △DHF.)
10. Therefore, *(per prop. 2. Elem. 6.)* . . . } DE . EF :: DG . GH.
11. Therefore out of 4° and 10°, by exchanging equal right-lines, ... } A . B :: C . GH.

Which was to be done.

Con-

Concerning the extraction of the Square Root.

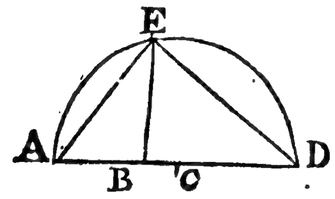
Probl. IX.

Between two right-lines given to find a mean Proportional; or, To transform a Long-square into a Square.

Suppos.

1. AB and BD are two right-lines given.

Req. to find



AB = 9
BD = 16
BE = 12
AD = 25
AC = 12½

2. BE a right-line, such, that . . . . . } AB . BE :: BE : BD.
3. Whence consequently, *(per prop. 17. Elem. 6.)* } □BE = □AB, BD.

Construction.

4. Joyn the given lines AB and BD so together in the point B, that they may make one right-line, as AD.
5. Upon AD as a Diameter, describe the Semicircle CAED.
6. Upon the point B where the given lines AB and BD are joyned together, erect a Perpendicular and extend it to the Circumference of the Semicircle CAED, as BE, which shall be the mean Proportional required.
7. . . . *Req. demonstr.* . . . . . AB : BE :: BE . BD.

Prepar.

8. Draw the right-lines, AE and DE.

Demonstration.

9. By *prop.* 31. *Elem.* 3. . . . . } $\angle$AED is $\perp$.
10. And by *Constr.* in 6°, ... } BE $\perp$ AD.
11. Therefore from 9° and 10°, *(per Coroll. of prop. 8. Elem. 6.)* . . . . . } AB : BE :: BE : BD.

Which was to be done.

Corollary.

12. Hence 'tis manifest, that a right-line drawn in a Circle from any point of the Diameter perpendicularly, and extended to the Circumference, is a mean proportional between the two segments of the Diameter which are made by the same Perpendicular.
13. Moreover, because from the preceding eleventh step, *(per prop. 17. Elem. 6.)* □BE = □AB, BD; therefore, if AB and BD be the sides of a Rectangle, a mean proportional BE between those sides shall be the side of a Square equal to that Rectangle or Long-square.
14. Here also by the way, the Learner may take notice, that a mean proportional number between two given numbers, is found out by extracting the square Root of the Product made by the mutual multiplication of the two numbers given: As, if it be required to find out a Mean between 16 (or BD,) and 9 (or AB,) the Rectangle or Product of 16 into 9 is 144, whose square Root 12, (or BE,) is the mean proportional sought; for 16 . 12 :: 12 . 9.

Probl. X.

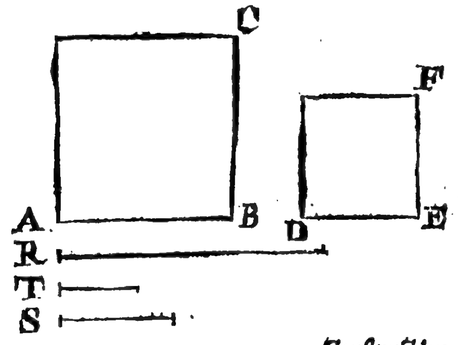
To find a right-line, to which a given right-line may be in the proportion of two Squares given.

Suppos.

1. AC is a Square given, whose side is AB.
2. DF is a Square given, whose side is DE.
3. R is a right-line given.

Req. to find

4. S a right-line, ſuch, that . . . . □AC . □DF :: R . S.



AB = 6
□AB = 36
DE = 4
□DE = 16
R = 9
T = 2⅔
S = 4

Conſtruction.

5. By *Probl.* 7. of this *Chapt.* let it be made as AB to DE, ſo DE to a third proportional, which ſuppoſe to be the line T, therefore . . . } : AB . DE :: DE : T.

6. Again, (by the foregoing *Probl.* 8.) let it be made as AB to T, ſo R to a fourth proportional, which ſuppoſe to be the line S, therefore . . . } : AB : T :: R : S.

I ſay the line S is that required by the *Problem*, therefore

7. . . . *Req. demonſtr.* . . . . □AC . □DF :: R : S.

Demonſtration.

8. Becauſe by *Conſtruction* in 5°, . . . } . AB : DE :: DE : T.
9. Therefore, (*per Coroll. Prop.* 20. *Elem.* 6.) . . . . . } . AB : T :: □AB . □DE.
10. And becauſe by *Conſtr.* in 6°, . . . } : AB : T :: R : S.
11. Therefore from 9° and 10°, (*per Prop.* 11. *Elem.* 5.) . . . . } □AB . □DE :: R . S.
12. That is, (*per Ax.* 7. *Chap.* 2.) . } □AC : □DF :: R . S.

Which was to be done.

Probl. XI.

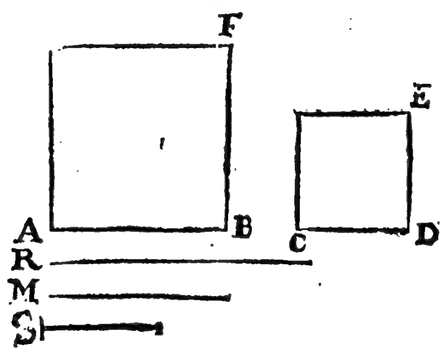
A Square being given, to find out another Square greater or leſs than that given, according to a Proportion given.

Suppoſ.

1. AF is a Square given, whoſe ſide is AB.
2. R and S are two right-lines expreſſing the given Proportion.

Req. to make

3. CE a Square, ſuch, that . . . . . : . R : S :: □AF . □CE.



AB = 6
□AB = 36
R = 9
S = 4
M = 6
CD = 4
□CD = 16

Con-

Conſtruction.

4. By *Probl.* 9. of this *Chapt.* find a mean proportional between the given lines R and S, which mean ſuppoſe to be the right-line M, therefore . . . . . } R . M :: M . S.
5. Again, (by *Probl.* 8. of this *Chapt.*) let it be made as R to M, ſo AB (the ſide of the given Square,) to a fourth proportional, which ſuppoſe to be the right-line CD, therefore } R . M :: AB . CD.
6. Upon the line CD deſcribe a Square, as CE, which ſhall be that required by the *Problem*, therefore
7. . . . *Req. demonſtr.* . . . . . R . S :: □AF . □CE.

Demonſtration.

8. Becauſe by *Conſtr.* in 4°, ... } R . M :: M . S.
9. Therefore, (*per Coroll. prop.* 20. *Elem.* 6.) } □R . □M :: R . S.
10. And becauſe by *Conſtr.* in 5°, ... } R . M :: AB . CD.
11. Therefore, (*per prop.* 22. *Elem.* 6.) . . } □R . □M :: □AB . □CD.
12. Therefore out of 9° and 11°, (*per prop.* 11. *Elem.* 5.) . . . . . } R . S :: □AB . □CD.
13. That is, (*per Ax.* 7. *Chap.* 2.) . . . } R : S :: □AF . □CE.

Which was to be done.

Probl. XII.

The mean of three proportional right-lines being given; as alſo the difference of the extremes, to find the extremes.

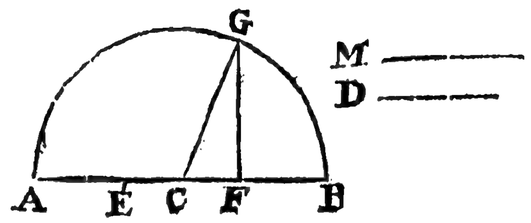
Suppoſ.

1. M = the mean of three proportionals is given.
2. D = the difference of the extremes is given.

Req. to find

3. AF and FB two right-lines, ſuch, that AF — FB = D; Alſo, that
4. AF . M :: M . FB.

M = 12 AF = 18
D = 10 FB = 8
EF = 10 AE = 8
FG = 12 CA = 13
EC = 5 CG = 13
CF = 5 CB = 13



Conſtruction.

5. Make EF = D (the given difference.)
6. Upon the point F erect FG ⊥ FE.
7. Make FG = M (the given mean.)
8. Divide EF into two equal parts in C.
9. Draw the right-line CG.
10. From C as a Center, at the diſtance of CG deſcribe the Semicircle CAGB
11. Draw the Diameter AECFB.
12. I ſay AF and FB are the extreme proportionals required.
13. . . . *Req. demonſtr.* . . . . { AF — FB = D, AF . M :: M : FB.

Demonſtration.

14. Becauſe by *Conſtruction* in 10°, CAGB is a Semicircle whoſe Center is C; therefore, (*per deſin.* 15. *Elem.* 1.) . . . . . } CA = CB = CG.
15. And becauſe by *Conſtr.* in 8°, ... } CE = CF.
16. Therefore the Equation in 15° being ſubtracted from that in 14°, gives (as is evident by the Diagram) . . . . . } EA = FB.
17. It is alſo evident by the Diagram that . . } AF — EA = EF.
18. Therefore out of 16° and 17°, (*per Ax.* 6. *Chap.* 2.) . . . . . } AF — FB = EF.
19. But by *Conſtr.* in 5°, ... } . . . D = EF.
Concluſ. 1. 20. Therefore from 18° and 19°, (*per Ax.* 1.) } AF — FB = D.
Which was to be dem.
21. Again, becauſe (*per Coroll. Probl.* 9. of this *Chapt.*) . . . . . } AF . FG :: FG . FB.
22. And by *Conſtr.* in 7°, ... } M = FG.
Concluſ. 2. 23. Therefore from 21° and 22°, by exchanging equal right-lines, ... } AF . M :: M . FB.
Which was alſo to be dem.

Therefore the *Problem* propounded is ſatisfied; and out of the premiſſes the following *Theorem* is deducible, for the ſolving of the ſame *Probl.* 12. Arithmetically.

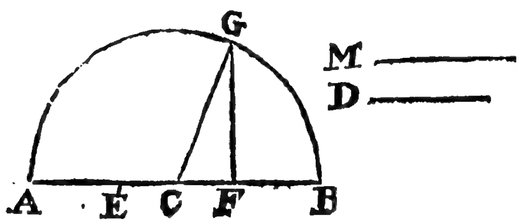
Theorem.

24. If half the difference of the extremes of three Proportionals, be added to the ſide (or ſquare Root) of that Square which is equal to the Square of half the difference of the extremes together with the Square of the mean, the ſumm ſhall be the greater extreme: But if the ſaid half difference be ſubtracted from the ſaid ſide, the remainder ſhall be the leſſer extreme. Hence,

The Arithmetical ſolution of Probl. 12.

Suppoſ.

25. AF, FG, FB are ÷÷, *viz.* AF . FG :: FG . FB
26. AF ⌐ FB.
27. FG = 12. . . . . } Given.
28. EF = 10 = AF — FB. }



Req. to find

29. AF and FB in numbers.

Operation Arithmetical.

30. By *Suppoſ.* in 28°, ... } . . . . EF = 10.
31. Therefore . . . . . } ½EF = CF = 5 = CE.
32. And conſequently, ... } . . . □CF = 25.
33. Alſo from 27°, ... } . . . □FG = 144.
34. The ſumm of the Equations in 32° and 33°, gives (*per prop.* 47. *Elem.* 1.) . . . . } . . . □CG = 169.
35. The ſquare Root of each part of the Equation in 34° gives . . . . . } . . . . CG = 13 = CA.
36. Therefore the ſumm of the Equations in 31° and 35° gives . . . . . } . . . . AF = 18. } Sought.
37. And by ſubtracting the Equation in 31° from that in 35°, ... } EA = FB = 8. }

The Proof.

38. It is manifeſt that theſe are Proportionals, for the Product of the extremes is equal to the Square of the mean, *viz.* . . . . . } 18 . 12 :: 12 . 8.
AF . FG :: FG . FB.

39. Alſo,

39. Alſo the mean proportional is 12, and the difference of the extremes is 10, as was preſcribed.

Coroll. 1.

40. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Equation ſo conſtituted, that the Square of an unknown right-line leſs by a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 12. But here is to be noted, that if the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *prop.* 14. *Elem.* 2.

41. As, in this Equation . . . . . $aa - da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a and ſome known right-line repreſented by d, and that the ſaid unknown Square aa leſs by the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide or right-line a may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

42. For (by *prop.* 14. *Elem.* 6.) the Equation above propos'd in 41° may be reſolved into theſe Proportionals,

$$a - d \;.\; m :: m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes a and $a - d$, (for a exceeds $a - d$ by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the greater whereof is the line a ſought.

43. Morover, if in the Equation above propounded, to wit, $aa - da = mn$, we ſuppoſe $m = 12$, and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the firſt part of the preceding *Theorem* in 24°,

$$a = \tfrac{1}{2}d + \sqrt{:\tfrac{1}{4}dd + mm:} = 18.$$

Coroll. 2.

44. It alſo follows from the preceding *Conſtruction* of *Probl.* 12. that if by the Algebraical Reſolution of a Geometrical *Problem*, an Equation be found out, ſuch, that the Square of a right-line ſought, together with a Rectangle contain'd under that unknown line and ſome known right-line, is equal to a known Plane; the ſaid unknown line ſhall be given by the Geometrical *Conſtruction* of the ſaid *Probl.* 12. But if (as before hath been ſaid) the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square.

45. As, in this Equation, ... $aa + da = mm$;
If we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a, alſo da a Rectangle contain'd under the ſaid unknown ſide a, and ſome known right-line repreſented by d; and that the ſaid unknown Square aa together with the ſaid Rectangle da is equal to ſome known Square, as mm, whoſe ſide is m; then the ſaid unknown ſide, or right-line a, may be found out by the Geometrical *Conſtruction* of the preceding *Probl.* 12.

46. For the Equation above propounded in 45° may be reſolved into theſe Proportionals, *viz.*

$$a + d \;.\; m :: m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo d the difference of the extremes $a + d$ and a, (for $a + d$ exceeds a by d,) therefore by the *Conſtruction* of the foregoing *Probl.* 12. the extremes ſhall be given ſeverally, the leſſer whereof is the line a ſought.

47. Laſtly, if in the Equation above propounded in 45°, to wit, $aa + da = mm$, we ſuppoſe $m = 12$ and $d = 10$, then the quantity of the line a ſhall be alſo given in number; for by the latter part of the preceding *Theorem* in 24°,

$$a = \sqrt{:\tfrac{1}{4}dd + mm:} - \tfrac{1}{2}d = 8.$$

Probl. XIII.

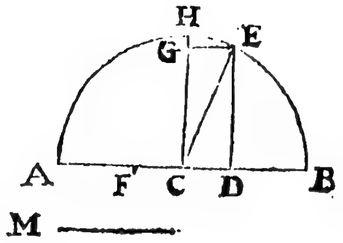
The mean of three proportional right-lines being given, as alſo the ſumm of the extremes, to find the extremes. But the given mean muſt not exceed half the given ſumm of the extremes.

Suppoſ.

1. M = the mean of three Proportionals is given.
2. AB = the ſumm of the extremes is given. Alſo M not ⊏ ½ AB.

Req. to find

3. AD and DB two right-lines, ſuch, that AD + DB = AB. Alſo, that
4. AD . M :: M . DB.



M = 12
AB = 26
AC = 13
CE = 13
CB = 13
AD = 18
DB = 8
DE = 12
CD = 5
CF = 5

Conſtruction.

5. Divide AB into two equal parts in C.
6. From C as a Center, with the diſtance CA, or CB, deſcribe the Semicircle CAHB.
7. Upon the Center C erect CH ⊥ AB.
8. From CH cut off CG = M, (the given mean Proportional,) which is poſſible to be done, for by the Determination annex'd to the *Problem* propounded, M not ⊏ CH, or ½ AB.
9. By the point G draw GE ‖ AB, which GE will either touch the Semicircle in H, when M = CH = ½ AB; or elſe cut the Semicircle, when M ⊐ CH, (or ½AB,) ſuppoſe then in this Example that M ⊐ CH, and conſequently that GE cuts the Circumference, as in E.
10. From the point E let fall ED ⊥ AB, then ſhall AD and DB be the extreme Proportionals required; for firſt their ſumm, in regard ADB is a right-line, (to wit, the Diameter of the Semicircle CAHB,) is equal to AB the given ſumm; it remains to prove that as AD is to M, ſo M to DB, therefore
11. . . . *Req. demonſtr.* . . . . . AD . M :: M . DB.

Demonſtration.

12. Becauſe by *Conſtruction* in 5° and 6°, . . . } CAEB is a Semicircle.
13. And by *Conſtr.* in 10°, ... } DE ⊥ AB.
14. Therefore from 12° and 13°, (*per Coroll.* in 12° of *Probl.* 9. *Chap.* 5.) . . . . . } AD . DE :: DE . DB.
15. And becauſe by *Conſtr.* in 7°, 9°, 10°, . . } DG is □.
16. Therefore (*per prop.* 34. *Elem.* 1.) . . . } DE = CG.
17. But by *Conſtr.* in 8°, ... } M = CG.
18. Therefore out of 16° and 17°, (*per Ax.* 1.) } DE = M.
19. Wherefore from 14° and 18°, (by taking M inſtead of DE,) . . . . . } AD . M :: M . DB.

Which was to be done.

20. The reaſon of the Determination annex'd to the *Problem*, to wit, that the line preſcribed for the mean muſt not exceed half the line given for the ſumm of the extremes, will be evident by this that follows. Firſt, if the right-line M be leſs than CH, or ½ AB, and at the diſtance of M a line be drawn parallel to the Diameter AB, as the parallel GE, it will neceſſarily cut the Semicircle, as in E, in which caſe the extreme Proportionals, to wit, the ſegments of the Diameter which are made by the falling of the Perpendicular ED, will always be unequal. Secondly, if the line M be equal to CH, or ½ AB, and at the diſtance of M or CH a line be drawn parallel to the Diameter AB, ſuch parallel will touch the Semicircle in H, and conſequently HC which is perpendicular to AB will be a mean between AC and CB, in which caſe, the three Proportionals AC, CH and CB are equal to one another, for each of them is the Semidiameter of the Semicircle. Laſtly, if the line M be greater than CH, or ½ AB, then 'tis eaſie to perceive, that a right-line drawn parallel to the Diameter AB at the diſtance of ſuch line M cannot poſſibly either touch or cut the Semicircle, but will lye altogether without the ſame, and conſequently ſuch line M cannot be a mean pro-

portional between any two ſegments of the Diameter; for a mean proportional line between two extremes is a right-line within a Circle, drawn perpendicularly to the Diameter, and extended only unto (not beyond) the Circumference: And therefore that there may be a poſſibility of ſolving this 13th *Problem*, the line given for the mean Proportional muſt nor be longer than half the line given for the ſumm of the extremes; which is the Determination annex'd to the *Problem.*

The premiſſes being well underſtood, it will not be difficult to apprehend how the following *Theorem* is thence deducible, for the ſolving of *Probl.* 13. Arithmetically.

Theorem.

21. If half the ſumm of the extremes of three Proportionals be increaſed with the ſide (or ſquare Root) of that Square which is equal to the exceſs whereby the Square of the ſaid half ſumm exceeds the Square of the mean; the ſaid half ſumm ſo increaſed ſhall be equal to the greater extreme: But if the ſaid halt ſumm be leſſened by the ſide (or ſquare Root) aforeſaid, the remainder ſhall be the leſſer extreme. Hence,

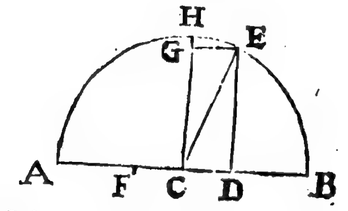
The Arithmetical ſolution of Probl. 13.

Suppoſ.

22. AD, DE, DB are ÷÷, *viz.* AD . DE :: DE . DB.
23. DE = 12. . . . . } Given.
24. AB = 26 = AD + DB. }

Req. to find

25. AD and BD in numbers.



Operation Arithmetical.

26. By *Suppoſ.* in 24°, ... } . . . AB = 26.
27. Therefore, ... } ½AB = CB = 13 = CE = CA.
28. And conſequently, ... } . . . □CE = 169.
29. And from 23°, ... } . . . □DE = 144.
30. Therefore by ſubtracting the Equation in 29°, from that in 28°, there will remain (*per prop.* 47. *Elem.* 1.) . . . . . } . . . □CD = 25.
31. And conſequently, by extracting the ſquare Root out of each part of the laſt Equation, } . . . CD = 5.
32. Therefore by adding together the Equations in 27° and 31°, ... } . . . AD = 18. } Sought.
33. And by ſubtracting the Equation in 31° from that in 27°, ... } . . . DB = 8. }

Coroll.

34. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Equation ſo conſtituted, that a Rectangle contain'd under ſome known right-line and a right-line ſought, leſs by the Square of the ſame line ſought, is equal to a known Plane; then the ſaid right-line ſought ſhall be given by the preceding Geometrical *Conſtruction* of *Probl* 13. But here is to be noted, that if the ſaid known Plane be not a Square, it muſt firſt be reduced to a Square, by *Probl.* 9. of this *Chapt.* or by *Prop.* 14. *Elem.* 2.
35. As, if this Equation be propoſed, ... $sa - aa = mm$;
We may ſuppoſe a in that Equation to repreſent a right-line unknown, aa the Square of that line, s a right line known, and that sa the Rectangle contain'd under thoſe lines, leſs by the ſaid unknown Square aa, is equal to ſome known Square, as mm, whoſe ſide is m; then ſhall the ſaid unknown right line a be given by the preceding *Conſtruction* of *Probl.* 13.
36. For the Equation before propos'd in 35°, may be reſolved into theſe Proportionals, *viz.*

$$s - a \;.\; m :: m \;.\; a;$$

Of which three Proportionals the mean m is known by ſuppoſition, as alſo s the ſumm

of

of the extremes $s-a$ and a; therefore, by the foregoing *Conſtruction* of *Probl.* 13. the extremes ſhall be given ſeverally, either of which may be taken for the line a ſought.

37. For (viewing the Diagram belonging to *Probl.* 13.) let us ſuppoſe that . . . . . } AD . DE :: DE . DB.
38. Then putting . . . . . } $m = DE$.
39. Alſo . . . . . } $s = AB$.
40. And . . . . . } $a = AD$.
41. It follows that . . . . . } $s-a = DB = AB - AD$.
42. And . . . . . } $a . m :: m . s-a$.
43. Wherefore, by comparing the Rectangle of the extremes to the Square of the mean, this Equation is produced, ... } $sa-aa = mm$.

Which is the ſame with the Equation before propos'd in 35°.

44. Again, the ſame Equation will ariſe if we put $a = DB$, the *Suppoſitions* in 37°, 38° and 39° remaining unalter'd; for,
45. Then it will follow that . . . . . } $s-a = AD$.
46. And . . . . . } $s-a . m :: m . a$.
47. That is, ... } AD . DE :: DE . DB.
48. Therefore from 46°, ... } $sa-aa = mm$.

Which is the ſame with the Equation before produced in 43°.

49. Whence 'tis manifeſt that the right line a ſought in all Quadratick Equations of the ſame form with that before propos'd in 35°, may be either of two right lines, repreſented in the Diagram by AD and DB, for which cauſe ſuch Equation is called *Ambiguous*.
50. Laſtly, if in the Equation above propounded, to wit, $sa - aa = mm$, we ſuppoſe $m = 12$, and $s = 26$, then the quantity of the line a ſhall be alſo given in number; for by the preceding *Theorem* in 21°,

$$a = \tfrac{1}{2}s + \sqrt{:\tfrac{1}{4}ss - mm:} = 18;$$
$$\text{Or,}\quad a = \tfrac{1}{2}s - \sqrt{:\tfrac{1}{4}ss - mm:} = 8.$$

Probl. XIV.

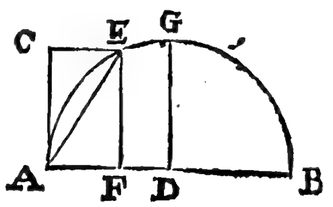
To divide a right-line given into two parts, ſuch, that another right-line may be a mean Proportional between the parts. But the line given for the mean muſt not exceed half the line given to be divided.

Suppoſ.

1. AB is a right line given to be cut into two parts.
2. AC is a right line given to be a mean Proportional between the parts.
3. AC not $\sqsubset \frac{1}{2}$ AB.

Req. to find

4. AF and FB ſuch parts of AB that AF + FB = AB; Alſo, that
5. AF . AC :: AC . FB.



AB = 26
AC = 12
AD = 13
DB = 13
AF = 8
FE = 12
FB = 18

Conſtruction.

6. Upon A one of the ends of AB, erect AC ⊥ AB.
7. Divide AB into two equal parts in D.
8. From D as a Center, with the diſtance DA, or DB, deſcribe the Semicircle DAEB.
9. By the point C draw CE ∥ AB, ſo ſhall CE (by what hath been ſaid in 20° of *Probl.* 13.) either touch or cut the Semicircle DAEB, for by *Suppoſ.* AC not $\sqsubset \frac{1}{2}$ AB, (or DG,) ſuppoſe then in this Example that AC $\sqsupset$ DG, or DA, and conſequently that CE cuts the Semicircle as in E.
10. From the point E let fall EF ⊥ AB, then ſhall AF and FB be the deſired parts of AB, between which parts, AC is a mean Proportional, as I ſhall in the next place demonſtrate.
11. . . . *Req. demonſtr.* . . . . . AF . AC :: AC . FB.

Preparat.

12. Draw the right lines AE and EB.

Demon-

Demonſtration.

13. Becauſe by *Conſtr.* in 8°, DAEB is a Semicircle, therefore (*per prop.* 31. *Elem.* 3.) . . . . } ∠AEB is ⌟.
14. And becauſe by *Conſtr.* in 10°, ... } EF ⊥ AB.
15. Therefore from 13° and 14°, (*per Coroll. prop.* 8. *Elem.* 6.) . . . . . } AF . FE :: FE . FB.
16. And becauſe by *Conſtr.* in 6°, 9° and 10°, . . . } CF is □.
17. Therefore, (*per prop.* 34. *Elem.* 1.) . . . . } AC = FE.
18. Therefore from 17° and 15°, by taking AC inſtead of FE, ... } AF . AC :: AC . FB.

Which was to be done.

Note. This Problem may be ſolved Arithmetically in the ſame manner as the preceding *Probl.* 13.

Probl. XV.

The mean of three Squares in continual proportion being given, as alſo a Square equal to the difference of the extremes, to find out the extremes.

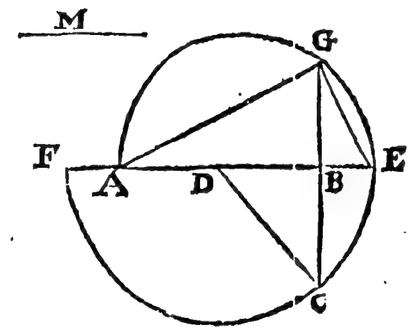
Or thus;

The Baſe of a right-angled Triangle being given, as alſo a mean proportional between the Hypothenuſal and Perpendicular, to find the Triangle.

Suppoſ.

1. AB = the Baſe of a right-angled Triangle is given.
2. M = a mean proportional between the Hypothenuſal and Perpendicular is given.

Req. to find out the Triangle.



3. Making the given Baſe AB the firſt of three Proportionals, and the given Mean M the ſecond, find out a third, (*per Probl.* 7. of this *Chapt.*) which ſuppoſe to be BC, therefore, } AB : M :: M : BC.
4. Upon B one of the ends of the given Baſe AB, make BC ⊥ AB, whence ∠ABC is ⌟.
5. Divide AB into two equal parts in D.
6. Draw the right-line DC.
7. From D as a Center, at the diſtance of DC, deſcribe the Semicircle DFCE having FADBE for its Diameter.
8. Upon AE as a Diameter deſcribe the Semicircle AGE.
9. Continue CB to the Circumference in G.
10. Draw the right lines AG and EG.
11. I ſay ABG is the right-angled Triangle required; now we muſt ſhew that it will ſatisfie the *Problem* propounded. Firſt then, by *Suppoſition* AB is equal to the given Baſe, but that the angle ABG is a right-angle, and that the given line M is a mean proportional between the Hypothenuſal AG and the Perpendicular BG, the following *Demonſtration* will make manifeſt.

12. . . . *Req. demonſtr.* . . . . } ∠ABG is ⌟; Alſo, AG . M :: M . BG.

Demonſtration.

13. Becauſe by *Conſtr.* in the fourth ſtep, . } ∠ABC is ⌟,
14. And by *Conſtr.* in 9°, ... } CBG is a right-line;
15. Therefore, (*per Coroll. Prop.* 13. *Elem.* 1.) } ∠ABG is ⌟. Which was to be Dem.
16. Again, becauſe by *Conſtr.* in 7°, . . } DF = DE = DC,
17. And by *Conſtr.* in 5°, ... } DA = DB;
18. Therefore by ſubtracting the Equation in 17° from that in 16°, ... } AF = BE.

19. And by adding AB to each part of the laſt preceding Equation, ... } . . FB = AE.
20. Again, out of 4° and 7° (*per Coroll. Probl.* 9. of this *Chapt.* . . . . . } FB . BC :: BC . BE.
21. Therefore from 20°, (*per prop.* 17. *Elem.* 6.) } □FB, BE = □BC.
22. And from 19°, (*per prop.* 1. *Elem.* 6.) by taking BE as a common altitude, . . . } □FB, BE = □AE, BE.
23. Therefore out of 21° and 22° (*per Ax.* 1. *Chap.* 2.) . . . . . } . . □BC = □AE, BE.
24. Again, becauſe by *Conſtr.* in 8°, . . . } AGE is a Semicircle.
25. Therefore, (*per prop.* 31. *Elem.* 3.) . . } ∠AGE is ⌟.
26. And becauſe by what hath been proved in 15°, } GB ⊥ BA.
27. Therefore from 25° and 26°, (*per Coroll.* 2. *prop.* 8. *Elem.* 6.) . . . . . } AE . GE :: GE . BE.
28. Therefore out of 27°, (*per prop.* 17. *Elem.* 6.) } □GE = □AE, BE.
29. But it hath been proved in 23°, that . . } □BC = □AE, BE.
30. Therefore from 28° and 29°, (*per Ax.* 1.) } □BC = □GE.
31. And conſequently, ... } BC = GE.
32. Again, out of 25° and 26°, (*per prop.* 8. *El.* 6.) } △ABG and △GBE are like.
33. Therefore from 32°, (*per prop.* 4. *Elem.* 6.) } AB . AG :: BG . GE.
34. And from 33°, (*per prop.* 16. *Elem.* 6.) } □AG, BG = □AB, GE.
35. And out of 31°, by taking AB for a common altitude, ... } □AB, BC = □AB, GE.
36. Therefore from 34° and 35°, (*per Ax.* 1.) } □AB, BC = □AG, BG.
37. But by *Conſtr* in 3°, and *per prop.* 17. *Elem.* 6. } □AB, BC = □M.
38. And conſequently from 36° and 37°, (*per Ax.* 1.) . . . . . } □AG, BG = □M.
39. Therefore out of 38°, (*per prop.* 14. *Eeml.* 6.) } AG . M :: M . BG.

Which was to be dem. And therefore the *Problem* is ſatisfied.

Coroll. 1.

40. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continued proportion, ſuch, that the greater extreme is a Square unknown, the mean a known Square, and the leſſer extreme the exceſs whereby that unknown Square exceeds ſome known Square; then the ſide of the ſaid unknown Square ſhall be given by the Geometrical *Conſtruction* of the preceding *Probl.* 15.

As, for example, let there be propos'd the following Analogy, where *aa* repreſents a Square unknown, and *a* its ſide; alſo *mm* and *bb* two known Squares, whoſe ſides are *m* and *b*;

$$aa - bb \;.\; mm :: mm \;.\; aa.$$

41. Then, (becauſe, by *prop.* 22. *Elem.* 6. the ſides of proportional Squares are proportionals alſo,) from the Analogy propos'd this ariſeth, *viz.*

$$\sqrt{:aa - bb:} \;.\; m :: m \;.\; a.$$

42. Which three Proportionals laſt expreſt being well conſider'd, it will be manifeſt that the greater extreme, to wit, *a*, may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is *b*, and Perpendicular $\sqrt{:aa - bb:}$: the leſſer extreme: Now in that right-angled Triangle the Baſe *b* is given by ſuppoſition, as alſo *m* a right line which is a mean proportional between the ſaid Hypothenuſal and Perpendicular, and therefore the Hypothenuſal and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 15. which Hypothenuſal ſhall be the line repreſented by *a* in the Analogy propos'd.

43. It is alſo manifeſt, that if the Terms of the Analogy propos'd for an Example in 40°, be ſuppos'd to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa - bbaa = mmmm;$$

Where,

where, if we ſuppoſe $bb = 6$; alſo $mm = \sqrt{27}$, and *a* to ſtand for ſome number unknown, that Biquadratick Equation may be expreſt thus, *viz.*

$$aaaa - 6aa = 27.$$

44. In which laſt Equation, the unknown number *a* may be found out either by an Arithmetical Operation deducible from the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals,

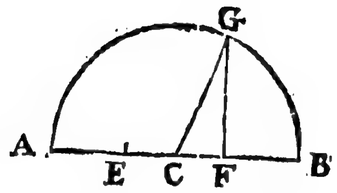
$$aa - 6 \;.\; \sqrt{27} :: \sqrt{27} \;.\; aa,$$

The greater extreme *aa* may firſt be made known after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. and then the ſquare Root of that number found out for the value of *aa* ſhall be the number *a* ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

45. AF, FG, FB ÷÷; *viz.* AF . FG :: FG . FB.
46. AF ⊏ FB.
47. □FG = 27. . . . . } given.
48. AF − FB = 6 = EF. }

Req. to find out



49. AF the greater extreme, ſignified by *aa* in the Analogy before expreſt in 44°.

Operation Arithmetical.

50. By *Suppoſition* in 48°, ... } . . . EF = 6.
51. Therefore . . . . . } ½EF = CF = 3 = CE.
52. And conſequently, ... } . . □CF = 9.
53. By *Suppoſ.* in 47°, ... } . . □FG = 27.
54. The ſumm of the two laſt Equations gives (*per prop.* 47. *Elem.* 1.) . . . . . } . . □CG = 36.
55. The ſquare Root of the laſt Equation gives } . . . CG = 6 = CA.
56. The ſumm of the Equations in 51° and 55°, gives . . . . . } . . . AF = 9 = *aa*.
57. Wherefore, ... } . . . $\sqrt{9} = 3 = a$.
58. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame form with that before expreſt in 43°, and here-under repeated, where *a* repreſents a number ſought, but *b* and *m* two numbers given; *viz.*

If . . . . . $aaaa - bbaa = mmmm$,

Then . . . . . $a = \sqrt{:\frac{1}{2}bb + \sqrt{\frac{1}{4}bbbb + mmmm}:}$

Coroll. 2.

59. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the leſſer extreme is a Square unknown, the mean a known Square, and the greater extreme is compoſed of the ſaid unknown Square and ſome known Square; then the ſide of the ſaid unknown Square ſhall be given by the Geometrical *Conſtruction* of the foregoing *Probl.* 15. As, for example,

If this Analogy be propos'd, where *aa* repreſents a Square unknown, and *a* its ſide; alſo, *mm* and *bb* two known Squares, whoſe ſides are *m* and *b*, ... } $aa + bb \;.\; mm :: mm \;.\; aa.$

60. Then (*per prop.* 22. *Elem.* 6.) theſe alſo ſhall be Proportionals, ... } $\sqrt{:aa + bb:} \;.\; m :: m \;.\; a.$

61. Which three laſt preceding Proportionals being well examined, it will be manifeſt that the greater extreme, to wit, $\sqrt{:aa + bb:}$ may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is *b*, and Perpendicular *a*, (the leſſer extreme.) Now in that right-angled Triangle the Baſe *b* is given by ſuppoſition, as alſo *m* a right-line which is a mean Proportional between the ſaid Hypothenuſal and Perpendicular, and therefore the Hypothenuſal and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 15. which Perpendicular ſhall be the line repreſented by *a* in the Analogy propos'd in 59°.

62. It is alſo manifeſt, that if the Terms of the Analogy in 59° be ſuppoſed to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this following Biquadratick Equation will thence be produced, *viz.*

$$aaaa + bbaa = mmmm.$$

Where, if we ſuppoſe $bb = 6$, alſo $mm = \sqrt{135}$, and a to repreſent ſome number unknown; that Biquadratick Equation may be expreſt thus, $aaaa + 6aa = 135$.

63. In which laſt Equation the unknown number a may be found out either by an Arithmetical Operation, deducible out of the preceding Geometrical *Conſtruction* of *Probl.* 15. or elſe after the ſaid Equation is reſolved into theſe three continual Proportionals, ... $aa + 6 : \sqrt{135} :: \sqrt{135} : aa.$

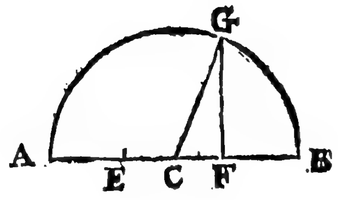
The leſſer extreme aa may firſt be diſcovered after the manner of the Arithmetical Solution of the foregoing *Probl.* 12. of this *Chapt.* and then the ſquare Root of that number found out for the value of aa ſhall be the number a ſought. All which will be manifeſt by the following Operation and Diagram.

Suppoſ.

64. AF,FG,FB, ÷ *viz.* AF . FG :: FG . FB.
65. AF ⊏ FB.
66. □FG = 135.
67. AF — FB = 6 = EF.

Req. to find out

68. FB the leſſer extreme, which anſwers to aa in the Analogy before expreſt in 63°.



Operation Arithmetical.

69. By *Suppoſ.* in 67°, ... EF = 6.
70. Therefore . . . . . ½EF = CF = 3 = CE.
71. And conſequently, ... □CF = 9.
72. By *Suppoſ.* in 66°, ... □FG = 135.
73. The ſumm of the two laſt preceding Equations gives (*per prop.* 47. *Elem.* 1.) . . . . . □CG = 144.
74. The ſquare Root of the Equation in 73° gives . . . . CG = 12 = CB.
75. And by ſubtracting the Equation in 70°, from that in 74°, ... FB = 9 = *aa*.
76. Wherefore . . . . . √9 = 3 = *a*.
77. Moreover, out of the premiſſes ariſeth the following *Canon*, for the Arithmetical Reſolution of all Biquadratick Equations falling under the ſame Form with that before expreſt in 62°, and here-under repeated; where a repreſents a number ſought, but b and m two given numbers; *viz.*

If . . . . . $aaaa + bbaa = mmmm$,

Then . . . . . $a = \sqrt{:\sqrt{\frac{1}{4}bbbb + mmmm} - \frac{1}{2}bb}$:

Corôll. 3.

78. From the premiſſes alſo it may be inferr'd, That if the Algebraical Reſolution of a Geometrical *Problem* diſcovers an Analogy conſiſting of three Planes in continual proportion, ſuch, that the mean is a Square known, and in each of the extremes one and the ſame unknown Square is found Affirmative, (that is, with the Sign + prefixt to it,) together with ſome known Square having either + or — prefixt to it, or when in each of the extremes one and the ſame unknown Square is found negative, (that is, with the ſign — ſet before it,) together with ſome known Square having the ſign + prefixt to it; then in either of thoſe Caſes, the ſide of that unknown Square may be found out Geometrically by the preceding *Conſtruction* of *Probl.* 15. as will be manifeſt by the four following Examples.

Example 1.

79. Suppoſe . . . . . $\overline{aa + bb} . mm :: mm . \overline{aa + cc}$;
And conſequently, $\sqrt{:aa + bb} : m :: m : \sqrt{:aa + cc}$:

In

In which Analogies, let aa repreſent an unknown Square whoſe ſide is a, but bb, mm and cc three known Squares, whoſe ſides are b, m and c; ſuppoſe alſo that b is greater than c, and conſequently $aa + bb \sqsubset aa + cc$. I ſay then, that the unknown ſide a ſhall be given by the preceding *Conſtruction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the firſt Analogy above propos'd is $bb - cc$, which by *Suppoſ.* is given, therefore the ſide of a Square equal to that difference, to wit, $\sqrt{:bb - cc}$: may be eſteem'd the Baſe of a right-angled Triangle, by the help of which given Baſe, and of the right-line m, which by *Suppoſition* is a given mean Proportional between the Hypothenuſal $\sqrt{:aa + bb}$: and the Perpendicular $\sqrt{:aa + cc}$: the Hypothenuſal and Perpendicular ſhall be given ſeverally (by the ſaid *Probl.* 15.) and may be repreſented by h and p, whoſe Squares are hh and pp; then by equating hh to the greater extreme $aa + bb$, or pp, to $aa + cc$, the ſide a will be found equal to $\sqrt{:hh - bb}$; or $\sqrt{:pp - cc}$: which Roots are given, and equal to one another.

Example 2.

80. Suppoſe . . . . . $\overline{aa - bb} . mm :: mm : \overline{aa - cc}$,
And conſequently, $\sqrt{:aa - bb} : m :: m : \sqrt{:aa - cc}$:

In which Analogies, if (as before in *Example* 1.) we ſuppoſe aa to repreſent an unknown Square whoſe ſide is a; alſo bb, mm and cc three known Squares, whoſe ſides are b, m and c, and b to be greater than c, whence conſequently $aa - cc \sqsubset aa - bb$; then the unknown ſide a ſhall be given by the preceding *Conſtruction* of *Probl.* 15. For the difference of the extremes of the three Proportionals in the firſt Analogy above propos'd is $bb - cc$, which by *Suppoſition* is given; then the ſide of a Square equal to that difference, to wit, $\sqrt{:bb - cc}$: may be eſteem'd the Baſe of a right-angled Triangle, by the help of which given Baſe and of the right-line m, which by *Suppoſition* is a given mean Proportional between the Hypothenuſal $\sqrt{:aa - cc}$: and the Perpendicular $\sqrt{:aa - bb}$: the Hypothenuſal and Perpendicular ſhall be given ſeverally by the ſaid *Probl.* 15. and may be repreſented by h and p, whoſe Squares are hh and pp; then by equating hh to $aa - cc$ (the greater extreme,) or pp to $aa - bb$; the unknown ſide or right-line a will be found equal to $\sqrt{:hh + cc}$: or $\sqrt{:pp + bb}$: which Roots are given, and equal to one another.

Example 3.

81. Suppoſe . . . . . $\overline{aa + bb} . mm :: mm . \overline{aa - cc}$,
And conſequently, $\sqrt{:aa + bb} : m :: m : \sqrt{:aa - cc}$:

Example 4.

82. Suppoſe . . . . . $\overline{bb - aa} : mm :: mm . cc - aa$,
And conſequently, $\sqrt{:bb - aa} : m :: m : \sqrt{:cc - aa}$:

By what hath been ſaid in the Explication of *Examples* 1. and 2. it will not be difficult to conceive how to find out in like manner the unknown Root or right-line repreſented by a in the third and fourth *Examples*, where b, m and c are ſuppoſ'd to be right-lines ſeverally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in any of the three preceding *Corollaries* of *Probl.* 15. ſuch known Squares or Rectangles muſt firſt of all be converted into a ſimple Square, *per Probl.* 2. *Chap.* 4.

Probl. XVI.

The mean of three Squares in Continual proportion being given, as alſo a Square equal to the ſumm of the extremes, to find out the extremes.

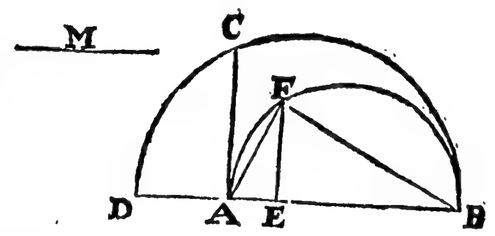
Or

Or thus,

The Hypothenuſal of a right-angled Triangle being given, as alſo a mean Proportional between the Baſe and Perpendicular, to find out the Triangle. But a right-~~angle~~ line ariſing out of the Application of the Square of the given Mean to the given Hypothenuſal muſt not exceed half the Hypothenuſal.

Suppoſ.

1. A B = the Hypothenuſal of a right-angled Triangle is given.
2. M = a mean proportional between the Baſe and Perpendicular is given.
3. $\frac{\square M}{AB}$ not $\sqsubset \frac{1}{2}$ A B.



Req. to find out the Triangle.

Conſtruction.

4. Upon the point A, (one of the ends of the given Hypothenuſal AB,) erect AC ⊥ AB, and make A C = M the given mean Proportional.
5. Let it be made *(per Probl. 7. of this Chapt.)* as AB to AC, ſo AC to a third Proportional, which ſuppoſe to be found AD, therefore, That is, (becauſe M = AC,) . . . ⁊ A B . A C :: A C . A D. ⁊ A B . M :: M . A D.
6. Upon A B as a Diameter deſcribe the Semicircle A F B.
7. Upon D B as a Diameter compos'd of A B and A D as one right-line, deſcribe the Semicircle D C B.
8. By *Probl.* 14. of this *Chapt.* divide the given Hypothenuſal AB into two ſuch parts, that the line A D may be a mean proportional between the parts; which is poſſible to be done, for by *Conſtruction* in 5°, A D = $\frac{\square M}{AB}$, and by *Suppoſition* in 3°, agreeable to the Determination annex'd to the *Problem*, $\frac{\square M}{AB}$ or A D not $\sqsubset \frac{1}{2}$ A B; ſuppoſe then the line A B to be cut in F, into two ſuch parts A E and E B, that E F is a mean proportional between A E and E B, and that E F is equal to A D, therefore,
A E . E F (or A D,) :: E F (or A D,) . E B.
9. Draw the lines A F and B F, then ſhall A F B be the Triangle ſought. Now we muſt ſhew that it will ſatisfie the *Problem.* Firſt then, by ſuppoſition A B is equal to the given Hypothenuſal; but that the angle A F B is a right-angle, and that the given right-line M is a mean proportional between A F and B F, (to wit, the Baſe and Perpendicular,) the following *Demonſtration* will make manifeſt.
10. . . *Req. demonſtr.* . . . . . ⁊ ∠A F B is ⊥; alſo, A F . M :: M . F B.

Demonſtration.

11. Becauſe by *Conſtr.* in 6°, ... ⁊ A F B is a Semicircle.
12. Therefore, *(per prop. 31. Elem. 3.)* . . ⁊ ∠A F B is ⊥. Which was to be Dem.
13. Again, becauſe by *Conſtr.* in 8°, . . . ⁊ E F = A D.
14. Therefore, by taking in A B as a common altitude, it follows from 13°, *(per prop. 1. Elem. 6.)* that . . . . . ⁊ □E F, A B = □A D, A B.
15. And becauſe by *Conſtr.* in 6° and 8°, . ⁊ F E ⊥ A B.
16. And it hath been proved in 12°, . . . ⁊ ∠A F B is ⊥.
17. Therefore out of 15° and 16° *(per prop. 8. Elem. 6.)* . . . . . ⁊ △A E F and △A F B are like.
18. And conſequently, *(per prop. 4. Elem. 6.)* ⁊ E F . A F :: F B . A B.
19. And from 18°, *(per prop. 17. Elem. 6.)* ⁊ □E F, A B = □A F, F B.
20. Therefore from 14° and 19°, *(per Ax. 1.)* ⁊ □A D, A B = □A F, F B.
21. And becauſe by *Conſtr.* in 5°, ... ⁊ A B . M :: M . A D.
22. And conſequently *(per prop. 17. Elem. 6.)* ⁊ □A D, A B = □M.
23. Therefore out of 20° and 22°, *(per Ax. 1.)* ⁊ □A F, F B = □M.

24. There-

24. Therefore from 23°, *(per prop. 14. Elem. 6.)* . . ⁊ A F . M :: M . F B.
Which was to be Dem. And therefore the *Problem* is ſatisfied.

Coroll. 1.

25. From the premiſſes it may be inferr'd, That if the Algebraical Reſolution of a Geometrical Problem diſcovers an Analogy conſiſting of three Planes in Continual proportion, ſuch, that one of the extremes is an unknown Square, the mean a known Square, and the other extreme the exceſs whereby ſome known Square exceeds the ſaid unknown Square; then the ſide of the ſaid unknown Square ſhall be given by the preceding Geometrical *Conſtruction* of *Probl.* 16.

As, for example, ſuppoſing *aa* to repreſent an unknown Square, whoſe ſide is *a*; alſo *mm* and *hh* two known Squares, whoſe ſides are *m* and *h*, let this Analogy be propos'd, *viz.* ⁊ *hh* — *aa* . *mm* :: *mm* . *aa*.

26. Then *(per prop. 22. Elem. 6.)* the ſides of proportional Squares are Proportionals alſo, therefore . . . . . ⁊ $\sqrt{:hh - aa:}$. *m* :: *m* . *a*.

27. Which three continual Proportionals laſt expreſt being well obſerved, it will be manifeſt that the extremes, to wit, $\sqrt{:hh - aa:}$ and *a* may be eſteem'd the Baſe and Perpendicular of a right-angled Triangle, whoſe Hypothenuſal is *h*. Now in that right-angled Triangle, the Hypothenuſal *h* is given by *Suppoſition*, as alſo *m* a right-line, which is a mean Proportional between the Baſe and Perpendicular; and therefore the Baſe and Perpendicular ſhall be given ſeverally by the preceding *Conſtruction* of *Probl.* 16. either of which right lines, *viz.* either the Baſe or Perpendicular ſo found out may be taken for the line repreſented by *a* in the Analogy propos'd in 25°. For, viewing the Diagram belonging to *Probl.* 16.

28. Suppoſe . . . . . ⁊ ∠A F B = ⊥ in △A F B.
29. Suppoſe alſo . . . . . ⁊ F A . M :: M . F B.
30. And conſequently *(per prop. 22. Elem. 6.)* ⁊ □F A . □M :: □M . □F B.
31. Then put . . . . . ⁊ *mm* = □M.
32. Alſo, ... ⁊ *hh* = □A B.
33. And . . . . . ⁊ *aa* = □F B.
34. Then from 28°, 32° and 33°, it will be manifeſt *(per prop. 47. Elem. 1.)* that ⁊ *hh* — *aa* = □F A (= □A B — □F B.)
35. Alſo from 30°, 31°, 33°, 34°, . . . ⁊ *hh* — *aa* . *mm* :: *mm* . *aa*.
36. Which Analogy is the ſame with that propos'd in 25°, and it will alſo be produced by putting *aa* = □F A, (the Poſitions in 31° and 32° remaining unalter'd;)
For then it follows that . . . . . ⁊ *hh* — *aa* = □F B (= □A B — □F A.)
37. And becauſe by *Suppoſ.* in 30°, . . . ⁊ □F B . □M :: □M . □F A.
38. Therefore from the premiſſes, ... ⁊ *hh* — *aa* . *mm* :: *mm* . *aa*.

Thus it appears, that from either of the two right-lines found out by *Probl* 16. one and the ſame Analogy may be conſtituted, in which reſpect 'tis ſaid to be Ambiguous.

39. It is alſo manifeſt, that if the Terms of the Analogy in 25° be ſuppos'd to repreſent numbers, then by comparing the Rectangle of the extremes to the Rectangle of the means, this Biquadratick Equation will be produced, *viz.* *hhaa* — *aaaa* = *mmmm*.
Where, if we ſuppoſe *hh* = 5, alſo *mm* = 2, and *a* to ſtand for ſome number unknown, that Biquadratick Equation may be expreſt thus, *viz.* . . . . . 5*aa* — *aaaa* = 4

40. In which laſt Equation the unknown number *a* may be found out either by an Arithmetical Operation deducible from the preceding Geometrical *Conſtruction* of *Probl.* 16. or elſe, after the ſaid Equation is reſolved into theſe three Continual Proportionals, . . 5 — *aa* . 2 :: 2 . *aa*.

The two values of *aa* may be found out after the manner of the Arithmetical Solution of the foregoing *Probl.* 13. of this *Chapt.* and conſequently the ſquare Root of each number

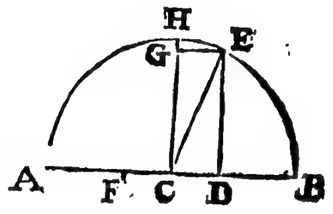
number found out for the value of aa being extracted, there will arise two numbers, each of which may be taken for the number a in the Equation in 39°. All which will be evident by the following Operation and Diagram.

Suppos.

41. AD, DE, DB ÷÷, *viz.* AD . DE :: DE . DB.
42. DE = 2, whence □DE = 4.
43. AD + DB = 5 = AB.

Req. to find

44. AD and DB in numbers, (signified by aa in the Equation in 39°.)



Operation Arithmetical.

45. By *Suppos.* in 43°, ... AB = 5.
46. Therefore, ... $\frac{1}{2}$AB = CE = $2\frac{1}{2}$ = AC = CB.
47. And consequently, ... □CE = $6\frac{1}{4}$.
48. By *Supposition* in 42°, ... □DE = 4.
49. And by subtracting the Equation in 48° from that in 47°, ... □CD = $2\frac{1}{4}$.
50. The square Root of the last Equation gives . . . . CD = $1\frac{1}{2}$.
51. The summ of the Equations in 46° and 50°, gives . . . AD = 4 = aa.
52. And consequently, ... $\sqrt{4}$ = 2 = a.
53. Again, by subtracting the Equation in 50, from that in 46°, ... DB = 1 = aa.
54. And consequently, ... $\sqrt{1}$ = 1 = a.

Whence 'tis manifest, that the number signified by a in the Analogy in 40° (or in the Equation in 39°,) may be either 2 or 1.

55. Moreover, out of the premisses ariseth the following *Canon*, for the Arithmetical Resolution of all Biquadratick Equations falling under the same Form with that before propos'd in 39°, and here-under repeated; where a represents a number sought, but h and m two numbers given, and subject to the Determination annex'd to *Probl.* 16. *viz.* $\frac{mm}{h}$ must not be greater than $\frac{1}{2}h$, and consequently mm not greater than $\frac{1}{2}hh$.

If . . . $hhaa - aaaa = mmmm$.

Then, . . $a = \sqrt{:\frac{1}{2}hh + \sqrt{\frac{1}{4}hhhh - mmmm}:}$

Or, . . $a = \sqrt{:\frac{1}{2}hh - \sqrt{\frac{1}{4}hhhh - mmmm}:}$

Coroll. 2.

56. From the premisses it may be also inferr'd, That if the Algebraical Resolution of a Geometrical *Problem* discovers an Analogy consisting of three Planes in Continual proportion, such, that the mean is a known Square, and also one of the extremes is the excess of some unknown Square above a known Square, and the other extreme is the excess of some known Square above the said unknown Square; or, when one of the extremes is the excess of a known Square above an unknown Square, and the other is the summ of the same unknown Square and a known Square; in each of those Cases, the side of that unknown Square shall be given by the preceding *Construction* of *Probl.* 16.

Example 1.

57. . . Suppose . . . . $aa - dd$: $4mm$ :: $4mm$: $cc - aa$;
And consequently, $\sqrt{:aa - dd}$: $2m$ :: $2m$: $\sqrt{:cc - aa}$:

In which Analogies, if we suppose aa to represent an unknown Square whose side is a; but dd, $4mm$ and cc three known Squares, whose sides are d, $2m$ and c; the unknown side or right line a shall be given by the preceding *Construction* of *Probl.* 16. For the summ of the extremes in the first of those Analogies is $cc - dd$, which is given by *Supposition*; then the side of a Square equal to that summ, to wit, $\sqrt{:cc - dd}$: may be esteem'd the Hypothenusal of a right-angled Triangle, by the help of which given Hypothenusal and the right-line $2m$, which by *Supposition* is a given mean Proportional between the Base and Perpendicular, (represented by $\sqrt{:aa - dd}$: and $\sqrt{:cc - aa}$:) the Base and Perpendicular shall be given severally by the said *Probl.* 16. and may be represented by f and g,

f and g, whose Squares are ff and gg; then by equating ff to $aa - dd$, or gg to $cc - aa$, one value of the right-line a sought shall be given; again, by equating gg to $aa - dd$, or ff to $cc - aa$, another value of the right-line a shall be given: So two lines are found out, either of which may be taken for the line a sought, and therefore the Analogy above-propos'd in *Example* 1. and all others of the same kind are said to be Ambiguous.

Example 2.

58. . . . Suppose . . . . $bb - aa$. mm :: mm . $aa + cc$,
And consequently, $\sqrt{:bb - aa}$: . m :: m . $\sqrt{:aa + cc}$:

By what hath been said in the Explication of *Example* 1. 'tis very easie to conceive how to find out in like manner the unknown Root or right-line represented by a in *Example* 2. where b, m and c are suppos'd to be right-lines severally given.

83. *Note.* When more than one known Square or Rectangle is found in any one of the three continual Proportionals mentioned in either of the three preceding *Corollaries* of *Probl.* 16. such known Squares or Rectangles must first of all be converted into a simple Square, *per Probl.* 2. *Chap.* 4.

Chap. VI.

The manner of finding out such Right-lines and Squares as are represented by Algebraical Fractions, the Quantities constituting those Fractions being given severally.

Probl. I.

Suppos.

1. $a = \frac{bb}{c}$. An Equation propounded.
2. b and c are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into these Proportionals, *viz.* . . . . . $c : b :: b . a$.

In which Analogy, the two first Terms are right lines given by *Supposition*, therefore a third Proportional to them shall be given also, (*per Probl.* 7. *Chap.* 5.) which third Proportional is the line a sought.

Probl. II.

Suppos.

1. $a = \frac{bd}{c}$. An Equation propos'd.
2. b, c and d are right lines given severally.

Req. to find the line a.

Construction.

3. The Equation propos'd may be resolved into this Analogy, *viz.* . . . . . $c : b :: d : a$.

In which Analogy, the three first Terms are right lines given by *Supposition*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth Proportional, to wit, the line a shall be given also.

Probl. III.

Suppos.

1. $a = \frac{bb + bd}{c - f}$. An Equation propos'd.
2. b, c, d and f are right lines given severally.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $c-f \;.\; b+d :: b \;.\; a.$

In which Analogy the three firſt Terms are given; for firſt, c and f being given by *Suppoſition*, a right line equal to $c-f$ ſhall be given alſo, (*per Probl.* 3. *Chap.* 5.) ſecondly, b and d being given by *Suppoſition*, a right line equal to $b+d$ ſhall be given, (*per Probl.* 1. *Chap.* 4.) and laſtly, the fourth Proportional line a ſought ſhall be given, *per Probl.* 8. *Chap.* 5.

Probl. IV.

Suppoſ.

1. $a = \frac{bb-dd}{c+f}$. An Equation propos'd.
2. b, c, d and f are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $c+f \;.\; b+d :: b-d \;.\; a.$
4. In which Analogy the three firſt Terms are given, and therefore the fourth proportional line a ſought ſhall be given alſo by *Probl.* 8. *Chap.* 5.
5. In like manner, if . . . . . $a = \frac{bb+2bc+cc}{d+f}$;
6. Then this Analogy will diſcover the line a ſought, *viz.* . . . . . $d+f \;.\; b+c :: b+c \;.\; a.$
7. Again, if . . . . . $a = \frac{bb-2bc+cc}{d+f}$.
8. Then ſuppoſing $b \sqsubset c$, the line a ſhall be given by this Analogy, *viz.* . . . . . $d+f \;.\; b-c :: b-c \;.\; a.$

Probl. V.

Suppoſ.

1. $a = \frac{bc+df}{g}$. An Equation propos'd.
2. b, c, d, f, g are right lines given.

Req. to find the line a.

Conſtruction.

3. Firſt reduce df to a Rectangle, which ſhall have b for one of its ſides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as b to d, ſo f to a fourth Proportional, which may be called h; therefore . . . . . $b \;.\; d :: f \;.\; h.$
4. Therefore by comparing the Rectangle of the extremes to the Rectangle of the means, ... $bh = df.$
5. Then by ſetting bh in the place of df in the Equation propos'd, it will be converted into this, *viz.* . . . . . $a = \frac{bc+bh}{g}.$
6. Which laſt Equation may be reſolved into this Analogy, ... $g \;.\; b :: c+h \;.\; a.$

But in the Analogy laſt expreſt, the three firſt Terms are given by *Suppoſition* and *Conſtruction*, therefore (*per Probl.* 8. *Chap.* 5.) the fourth proportional line a ſought ſhall be given alſo. The ſame line a may be found out divers other ways, as the induſtrious Learner will eaſily perceive.

Probl. VI.

Suppoſ.

1. $a = \frac{bcdf}{gcd-ghk}$. An Equation propos'd.
2. b, c, d, f, g, h and k are right lines given.

Req. to find the line a.

Conſtruction.

3. Firſt, reduce bc to a Rectangle which ſhall have g for one of its ſides, *viz.* let it be made (*per Probl.* 8. *Chap.* 5.) as g to b, ſo c to a fourth Proportional, call it l, therefore, ... $g \;.\; b :: c \;.\; l.$
4. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, . . . $gl = bc.$
5. Therefore from 1° and 4°, by ſetting gl in the place of bc in the Equation propos'd, it gives . . . $a = \frac{gldf}{gcd-ghk} = \frac{ldf}{cd-hk}.$
6. Again, reduce hk to a Rectangle that ſhall have d for one of its ſides, *viz.* let it be made as d to h, ſo k to a fourth, which may be called m, therefore $dm = hk.$
7. Therefore from 5° and 6°, by ſetting dm in the place of hk in the Fraction in the latter part of the 5th ſtep, it gives . . . . . $a = \frac{ldf}{cd-dm} = \frac{lf}{c-m}.$
8. Therefore by reſolving the latter Fraction in the 7th ſtep into Proportionals, this Analogy ariſeth, *viz.* . . . . . $c-m \;.\; l :: f \;.\; a.$

In which Analogy the three firſt Terms are given by *Suppoſition* and *Conſtruction*, therefore the fourth Proportional, to wit, the line a ſought ſhall be given alſo, by *Probl.* 8. *Chap.* 5. of this Book.

Probl. VII.

Suppoſ.

1. $a = \frac{dbb}{cc}$. An Equation propos'd.
2. b, c, d are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, *viz.* . . . . . $cc \;.\; bb :: d \;.\; a.$

In which Analogy, the three firſt Terms are given by *Suppoſition*, and qualified according to the tenour of *Probl.* 10. *Chap.* 5. therefore the fourth Term, that is, the line a ſought ſhall be given alſo by that *Problem.*

Probl. VIII.

Suppoſ.

1. $a = \frac{bcd}{ff}$. An Equation propos'd.
2. b, c, d, f, are right lines given.

Req. to find the line a.

Conſtruction.

3. Firſt, the Equation propos'd may be reſolved into this Analogy, ... $ff \;.\; bc :: d \;.\; a.$
4. Then let ff be reduced to a Rectangle that ſhall have b for one of its ſides; *viz.* let it be made (*per probl.* 7. *Chap.* 5.) as b to f, ſo f to a third Proportional, which may be called g, therefore $bg = ff.$
5. Then by taking bg inſtead of ff for the firſt Term of the Analogy in the third ſtep, that Analogy will be converted into this, ... $bg \;.\; bc :: d \;.\; a.$
6. Whence, by caſting away the common altitude b, this Analogy ariſeth, ... $g \;.\; c :: d \;.\; a.$

In which laſt Analogy the three firſt Terms are right lines given by *Suppoſition* and *Conſtruction*, therefore the fourth proportional line a ſhall be given alſo.

Probl. IX.

Suppoſ.

1. $a = \frac{2bbc}{bb-cc}$. An Equation propos'd.
2. b and c are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $bb-cc \;.\; bb :: 2c \;.\; a.$
4. Then (by *Probl.* 4. *Chap.* 5.) find a Square equal to $bb-cc$, which Square may be called dd; this being ſet in the place of $bb-cc$, (the firſt Term of the preceding Analogy,) will give theſe Proportionals, *viz.* $dd : bb :: 2c \;.\; a.$
5. Reduce dd to a Rectangle that ſhall have b for one of its ſides, *viz.* let it be made as b to d, ſo d to a third Proportional, which may be called f, therefore . . $bf = dd.$
6. Then by taking bf inſtead of dd, the Analogy in the 4th ſtep will be converted into this, ... $bf \;.\; bb :: 2c \;.\; a.$
7. Whence, by rejecting the common altitude b, this Analogy ariſeth, ... $f \;.\; b :: 2c \;.\; a.$

In which laſt Analogy the three firſt Terms are right lines given by *Suppoſition* and *Conſtruction*; therefore the fourth Proportional line a ſhall be given alſo, *per Probl.* 8. *Ch.* 5.

Probl. X.

Suppoſ.

1. $a = \frac{bbcc-ddff}{dff-bcc}$. An Equation propos'd.
2. $b, c, d, f,$ are right lines given.

Req. to find the line a.

Conſtruction.

3. Let it be made (*per Probl.* 8. *Chap.* 5.) as c to f, ſo d to a fourth proportional line, call it g, therefore, . . $cg = df.$
4. Then ſetting cg in the place of df in the Equation propos'd, and expunging c out of the Numerator and Denominator, it gives . . . . . $a = \frac{bbc-gdf}{gf-bc}.$
5. Again, let it be made as c to f, ſo g to a fourth proportional line, call it k, therefore, ... $ck = gf.$
6. Then ſetting ck in the place of gf in the Fraction in the fourth ſtep, and caſting c out of the Numerator and Denominator, there ariſeth . . . . . $a = \frac{bb-kd}{k-b}.$
7. Again, let it be made as k to b, ſo b to a third proportional line, call it m, therefore, ... $km = bb.$
8. Then ſetting km in the place of bb in the Numerator of the Fraction in the 6th ſtep, there will thence ariſe . . $a = \frac{km-kd}{k-b}.$
9. Again, let it be made as $k-b$ to $m-d$, ſo k to a fourth Proportional, which ſhall be the line a ſought, *viz.* . $k-b \;.\; m-d :: k \;.\; a.$
10. From the preceding *Conſtruction* 'tis evident that the deſired line a may be found out by theſe four following Rules of Three, *viz.*

1. $c \;.\; f :: d \;.\; g.$
2. $c \;.\; f :: g \;.\; k.$
3. $k \;.\; b :: b \;.\; m.$
4. $k-b \;.\; m-d :: k \;.\; a.$

11. But the Numerator and Denominator of the Algebraical Fraction in the Equation propos'd in this *Problem* do manifeſtly ſhew, that the lines b, c, d, f muſt be given with this Caution or Determination, *viz.*

$$f \sqsupset \frac{bc}{d}; \text{ but } f \sqsubset \sqrt{\frac{bcc}{d}}.$$

That

That is to ſay, if it be made as d to b, ſo c to a fourth Proportional, the line f muſt be ~~greater~~ leſſer than that fourth Proportional; but if (by *Probl.* 11. *Chap.* 5.) it be made as d to b, ſo cc to another Square, then the line f muſt be greater than the ſide of that latter Square.

Now if the line f be given within thoſe limits, then k will be greater than b, and m greater than d, as the laſt Analogy in the 10th ſtep requires. The ſame limits of f may be eaſily inferr'd alſo from the four Analogies in the 10th ſtep.

An Example in Numbers.

If $\left.\begin{matrix} b=40, \\ c=24, \\ d=16, \\ f=48; \end{matrix}\right\}$ then by the Analogies in the 10th ſtep you will find $\left\{\begin{matrix} g=32, \\ k=64, \\ m=25, \\ a=24. \end{matrix}\right.$

Thence it follows that $a = 24 = \frac{bbcc-ddff}{dff-bcc}$, the Equation propos'd.

Probl. XI.

Suppoſ.

1. $aa = \frac{bbd}{c}$. An Equation propos'd.
2. b, c, d are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $c : d :: bb \;.\; aa.$

In which Analogy, the three firſt Terms are given, and qualified according to the tenour of *Probl.* 11. *Chap.* 5. therefore the line a ſhall be given alſo by that *Problem.*

Probl. XII.

Suppoſ.

1. $aa = \frac{bdf}{c}$. An Equation propos'd.
2. b, c, d and f are right lines given.

Req. to find the line a.

3. The Fraction $\frac{bdf}{c}$ ſignifies a Rectangle contain'd under a right line equal to $\frac{bd}{c}$ and the line f, therefore firſt, according to the Conſtruction of *Probl.* 2. of this Chapter, find a right line equal to $\frac{bd}{c}$, which line may be called g, therefore,

$$aa = fg = \frac{bdf}{c}.$$

4. Then by *Probl.* 9. *Chap.* 5. reduce the Rectangle fg to a Square, which may be called mm, whoſe ſide is m, therefore,

$aa = mm$ $(= fg.)$ And conſequently, $m = a$ the line ſought

Probl. XIII.

Suppoſ.

1. $aa = \frac{bbcc}{dd}$. An Equation propos'd.
2. b, c and d are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... $dd : bb :: cc \;.\; aa.$
4. But the ſides of proportional Squares are alſo Proportionals, therefore from 3°, ... $d : b :: c : a.$

In which laſt Analogy the three firſt Terms are right lines given, therefore the fourth proportional line a ſought ſhall be given alſo.

Probl. XIV.

Suppoſ. *Probl.* XIV.

1. $aa = \frac{2bcdf}{gg}$. An Equation propos'd.
2. b, c, d, f, g are right lines given.

Req. to find the line a.

Conſtruction.

3. The Equation propos'd may be reſolved into this Analogy, ... } $gg . 2bc :: df . aa$.
4. By *Probl.* 9. *Chap.* 5. reduce $2bc$ to a Square, which may be called hh; reduce likewiſe df to a Square, which you may call kk, then the preceding Analogy will be converted into this, ... } $gg . hh :: kk . aa$.
5. But the ſides of proportional Squares are alſo Proportionals, therefore, from 4°, ... } $g . h :: k . a$.

In which laſt Analogy the three firſt Terms are right lines given, and therefore the fourth proportional line a ſought ſhall be given alſo.

Many other ways might be ſhewn to conſtruct (or effect) moſt of the preceding *Problems* of this Chapter; but for brevity ſake, I leave them to be found out by the induſtrious Learner, who by the help of thoſe before deliver'd will alſo eaſily perceive how to ſolve other *Problems* of like nature: And now having explain'd all ſuch things as are materially neceſſary by way of Preparation to the Reſolution and Compoſition of *Plane Problems*, I ſhall proceed to *Examples*, which I have divided into four *Claſſes* or Forms, contain'd in the four following Chapters.

CHAP. VII.

The firſt Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems, to wit, ſuch whoſe Conſtruction may be perform'd by drawing only Right and Circular lines.*

IN which Examples, the Reſolution ends either in an Analogy whoſe three firſt Terms are right lines known, and the fourth gives the right line ſought; or elſe it ends in a ſimple Equation between the right line ſought, and one or more right lines known. What is meant by Mathematical *Reſolution* and *Compoſition*, I have hinted by Definitions in the beginning of *Chap.* 1. *Book* I. of this Treatiſe, and now I come to expound and illuſtrate the ſame by Examples, after I have recommended a few things by way of Caution and Direction to Learners.

Firſt, Let the Analyſt take care to underſtand the import and meaning of a Problem propounded, leſt by too much haſt he loſe his labour, or be too forward in cenſuring the Propoſer, when the fault is in himſelf; for many undertake to be Correctors of others, when they themſelves have indeed more need of correction.

Secondly, Foraſmuch as the moſt part of Problems propos'd in Geometrical Figures have need of Preparation, let the Analyſt endeavour, before he begins the Algebraical Reſolution, to find out as much as he can by the Synthetical Method, which proceeds by a Series of Conſequences deduced altogether from known Quantities; and ſometimes it will be convenient to premiſe one or more preparatory Propoſitions to render the Reſolution of a Problem propos'd, the more ſimple and intelligible.

Thirdly, When the Reſolution of a Geometrical Problem is begun, the like care muſt be taken to keep every ſtep thereof in the ſimpleſt Terms, for avoiding Equations of higher Powers than the nature of the Problem requires, eſpecially ſuch as exceed Geometrical Dimenſions; for example, in the Reſolution of a Plane Problem, no Term of any Analogy or Equation ought to exceed two Dimenſions, *viz.* every Term muſt be either a right Line or a Plane, for 'tis improper to introduce Solids in the Reſolution or Compoſition of a Plane Problem.

Fourthly

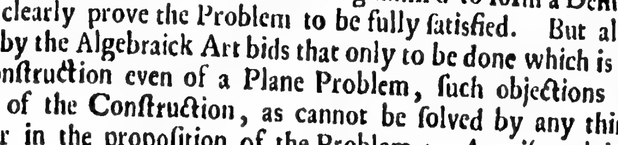
Fourthly, The Scope or aim of the Analyſt in ſolving a Problem muſt be, firſt, to find out a Canon to direct how the Conſtruction of the Problem may be effected by the Quantities given, and then the Conſtruction being finiſh'd to form a Demonſtration Synthetically, that may clearly prove the Problem to be fully ſatisfied. But although a Canon rightly found out by the Algebraick Art bids that only to be done which is poſſible, yet oftentimes in the Conſtruction even of a Plane Problem, ſuch objections will ſtart up againſt the poſſibility of the Conſtruction, as cannot be ſolved by any thing apparent either in the Canon, or in the propoſition of the Problem: As, if a plain Triangle be to be made of three right lines, whereof one is rightly found out by *Conſtruction* according to the direction of the Canon, and the other two are alſo diſcovered by the help of Quantities given and found out, yet before a Triangle can be made of thoſe three right lines, it muſt be proved, that every two of them being joyned together as one right line, are longer than the third, which Proof may happen to be a more difficult work than the invention of the Canon: And therefore when any doubt ariſeth concerning the poſſibility of any particular Conſtruction, and it doth not clearly appear whether ſuch Conſtruction can be done or not, an induſtrious Enquiry muſt be made to diſcover what is abſolutely neceſſary to be given or granted to make that poſſible to be done, which the Problem requires, or the Canon bids to be done: Upon which Search, oftentimes one or more Determinations or Cautions will be found neceſſary to limit the Quantities given in the Problem, that its Conſtruction may meet with no Impediment. Examples of *Determinations* will appear in divers Problems in this and the following Chapters.

Fifthly, After all neceſſary Determinations are premis'd, and the Conſtruction of a Problem is finiſh'd, it remains to demonſtrate that the Quantity or Quantities found out by the *Conſtruction* will ſatisfie the Problem: But the Demonſtration of the Solution of a Plane Problem, if its Conſtruction be Algebraically found out in ſuch manner that no Term of any Analogy or Equation in the Reſolution exceeds Geometrical dimenſions, may be formed by a repetition of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution; and the Demonſtration of a Theorem may be formed by the ſteps of the Algebraick Reſolution in a direct order, that is, by proceeding forward from the beginning to the end of the Reſolution. All which will be copiouſly illuſtrated by the *Reſolutions* and *Compoſitions* of Problems in this and the following Chapters.

Sixthly and laſtly, I deſire the Reader to take notice that in the *Reſolution* of a Problem, I uſe the ſmall *Italian* letters, a, b, c, d, &c. aſſuming always ſome Vowel, as a, or e, &c. to repreſent a line ſought, and Conſonants, as b, c, d, &c. to ſignifie lines given or known: But in the *Compoſition* of a Problem, that is, in its Conſtruction and Demonſtration, I uſe the *Roman* Capital letters, A, B, C, D, &c. to expreſs lines known, that the *Reſolution* and *Compoſition* may be compared to one another without Confuſion.

Problem I.

To divide a given right line into two parts which ſhall be in a given Reaſon, that is, one part to the other as two right lines given.



Suppoſ.

1. $b =$ AB a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ the Terms of the given Reaſon of the parts ſought.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB; alſo,
4. AC . CB :: R . S.

Reſolution.

5. For one of the parts ſought put . . . . . } a.
6. Therefore from 1° and 5° the other part is . . . } $b - a$.
7. And according to the tenour of the Problem propounded, ... } $r . s :: a . b - a$.

8. There-

8. Therefore by *Compoſition of Reaſon converſe*, (defin'd in *Sect.* 4. *Chap.* 3.) . . . . . } $r + s \;.\; r :: b \;.\; a.$

Which laſt Analogy gives this

C A N O N.

9. As the ſumm of the Terms of the given Reaſon is to the firſt Term; (that is, which of the two you pleaſe;) ſo is the line given to be divided, to one of the parts deſired, which part ſubtracted from the line given to be divided leaves the other part.

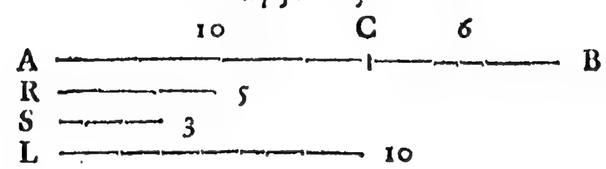
10. *Note.* Although the Analogy in the ſeventh ſtep may be converted into an Equation, (by comparing the Rectangle of the extremes to the Rectangle of the means,) from which, after due Reduction, the Analogy in the eighth ſtep will ariſe, yet in Geometrical Demonſtrations, which require a Contemplation upon Schemes or Figures, an Analogy in right lines is more ſimple, and eaſier to be underſtood than an Equation between Planes, or Solids; and therefore 'tis more uſual with Geometricians in their Argumentations, to proceed as much as is poſſible from one Analogy to another, by Compoſition, Diviſion, and other ways of arguing about Proportions, (defin'd in *Chap.* 3. of this *Book*,) that at length an Analogy may ariſe, when there is a poſſibility, wherein the three firſt Terms are given to find a fourth Proportional, which gives the Quantity ſought; but there will be very often a neceſſity of converting an Analogy into an Equation, when known Quantities cannot be otherwiſe ſeparated from unknown, as will hereafter appear by variety of Examples.

Concerning the Compoſition of a Geometrical Problem.

11. The *Compoſition* of a *Problem* conſiſts of two parts, to wit, *Conſtruction*, (or *Delineation*,) and *Demonſtration*; the former finds out that which is required to be done or found out, and the latter proves that that which is done or found out will ſatisfie the *Problem* propounded.

But before the *Conſtruction* be begun, if the Problem be not univerſal, ſuch Determinations (or Cautions) as are needful to limit the given Quantities, that the Problem may be poſſible muſt be annext to it, and the truth and reaſon of ſuch Determinations made manifeſt; for 'tis the Office of him that undertakes to ſolve a Problem to determine what can, and what cannot be done; and if that which is required be poſſible, then to ſhew how, and how many ways it may be done: Now the Algebraical Art is an excellent Guide to ſhew the way leading to thoſe ends; for firſt, the *Canon* reſulting from the Reſolution doth for the moſt part diſcover all ſuch Determinations as are neceſſary to limit the given Quantities that the Problem may be poſſible, and directs alſo how its *Conſtruction* may be made by working only with given Quantities. And laſtly, if no Term of any Analogy or Equation in the Reſolution exceeds Geometrical Dimenſions, a Demonſtration of the Solution of the Problem may be form'd out of the ſteps of the Reſolution in a retrograde order, that is, by returning backwards from the end of the Reſolution to its beginning. But theſe things will beſt appear by Examples, and therefore I ſhall proceed to

The Compoſition of Probl. 1.



Suppoſ.

12. A B is a right line given to be cut into two parts.

13. R and S are the Terms of the given Reaſon of the parts ſought.

Req. to find

14. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,

15. A C . C B :: R . S.

Conſtruction.

16. Let it be made (*per Probl.* 8. *Chap.* 5.) as R + S to R, ſo A B to a fourth Proportional line, which may be called L, therefore

R + S . R :: A B . L.

17. From A B cut off A C = L, which is poſſible to be done if A B be greater than L, but

but R + S the firſt Term of the laſt preceding Analogy is evidently greater than R the ſecond Term, therefore (*per Schol Prop.* 14. *Elem.* 5.) the third Term A B ſhall be greater than the fourth L; and conſequently A C = L may be cut off from A B. That done, the given line A B is divided in the point C into two parts A C, C B, which will ſatisfie the Problem For firſt, A C + C B = A B, and that A C is to C B as R to S, I ſhall demonſtrate by a retrograde repetition of the ſteps of the Reſolution in manner following.

18. . . . *Req. demonſtr.* . . . . . R . S :: A C . C B.

Demonſtration.

19. Becauſe by *Conſtr.* in 16°, ... } R + S . R :: A B . L.

20. And by *Conſtr.* in 17°, ... } A C = L.

21. Therefore from 19°, by taking A C inſtead of L, } R + S . R :: A B . A C.

That is, in 8°, (the laſt ſtep of the Reſolution,) } $r + s \;.\; r :: b \;.\; a.$

22. Therefore from the Analogy in 21°, by *Diviſion of Reaſon converſe*, (defin'd in *Sect.* 8. *Ch.* 3.) } . R . S :: A C . C B. *Concluſ.*

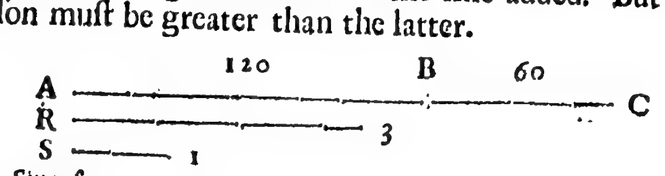
That is, in 7°, ... } . $r \;.\; s :: a \;.\; b - a.$

Which was to be done.

Note In forming a *Demonſtration* by a repetition of the ſteps of the Reſolution in a backward order, it muſt be obſerved as a perpetual Rule, That when in the Reſolution you paſs forward from one ſtep to another by *Compoſition of Reaſon*, in the Demonſtration you are to return backward by *Diviſion of Reaſon*; and when you paſs by *Diviſion of Reaſon* in the Reſolution, you are to return by *Compoſition of Reaſon* in the Demonſtration: alſo, *Addition* in the one, anſwers to *Subtraction* in the other. All which will be evident in the following *Problems.*

Probl. II.

To a given right line to add another right line, that the given with the added may have a given Reaſon to the line added. But the firſt Term of the Reaſon muſt be greater than the latter.



Suppoſ.

1. b = A B a right line given to be increaſed.

2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ the Terms of the given Reaſon.

3. R ⊏ S.

Req. to find

4. B C a right line, ſuch, that A B + B C : B C :: R : S.

Reſolution.

5. For the line ſought put . . . . . } a.

6. Which added to the given line b makes . . } $b + a$.

7. Then according to the import of the Problem, } $r \;.\; s :: b + a \;.\; a.$

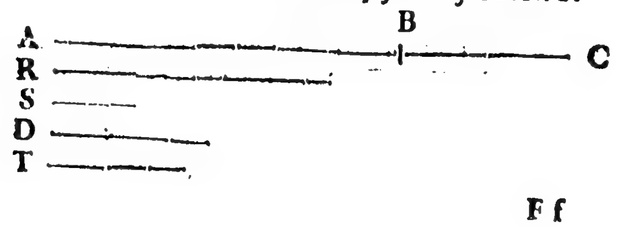
8. Therefore by *Diviſion of Reaſon*, (defin'd in *Sect.* 7. *Chap.* 3.) . . . . . } $r - s \;.\; s :: b \;.\; a.$

Hence this

Canon.

9. As the difference of the Terms of the given Reaſon is to the leſſer Term, ſo is the line given to be increaſed, to the increaſe ſought.

The Compoſition of Probl. 2.



A B = 120
R = 3
S = 1
D = 2
T = 60
B C = 60

Suppoſ.

10. A B is a right line given.
11. R and S are the Terms of a given Reaſon.
12. R ⊏ S.
13. D = R — S.

Req. to find

14. B C a right line, ſuch, that . . . A B + B C . B C :: R . S.

Conſtruction.

15. Let it be made (*per Probl.* 8. *Chap.* 5.) as D (or R — S) to S, ſo A B to a fourth proportional line, call it T, therefore, . . . } D . S :: A B . T.
16. Let A B be continued to C, ſo, that . . } B C = T.

Now the line B C (or T) being found out by the help of the given lines, A B, R and S, according to the direction of the *Canon*, we muſt ſhew that it will ſatisfie the *Probl.* therefore,

17. . . . *Req. demonſtr.* . . . . . R . S :: A C . (A B + B C.) B C.

Demonſtration.

18. Becauſe by *Conſtr.* in 15° and 16°, . . } R — S . S :: A B . B C.
19. Therefore by *Compoſ. of Reaſon*, . . . } R . S :: A C . B C.

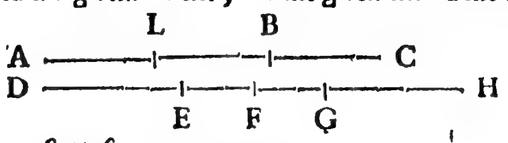
Which was to be done.

Note. In paſſing from the firſt ſtep of this Demonſtration, (which is the laſt ſtep in the Reſolution,) to the ſecond; the Argumentation is made by *Compoſition of Reaſon*, becauſe in paſſing to the laſt ſtep of the Reſolution from the laſt but one, it was argued by *Diviſion of Reaſon*; agreeable to the *Note* at the end of the preceding *Probl.* 1.

Probl. III.

To a given right line to add another right line, that the Difference of the given and added may have a given Reaſon to their Summ. But the firſt Term of the Reaſon muſt be leſs than the latter Term.

This Problem hath two Caſes; for either the given right line ſhall exceed the added, or the added the given. Firſt, let the given exceed the added.



A B = 120
F G = 1
D F = 3
B C = 60

Suppoſ.

1. b = A B a right line given.
2. r = F G, s = D F } the Terms of the given Reaſon.
3. $r \sqsupset s$.

Req. to find

4. B C a right line, ſuch, that A B — B C . A B + B C :: F G . D F.

Reſolution.

5. For the line ſought to be added put . . . } a.
6. Therefore the exceſs of the given line b above the line ſought ſhall be . . . . . } $b - a$.
7. And the ſumm of the given line and the line ſought ſhall be . . . . . } $b + a$.
8. Therefore, according to the tenour of the Problem propounded, this Analogy ariſeth, . . } $r . s :: b - a . b + a$.
9. Therefore by *Compoſition of Reaſon*, . . . } $s + r . s :: 2b . b + a$.
10. And by doubling the Conſequents, . . . } $s + r . 2s :: 2b . 2b + 2a$.
11. And inverſly, ... } $2s . s + r :: 2b + 2a . 2b$.
12. And by *Diviſion of Reaſon*, ... } $s - r . s + r :: 2a . 2b$.
13. And inverſly, ... } $s + r . s - r :: 2b . 2a$.
14. But a ſimple quantity is to a ſimple, as the double of the former to the double of the latter, therefore . . . . . } $b . a :: 2b . 2a$.

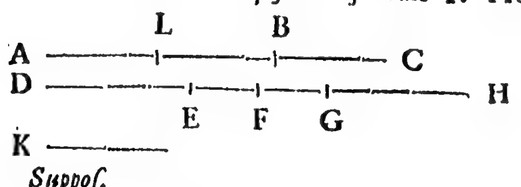
15. And

15. And out of 13° and 14°, (*per prop.* 11. *Elem.* 5.) } $s + r . s - r :: b . a$.

Hence this *CANON.*

16. As the ſumm of the Terms of the given Reaſon is to their difference, ſo is the given line to the line ſought. Therefore the line required to be added to the given line, is given alſo.
17. *Note.* The line ſought may eaſily be diſcovered by the Analogy in the ninth ſtep, where the three firſt Terms being known, the fourth is known by Conſequence; and ſince that fourth Term is evidently compos'd of the given line and the line ſought, the given line ſubtracted from that known fourth Proportional ſhall neceſſarily give the line ſought; whence 'tis manifeſt, that the Argumentation continued from the ninth ſtep to the end of the Reſolution is not of neceſſity, but only to ſhew how the line ſought may be purely the fourth Proportional of an Analogy whoſe three firſt Terms are known, and conſequently the line ſought is known alſo: Which way of arguing by Analogies is more proper, (when it may be uſed,) than that by Equations, as hath before been hinted in *Sect.* 10. *Probl.* 1. of this Chapter.

The Compoſition of Caſe 1. Probl. 3.



A B = 120
F G = 1
D F = 3
K = 60
B C = 60

Suppoſ.

18. A B is a right line given.
19. F G and D F are the Terms of a given Reaſon.
20. F G ⊐ D F.

Req. to find

21. B C a right line, ſuch, that . . . A B — B C . A B + B C :: F G . D F.

Conſtruction.

22. Let it be made as D F + F G (that is, D G,) to D F — F G, ſo A B to a fourth proportional line, call it K, therefore, ... } D F + F G . D F — F G :: A B . K.
23. Let A B be continued to C, ſo, that . } B C = K.
24. Now the line B C, or K being found out by the help of the given lines A B, D F, F G, (according to the direction of the *Canon*,) we muſt ſhew that it will ſatisfie the Problem, *viz.* that the difference of A B and B C is to their ſumm, as F G to D F; but this Analogy, (after I have premiſ'd a few things to contract the *Demonſtration*,) I ſhall make manifeſt by a repetition of the ſteps of the Reſolution in a retrograde order; that is, by returning backwards from the end to the beginning of the Reſolution.

Prepar.

25. From A B cut off A L = B C = K, which is poſſible to be done, for the firſt Term of the Analogy in 22° is evidently greater than the ſecond, and therefore (*per Schol. Prop.* 14. *Elem.* 5.) the third Term A B ſhall be greater than the fourth K, or B C; ſuppoſe therefore } A L = B C.
26. Thence it follows that . . . . . } L B = A B — B C.
27. Let D F be continued to H, ſo, that . } D F = F H.
28. From D F cut off F E = F G, which is poſſible to be done, for by *Suppoſition* in 20°, D F ⊏ F G; ſuppoſe therefore . . . . } F E = F G.
29. Therefore by ſubtracting the laſt Equation from that in 27°, ... } D E = G H = D F — F G.
30. . . . *Req. demonſtr.* . . } F G . D F :: A B — B C . A B + B C :: L B . A C.

Demonſtration.

31. By *Conſtr.* in 22° and 23°, ... } D F + F G . D F — F G :: A B . B C.
That is in 15°, (the laſt ſtep of the Reſolution,) } $s + r . s - r :: b . a$.
32. But there is the ſame Reaſon of the double to the double, as of the ſimple to the ſimple, therefore, ... } 2 A B . 2 B C :: A B . B C.

That is, in the 14th step, ... : $2b$. $2a$ :: b . a.

33. Therefore out of 31° and 32°, (*per Prop.* 11. *Elem.* 5.) . . DF+FG . DF−FG :: 2AB . 2BC.

That is, in the 13th step, ... $s+r$. $s-r$ :: $2b$. $2a$.

34. Therefore inversly, ... DF−FG . DF+FG :: 2BC . 2AB.

That is, in the 12th step, ... $s-r$. $s+r$ :: $2a$. $2b$.

35. Therefore by *Compos. of Reason*, . . 2DF . DF+FG :: 2AB+2BC . 2AB.

That is, in the 11th step, ... $2s$. $s+r$ :: $2b+2a$. $2b$.

36. Therefore inversly, ... DF+FG . 2DF :: 2AB . 2AB+2BC.

That is, in the 10th step, ... $s+r$. $2s$ :: $2b$. $2b+2a$.

27. Therefore by halving the Consequents in the 36th step, . . . DF+FG . DF :: 2AB . AB+BC.

That is, in the 9th step, ... $s+r$. s :: $2b$. $b+a$.

38. Therefore by *Division of Reason*, . FG . DF :: AB−BC . AB+BC.

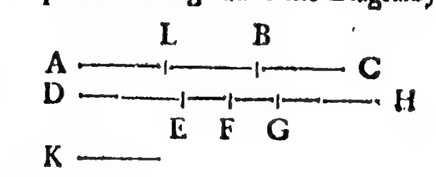
That is, in the 8th step, ... r . s :: $b-a$. $b+a$.

39. Therefore from 38° and 36°, (*per Ax.* 6. *Chap.* 2.) . . . . FG . DF :: LB . AC.

Which was to be Dem. Therefore the Problem is satisfied.

40. *Note.* Under every step of this Demonstration, I have set the correspondent step of the Resolution, that the Learner having respect to the *Note* at the end of *Probl.* 1. of this Chapter, may clearly perceive how the Demonstration is form'd out of the steps of the Resolution in a Retrograde order, that is, by returning backwards from the end to the beginning of the Resolution; for the first step in the Demonstration answers to the last in the Resolution, the second in the Demonstration, to the last but one in the Resolution, and so backwards in the Resolution, until the Analogy that was first assumed in the Resolution be positively and infallibly proved to be true. But after the Demonstration is in that manner discovered, the Algebraical steps must be omitted: So when the foregoing Demonstration beginning at the 31st step, is freed from the Analogies expresst by the small *Italian* letters belonging to the Resolution, and contracted by the help of the preparatory Equations in the 26th and 29th steps, respect also being had to the Diagram, there will arise this following

Demonstration.



41. Because by *Constr.* in 22° and 23°, . . DG . DE :: AB . BC.

42. And by *prop.* 15. *Elem.* 1. . . . 2AB . 2BC :: AB . BC.

43. Therefore, *per prop.* 11. *Elem.* 5. . DG . DE :: 2AB . 2BC.

44. And inversly, ... DE . DG :: 2BC . 2AB.

45. And by *Composition*, ... 2DF . DG :: 2AC . 2AB.

46. And inversly, ... DG . 2DF :: 2AB . 2AC.

47. And by halving the Consequents, . DG . DF :: 2AB . AC.

48. Wherefore by *Division of Reason*, . FG . DF :: LB . AC.

Which in 30° was *Req. dem.*

49. Thus you have seen the first Case of *Probl.* 3. effected and demonstrated Synthetically, or by way of Composition, which argues altogether with known quantities; but the substance of the Composition, to wit, the *Construction* and *Demonstration*, was found out Analytically, or by way of Resolution, which from an Assumption of the quantity sought as if it were known or granted, together with the help of one or more known quantities, proceeds by Consequences, until in Conclusion the quantity so assumed or feigned to be known, is found equal to some quantity certainly known, and is therefore known also.

But it may be objected, that *Demonstrations* formed by the steps of Algebraical Resolution are for the most part rude and prolix; this I grant, but experience shews, that a *Demonstration* so found out may oftentimes be easily contracted, or, at least, give light to find out others more succinct and elegant. And since my purpose is, to shew the Learner a general and ready way of forming the *Demonstrations* of such *Theorems*, and *Solutions* of *Problems* as he finds out by Algebra, when no Term of any Analogy or Equation

Equation in any step of the Resolution exceeds Geometrical Dimensions, I shall very seldom digress from the steps of the Resolution.

The Resolution of Case 2. Probl. 3.

In this *Case* the line sought is suppos'd to exceed the given line AB.

50. For the given line AB put (as before) . . b.

51. And for the line sought, ... a.

52. Therefore the excess of the line sought above the given is . . . . . $a-b$.

53. And the summ of both lines is . . . $a+b$.

54. Therefore, according to the tenour of *Probl.* 3. this Analogy ariseth, . . . : r . s :: $a-b$. $a+b$.

55. Therefore by *Composition of Reason*, . $s+r$. s :: $2a$. $a+b$.

56. And by doubling the Consequents, . $s+r$. $2s$ :: $2a$. $2a+2b$.

57. And inversly, ... $2s$. $s+r$ :: $2a+2b$. $2a$.

58. And by *Division of Reason*, . . . $s-r$. $s+r$ :: $2b$. $2a$.

59. And because there is the same Reason of the simple to the simple, as of the double to the double, therefore . . . b . a :: $2b$. $2a$.

60. And out of 58° and 59°, (*per prop.* 11. *Elem.* 5.) . . . . . $s-r$. $s+r$ :: b . a.

Hence this

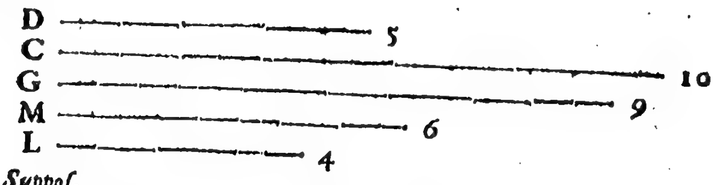
CANON.

61. As the Difference of the Terms of the given Reason is to their Summ; so is the given line to the line sought. Therefore the line required to be added to the given line, is given also.

The Composition of this latter *Case* differing but little from the former, I shall leave it as an exercise to the Learner.

Probl. IV.

The difference of the extremes of three proportional right lines being given, as also the summ of the mean and lesser extreme; to find the Proportionals.



Suppos.

1. G, M, L ∴, *viz.* G . M :: M . L.
2. G ⊏ L.
3. d = G − L is given.
4. c = M + L is given.

Req. to find G, M, L.

Resolution.

5. For the lesser extreme Proportional put . . . a.
6. Therefore out of 3° and 5°, the greater extreme is $d+a$.
7. And from 4° and 5°, the mean is . . . . $c-a$.
8. Therefore according to the tenour of the Problem, $d+a$. $c-a$ :: $c-a$. a.
9. Therefore by *Composition of Reason*, . . . $d+c$. $c-a$ :: c . a.
10. And alternately, ... $d+c$. c :: $c-a$. a.
11. Wherefore by *Composition*, ... $d+2c$. c :: c . a.

Hence this

CANON.

12. As the summ of the difference of the extremes and the double summ of the mean and lesser extreme, is to the summ of the mean and lesser extreme; so is the last mentioned summ to the lesser extreme. Therefore the lesser extreme sought is given.

13. After

13. After a Canon is found out by the Algebraical Art, it may be propounded in the form of a Theorem, whoſe Demonſtration may be made by a repetition of the ſteps of the Reſolution in a direct order, (not in a retrograde,) *viz.* by proceeding from the beginning to the end of the Reſolution; as for example, the laſt preceding Canon may be propos'd in the form of a Theorem, and demonſtrated thus.

THEOREM 1.

14. If three right lines be Proportionals, the ſumm of the mean and leſſer extreme, ſhall be a mean proportional between the leſſer extreme, and the ſumm of the difference of the extremes and the double ſumm of the mean and leſſer extreme.

G —————— 9
M ———— 6
L ——— 4
D ———— 5
C ——————— 10
F ——————————————————— 25

Suppoſ.

15. G, M, L ∴; *viz.* G . M :: M . L.
16. G ⊏ L.

Prepar.

17. Make D = G — L, therefore D + L = G.
18. Make C = M + L, therefore C — L = M.
19. Make F = D + 2C = G + 2M + L.
20. ∴ . . *Req. demonſtr.* . . . . . F . C :: C . L.

Demonſtration.

21. By *Suppoſ.* in 15°, ... } G . M :: M . L.
22. Therefore out of 17°, 18° and 21°, by exchanging equal right lines, . . . } D+L . C—L :: C—L . L.
That is, in 8°, (the firſt Analogy in the Reſolution,) . . . . . } *d*+*a* . *c*—*a* :: *c*—*a* . *a*.
23. Therefore from 22°, by *Compoſition of Reaſon*, ... } D+C : C—L :: C : L.
That is, in 9°, ... } *d*+*c* . *c*—*a* :: *c* . *a*.
24. Therefore alternately, ... } D+C . C :: C—L . L.
That is, in 10°, ... } *d*+*c* . *c* :: *c*—*a* . *a*.
25. Therefore by *Compoſition*, ... } D+2C . C :: C . L.
26. But by *Conſtr.* in 19°, ... } F = D + 2C.
27. Therefore from 25° and 26°, . . . } . F . C :: C . L.

Which was to be Dem.

But that Demonſtration, after the letters of the Reſolution are caſt away, may be compendiouſly reduced unto this that follows, reſpect being had to the *Suppoſitions* and *Preparation* in 15°, 16°, 17°, 18°, 19°.

28. ∴ . . *Req. demonſtr.* . . . . . G+2M+L . M+L :: M+L . L.

Demonſtration.

29. By *Suppoſ.* in 15°, ... } . G . M :: M . L.
30. Therefore by *Compoſition*, ... } G+M . M :: M+L . L.
31. And alternately, ... } G+M . M+L :: M . L.
32. Wherefore by *Compoſ.* . . . . } G+2M+L . M+L :: M+L . L.

Which was to be Dem.

Hence ariſeth

THEOREM 2.

33. If three right lines be Propoſitionals, the ſumm of the mean and leſſer extreme ſhall be a mean Proportional between the leſſer extreme, and the aggregate of the ſumm of the extremes and double ſumm of the mean.

The

The Compoſition of Probl. 4.

D ——————— 5
C ——————————— 10
L ——— 4
M ———— 6
G ——————————— 9

Suppoſ.

34. D = the difference of the extremes of three Proportionals is given.
35. C = the ſumm of the mean and leſſer extreme is given.
Req. to find the Proportionals.

Conſtruction.

36. Let it be made (by *Probl. 7. Chap. 5.*) as D+2C to C, ſo C to a third proportional line, which ſuppoſe to be found L, therefore, } D+2C . C :: C . L.
37. By which Analogy, (and *per Schol. Prop.* 14. *Elem.* 5.) the mean C is greater than L, therefore find a right line M equal to C—L, thence it follows that . . . . . } M + L = C.
38. Find a right line G equal to D+L, therefore, } G — L = D.
39. So by the help of the given lines D and C, according to the direction of the Canon in the preceding twelfth ſtep, three right lines are found out, to wit, L, M and G, which ſhall be the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem. Firſt then, 'tis manifeſt by *Conſtruction* in 38°, that the difference of the extremes G and L is equal to the given difference D. Secondly, by *Conſtruction* in 37°, the ſumm of the mean M and the leſſer extreme L is equal to the given ſumm C. It remains only to prove that the ſaid G, M and L are Proportionals, *viz.* that as G is to M, ſo M to L; but this Analogy may be made manifeſt by a Repetition of the ſteps of the preceding Reſolution in a retrograde order, that is, by returning backwards from the end to the beginning of the Reſolution, in manner following.

40. . . . *Req. demonſtr.* . . . . . G . M :: M . L.

Demonſtration.

41. Becauſe by *Conſtr.* in 36°, ... } D+2C : C :: C . L.
That is, in 11°, (the laſt ſtep of the Reſolution,) } *d*+2*c* . *c* :: *c* . *a*.
42. Therefore by *Diviſion of Reaſon*, ... } D+C . C :: C—L . L.
That is, in the tenth ſtep, ... } *d*+*c* . *c* :: *c*—*a* . *a*.
43. Therefore alternately, ... } D+C . C—L :: C . L.
That is, in the ninth ſtep, ... } *d*+*c* . *c*—*a* :: *c* . *a*.
44. Therefore, by *Diviſion of Reaſon*, . . . } D+L . C—L :: C—L . L.
That is, in the eighth ſtep, ... } *d*+*a* . *c*—*a* :: *c*—*a* . *a*.
45. And becauſe by *Conſtr.* in 38° and 37°, . . } G=D+L. Alſo, M=C—L.
46. Therefore out of 44° and 45°, by exchanging equal right lines, ... } . G . M :: M . L.

Which was to be Dem. And therefore the Problem is ſatisfied.

Another way of reſolving the foregoing Probl. 4.

47. The ſame things being ſuppoſed and given as before in 1°, 2°, 3° and 4° of this *Probl.* put *a* for the mean proportional ſought; *viz.* ſuppoſe . . . . . } *a* = M.
48. Therefore out of 4° and 47° the leſſer extreme ſhall be } *c* — *a* (= L.)
49. And by adding *d* the given difference of the extremes to the ſaid leſſer extreme *c* — *a*, the greater extreme ſhall be . . . . . } *d*+*c*—*a* (= G.)
50. Therefore according to the tenour of the *Probl.* this Analogy ariſeth out of 47°, 48°, 49°, ... } *d*+*c*—*a* . *a* :: *a* . *c*—*a*.
51. Therefore by *Compoſition of Reaſon*, . . . } *d*+*c* . *a* :: *c* . *c*—*a*.
52. And alternately, ... } *d*+*c* . *c* :: *a* . *c*—*a*.
53. And inverſly, ... } *c* . *d*+*c* :: *c*—*a* . *a*.
54. Therefore by *Compoſition of Reaſon*, . . . } *d*+2*c* . *d*+*c* :: *c* . *a*.

Which

Which laſt Analogy gives this *CANON.*

55. As the aggregate of the difference of the extremes and the double ſumm of the mean and leſſer extreme, is to the aggregate of the difference of the extremes and the ſumm of the mean and leſſer extreme; ſo is the ſumm of the mean and leſſer extreme to the mean.

Therefore the difference of the extremes of three Proportionals being given, as alſo the ſumm of the mean and leſſer extreme, the mean ſhall be alſo given by the Canon laſt expreſt. The Demonſtration whereof, and the *Compoſition* of the Problem according to this latter way of Reſolution being very eaſie, I ſhall leave the ſame to the Learners exerciſe.

Probl. V.

The ſumm of the firſt and ſecond of three Proportionals being given, as alſo the ſumm of the ſecond and third, to find the Proportionals.

Suppoſ.

1. L, M, N are ÷÷; *viz.* L . M :: M . N.
2. b = L ⊣⊢ M is given.
3. c = M ⊣⊢ N is given.

Req. to find L, M, N.

B ——————————————— 15
C ——————————— 10
L ———————— 9
M ————— 6
N ——— 4

Reſolution.

4. Put a for the firſt Proportional ſought, *viz.* ſuppoſe . . . . . } a = L.
5. Therefore out of 2° and 4°, the mean is . } $b-a$ (= M.)
6. And by ſubtracting the ſaid mean $b-a$ from the given ſumm c, the remainder gives the third Proportional, to wit, ... } $c+a-b$ (= N.)
7. Therefore (according to the *Probl.*) theſe muſt be Proportionals, *viz.* . . . . } $a \,.\, b-a :: b-a \,.\, c+a-b.$
8. Therefore inverſly, ... } $b-a \,.\, a :: c+a-b \,.\, b-a.$
9. And by *Compoſition of Reaſon*, ... } $b \,.\, a :: c \,.\, b-a.$
10. And alternately, ... } $b \,.\, c :: a \,.\, b-a.$
11. And inverſly, ... } $c \,.\, b :: b-a \,.\, a.$
12. Therefore by *Compoſition of Reaſon*, . . } $b+c \,.\, b :: b \,.\, a.$

Which laſt Analogy gives this *CANON.*

13. As the aggregate of the ſumm of the firſt and ſecond Proportionals and ſumm of the ſecond and third, is to the ſumm of the firſt and ſecond; ſo is the laſt mentioned ſumm to the firſt Proportional.

Therefore if the ſumm of the firſt and ſecond of three Proportionals be given, as alſo the ſumm of the ſecond and third, the mean ſhall be alſo given by the ſaid Canon; whence alſo this

THEOREM.

14. If three right lines be Proportionals, the ſumm of the firſt and ſecond is a mean Proportional, between the firſt, and the aggregate of the ſumm of the firſt and ſecond, and ſumm of the ſecond and third.

Which Theorem may eaſily be demonſtrated by a repetition of the ſteps of the Reſolution in a direct order, after the manner of demonſtrating the Theorem in 14° of the foregoing *Probl.* 4. but for brevity ſake I ſhall leave the Demonſtration to the Learners practice, and proceed to the *Compoſition* of *Probl.* 5.

The Compoſition of Probl. 5.

Suppoſ.

15. B = the ſumm of the firſt and ſecond of three Proportionals is given.
16. C = the ſumm of the ſecond and third Proportionals is given.

Req.

Req. to find the Proportionals. [*In the preceding Diagram.*]

17. By *Probl.* 7. *Chap.* 5. let it be made as B ⊣⊢ C to B, ſo B to a third Proportional, ſuppoſe it be L, therefore

B ⊣⊢ C . B :: B . L.

18. Make M = B — L, whence L ⊣⊢ M = B; but that B ⊏— L, as that effection requires, is manifeſt by *Conſtruction* in 17°; for B ⊣⊢ C the firſt Term of the Analogy in 17° is greater than B in the ſecond, and therefore (*per Schol. Prop.* 14. *Elem.* 5.) the third Term, which is alſo B, ſhall be greater than L the fourth, therefore 'tis poſſible to cut off from B the right line L, and a right line will remain, which may be called M.

19. Make N = C ⊣⊢ L — B, which is poſſible to be done if C ⊣⊢ L ⊏— B; but that C ⊣⊢ L ⊏— B, I prove thus,

By *Conſtr.* in 17°, ... B ⊣⊢ C . B :: B . L.
Therefore (*per Schol. Prop.* 25. *Elem.* 5.) B ⊣⊢ C ⊣⊢ L ⊏— 2 B.
And conſequently, by equal ſubtraction of B, . C ⊣⊢ L ⊏— B.

Which was to be proved. Therefore 'tis poſſible from the ſumm of the right lines C and L to cut off the right line B, and a right line will remain, which may be called N.

20. I ſay L, M and N are the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem.

21. Firſt then, the ſumm of the right lines L and M is (by *Conſtruction* in 18°,) equal to the given ſumm B.

22. Secondly, that the ſumm of the right lines M and N is equal to the given ſumm C, I prove thus,

By *Conſtr.* in 18°, ... M = B — L.
And by *Conſtr.* in 19°, ... N = C ⊣⊢ L — B.
Therefore by adding the two laſt Equations together, M ⊣⊢ N = C.

Which was to be proved. It remains to ſhew that the ſaid three right lines L, M and N are Proportionals, but that will be made manifeſt by the following Demonſtration, which is formed out of the preceding Reſolution by a repetition of the ſteps thereof in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Reſolution.

23. . . . *Req. demonſtr.* . . . . . L . M :: M . N.

Demonſtration.

24. Foraſmuch as by *Conſtruction* in 17°, . . } B ⊣⊢ C . B :: B . L.
That is, in 12°, (the laſt ſtep of the Reſolut.) } $b+c \,.\, b :: b \,.\, a.$
25. Therefore by *Diviſion of Reaſon*, ... C . B :: B—L . L.
That is, in 11°, ... $c \,.\, b :: b-a \,.\, a.$
26. Therefore inverſly, ... B . C :: L . B—L.
That is, in 10°, ... $b \,.\, c :: a \,.\, b-a.$
27. Therefore alternately, ... B . L :: C . B—L.
That is, in 9°, ... $b \,.\, a :: c \,.\, b-a.$
28. And becauſe by *Conſtruction* in 18°, ... M = B — L.
29. Therefore out of 27° and 28°, ... B . L :: C . M.
30. And becauſe it hath been proved in 18°, that . . . . B ⊏— L.
31. Therefore out of 29°, by *Diviſion of Reaſon*, B—L . L :: C—M . M.
That is, in 8°, ... $b-a \,.\, a :: c+a-b \,.\, b-a.$
32. Therefore inverſly, ... L . B—L :: M . C—M.
That is, in 7°, ... $a \,.\, b-a :: b-a \,.\, c+a-b.$
33. But by *Conſtr.* in 18°, ... M = B — L.
34. And by what hath been proved in 22°, ... N = C — M.
35. Therefore out of 32°, 33° and 34°, . . . L . M :: M . N.

Which was to be Dem. and therefore the Problem is ſatisfied.

Another way of reſolving the foregoing Probl. 5.

36. The ſame things being given and ſuppoſed, as before in 1°, 2°, 3°, put a for the mean Proportional ſought, *viz.* . . } a = M.
37. Therefore out of 2° and 36°, the firſt Proportional ſhall be } $b-a$ (= L.)
38. And out of 3° and 36°, the third Proportional ſhall be . } $c-a$ (= N.)

39. Therefore out of 36°, 37° and 38°, according to the tenour of *Probl.* 5. . . . . } $b-a \;.\; a :: a \;.\; c-a.$
40. Therefore inverſly, ... $a \;.\; b-a :: c-a \;.\; a.$
41. And by *Compoſition of Reaſon*, ... $b \;.\; b-a :: c \;.\; a.$
42. And alternly, ... $b \;.\; c :: b-a \;.\; a.$
43. Wherefore by *Compoſition of Reaſon*, ... $b+c \;.\; c :: b \;.\; a.$

Which laſt Analogy gives this

CANON.

44. As the aggregate of the given ſumm of the firſt and ſecond Proportionals, and the given ſumm of the ſecond and third, is to the ſumm of the ſecond and third; ſo is the ſumm of the firſt and ſecond, to the mean Proportional ſought.

Which Canon, if it be propounded in the form of a Theorem, may be demonſtrated by a repetition of the ſteps of the Reſolution in a direct order. But leaving that and the Compoſition of *Probl.* 5. according to the latter Reſolution, to the Learners exerciſe, I ſhall demonſtrate the following Theorem by a repetition of the ſteps of the latter Reſolution in a retrograde order.

THEOREM.

45. If three right lines be ſuch, that the aggregate of the ſumm of the firſt and ſecond and ſumm of the ſecond and third, is to the ſumm of the ſecond and third; as the ſumm of the firſt and ſecond, to the ſecond: thoſe three lines ſhall be Proportionals, *viz.* As the firſt is to the ſecond, ſo is the ſecond to the third.

Suppoſ.

46. L, M, N, are three right lines.
47. B = L + M, whence B — M = L.
48. C = M + N, whence C — M = N.
49. B + C . C :: B . M.

L —— 9
M —— 6
N —— 4
B ———— 15
C ———— 10

50. . . *Req. demonſtr.* L, M, N are ÷÷, *viz.* L . M :: M . N.

Demonſtration.

51. Becauſe by *Suppoſ.* in 49°, ... B+C . C :: B . M.
52. Therefore by *Diviſion of Reaſon*, ... B . C :: B—M . M.
53. And alternately, ... B . B—M :: C . M.
54. Therefore by *Diviſion of Reaſon*, ... M . B—M :: C—M . M.
55. And inverſly, ... B—M . M :: M . C—M.
56. But by *Suppoſ.* in 47°, ... L = B — M.
57. And by *Suppoſ.* in 48°, ... N = C — M.
58. Therefore out of 55°, 56° and 57°, ... L . M :: M . N.

Which was to be dem.

Probl. VI.

The difference of the greater extreme and mean of three Proportionals being given, as alſo the difference of the mean and leſſer extreme; to find the Proportionals. But the firſt difference muſt be greater than the latter.

Suppoſ.

1. Q, R, S are ÷÷, *viz.* Q . R :: R . S.
2. Q ⊏ R.
3. b = Q — R is given.
4. c = R — S is given.

Req. to find Q, R, S.

Q ——

Q —————— 25
R ———————— 20
S ———————— 16
B ——— 5
C —— 4

Reſolution.

5. Put a for the mean Proportional ſought, *viz.* . . } a = R.
6. Therefore out of 3° and 5°, the greater extreme ſhall be } $a+b$ (= Q.)
7. And out of 4° and 5°, the leſſer extreme ſhall be . . } $a-c$ (= S.)
8. Therefore out of 6°, 5° and 7°, this Analogy will ariſe, (according to the import of the Problem,) *viz.* . } $a+b \;.\; a :: a \;.\; a-c.$
9. Therefore by *Diviſion of Reaſon*, ... $b \;.\; a :: c \;.\; a-c.$
10. And alternately, ... $b \;.\; c :: a \;.\; a-c.$
11. And by *Converſion of Reaſon*, ... $b \;.\; b-c :: a \;.\; c.$
12. And inverſly, ... $b-c \;.\; b :: c \;.\; a.$

Which laſt Analogy gives this

CANON.

13. As the exceſs by which the given difference of the greater extreme and mean exceeds the given difference of the mean and leſſer extreme, is to the difference of the greater extreme and mean; ſo is the difference of the mean and leſſer extreme, to the mean Proportional ſought, whence the extremes will be eaſily diſcovered.

Which Canon, if it be propounded in the form of a Theorem, may be eaſily demonſtrated by a repetition of the ſteps of the Reſolution in a direct order; but leaving that to the Learner's practice, I ſhall demonſtrate the following Theorem by a retrograde repetition of the ſteps of the Reſolution.

THEOREM 1.

14. If three right lines be ſuch, that the exceſs by which the exceſs of the firſt above the ſecond exceeds the exceſs of the ſecond above the third, be to the exceſs of the ſecond above the third, as the exceſs of the firſt above the ſecond is to the ſecond; then thoſe three right lines ſhall be Proportionals, *viz.* As the firſt is to the ſecond, ſo the ſecond to the third.

Suppoſ.

15. Q, R, S are three right lines.
16. Q ⊏ R.
17. R ⊏ S.
18. B = Q — R; whence Q = B + R.
19. C = R — S, whence S = R — C.
20. B — C . C :: B . R.
21. . . *Req. demonſtr.* Q, R, S are ÷÷, *viz.* Q . R :: R . S.

Demonſtration.

22. Becauſe by *Suppoſ.* in 20°, ... B—C . C :: B . R.
23. Therefore alternately, ... B—C . B :: C . R.
24. And inverſly, ... B . B—C :: R . C.
25. And by *Converſion of Reaſon*, ... B . C :: R . R+C.
26. And alternately, ... B . R :: C . R—C.
27. And by *Compoſ. of Reaſon*, ... B+R . R :: R . R+C.
28. But by *Suppoſition* in 18°, ... Q = B + R.
29. Alſo by *Suppoſ.* in 19°, ... S = R — C.
30. Therefore out of 27°, 28° and 29°, by exchange of equal right lines, ... } Q . R :: R . S.

Which was to be Dem.

31. The Determination annex'd to *Probl.* 6. to wit, That the given difference of the greater extreme and mean muſt be greater than the given difference of the mean and leſſer extreme, is diſcovered by the preceding Canon in 13°, and is neceſſarily to be preſcribed for limiting the ſaid differences, that they may be capable of conſtructing the Problem; as will be manifeſt by the ſubſequent

LEMMA.

32. If three right lines be Proportionals, the difference of the greater extreme and mean, is greater than the difference of the mean and lesser extreme.

Suppos.

33. Q, R, S are ∴; *viz.* Q . R :: R . S.
34. Q ⊏ R, and consequently, R ⊏ S.
35. B = Q — R, whence B + R = Q.
36. C = R — S, whence R — C = S.

Q —————————— 25
R —————————— 20
S ———————— 16
B ——— 5
C ——— 4

37. . . *Req. demonstr.* . . B ⊏ C.

Demonstration.

38. Because by *Suppos.* in 33°, ... } . Q . R :: R . S.
39. And by *Suppos.* in 35° and 36°, ... } B + R = Q. And R − C = S.
40. Therefore out of 38° and 39°, ... } B + R . R :: R . R − C.
41. And by *Division of Reason*, ... } . B . R :: C . R − C.
42. But R ⊏ R — C, therefore from 41°, (*per Schol. prop.* 14. *Elem.* 5.) . . . . . } . . B ⊏ C.

Which was to be Demonstr.

The Determination being demonstrated, I shall proceed to

The Composition of Probl. 6.

Suppos.

43. B = the difference of the greater extreme and mean of three Proportionals is given.
44. C = the difference of the mean and lesser extreme is given.
45. B ⊏ C. (Determination.)

Req. to find the Proportionals.

B ——— 5
C ——— 4
Q —————————— 25
R ———————— 20
S ———————— 16

Construction.

46. By *Probl.* 8. *Chap.* 5. let it be made as B — C to B, so C to a fourth Proportional which may be called R, therefore,

B — C . B :: C . R.

which fourth Proportional R shall be greater than the third C, because the second B is greater than the first B — C.

47. Make Q = R + B; whence, Q — R = B.
48. Make S = R — C; whence, R — S = C, which effection is possible, for by the Analogy in 46°, it is manifest that R ⊏ C.
49. I say Q, R and S are the three Proportionals required: Now we must shew that they will satisfie the Problem; First then, by *Construction* in 47°, the excess of Q above R is equal to the given difference B; secondly, by *Constr.* in 48°, the excess of R above S is equal to the given difference C. So it remains only to prove that Q, R and S are Proportionals, in this order, *viz.* Q . R :: R . S, but that is made manifest by the subsequent Demonstration, which is form'd out of the preceding Resolution, by a repetition of the steps thereof in a retrograde (not in a direct) order.
50. . . *Req. demonstr.* . . . . . Q . R :: R . S.

Demonstration.

51. Because by *Constr.* in 46°, which answers to the last step of the Resolution, to wit, 12°, ... } B — C . B :: C . R.
52. Therefore inversly, ... } . B . B − C :: R . C.
53. And by *Conversion of Reason*, ... } . B . C :: R . R − C.
54. And alternately, ... } . B . R :: C . R − C.
55. Therefore by *Composition of Reason*, ... } B + R . R :: R . R − C.
56. But by *Constr.* in 47°, ... } . . . Q = B + R.
57. And by *Constr.* in 48°, ... } . . . S = R — C.
58. Therefore out of 55°, 56°, 57°, ... } . Q . R :: R . S.

Which was to be dem. therefore that is done which was required by *Probl.* 6.

Another way of resolving the foregoing Probl. 6.

59. The same things being given and supposed as before in 1°, 2°, 3°, 4° of *Probl.* 6. for the lesser extreme of the three Proportionals sought, put . . . . } *a.*
60. To which lesser extreme if you add the given difference *c*, it makes the mean, to wit, ... } *a* + *c*.
61. And by adding the given difference *b* to the mean Proportional, it gives the greater extreme, to wit, } *a* + *c* + *b*.
62. Therefore according to the import of the Problem, these must be Proportionals, *viz.* . . . . . } *a* . *a* + *c* :: *a* + *c* . *a* + *c* + *b*.
63. Therefore inversly, ... } *a* + *c* . *a* :: *a* + *c* + *b* . *a* + *c*.
64. And by *Division of Reason*, ... } *c* . *a* :: *b* . *a* + *c*.
65. And by *altern* and *inverse Reason*, ... } *b* . *c* :: *a* + *c* . *a*.
66. Wherefore by *Division of Reason*, ... } *b* − *c* . *c* :: *c* . *a*.

Which last Analogy gives this

CANON.

67. As the excess whereby the given difference of the greater extreme and mean exceeds the given difference of the mean and lesser extreme, is to the difference of the mean and lesser extreme; so is the difference last mentioned, to the lesser extreme sought: whence the mean and greater extreme are easily discovered.

Probl. VII.

The difference of the extremes of three Proportionals being given, as also a right line whose Square is equal to the difference between the Square of the mean and the Square of one of the extremes, to find the Proportionals.

Suppos.

1. N, M, L are ∴; *viz.* N . M :: M . L.
2. N ⊏ L.
3. *d* = N — L is given.
4. *c* = √: □ M — □ L : is given; and consequently,
5. *cc* = □ M — □ L is given.

Req. to find N, M, L.

N —————————— 25
M —————————— 20
L ———————— 16
D ——— 9
C ——————— 12

Resolution.

6. Put *a* for the lesser extreme sought, *viz.* . . . . . } *a* = L.
7. Therefore out of 3° and 6°, the greater extreme shall be . . } *a* + *d* (= N.)
8. And out of 6° and 7°, the Rectangle contained under the extremes, or the Square of the mean, is equal to . . . . } *aa* + *da*.
9. And out of 6°, the Square of the lesser extreme is . . . . } *aa*.
10. Which Square *aa* being subtracted from *aa* + *da*, (to wit, from the Square of the mean,) the remainder shall be the difference of the said Squares, *viz.* . . . . . } *da*.
11. But the difference last mentioned must be equal to the given difference *cc*, therefore . . . . . } *da* = *cc*.
12. Which Equation may be resolved into this Analogy, . . } *d* . *c* :: *c* . *a*.

Hence

Hence this *CANON.*

13. As the given difference of the extremes, is to the given right line whoſe Square is equal to the difference of the Squares of the mean and leſſer extreme; ſo is the ſame right line to the leſſer extreme ſought.

Or thus,

As the given difference of the extremes, is to the given right line whoſe Square is equal to the difference of the Squares of the mean and greater extreme; ſo is the ſame right line to the greater extreme ſought.

Hence this *THEOREM.*

14. If three right lines be Proportionals, the difference of the Squares of the mean and leſſer extreme is equal to the Rectangle contained under the difference of the extremes and the leſſer extreme.

Or thus,

The difference of the Squares of the mean and greater extreme, is equal to the Rectangle contained under the difference of the extremes and greater extreme.

The Compoſition of Probl. 7.

Suppoſ.

15. D = the difference of the extremes of three Proportionals is given.
16. C = a right line, whoſe Square is equal to the difference of the Squares of the mean and leſſer extreme is given.

Req. to find the Proportionals.

N —————————— 25
M ———————— 20
L ——————— 16
D ———— 9
C ———— 12

Conſtruction.

17. By *Probl.* 7. *Chap.* 5. let it be made as D to C, ſo C to a third Proportional, which ſuppoſe to be the right line L, therefore,

D . C :: C . L.

18. By *Probl.* 2. *Chap.* 5. find a right line, as M, ſuch, that its Square may be equal to the Square of L together with the Square of C, therefore,

□M = □L -|- □C;

And conſequently, □M — □L = □C.

19. Make N = L -|- D; whence, N — L = D.
20. I ſay N, M and L are the three Proportionals required. Now we muſt ſhew that they will ſatisfie the Problem.
21. Firſt then, by *Conſtruction* in 19°, the exceſs of N above L is equal to D the given difference of the extremes.
22. Secondly, the exceſs by which the Square of the mean M exceeds the Square of the leſſer extreme L, is (by *Conſtr.* in 18°,) equal to the Square of C, to wit, the given difference of the Squares of the mean and leſſer extreme.
23. It remains only, to prove that the ſaid three right lines N, M and L are Proportionals, in this order, *viz.* As N is to M, ſo M to L; But that is made manifeſt by the ſubſequent Demonſtration, which is formed by a retrograde repetition of the ſteps of the preceding Reſolution.

18. . . *Req. demonſtr.* . . . . . N . M :: M . L.

Demonſtration.

25. Becauſe by *Conſtruction* in 17°, ... D . C :: C . L.
That is, in 12°, (the laſt ſtep of the *Reſolut.*) . . . *d* . *c* :: *c* . *a*.
26. Therefore, *per* 17. *prop.* 6. *Elem.* . . . . . □D, L = □C.
That is, in 11°, ... *da* = *cc*.
27. Again, becauſe by *Conſtr.* in 19°, ... N = L -|- D.
28. Therefore (*per prop.* 1. *Elem.* 6.) by drawing L as a common altitude into each part of the Equation in 27°, □NL = □L -|- □D, L
29. And conſequently out of 28° and 26°, by exchange of equal Rectangles, ... □NL = □L -|- □C.

30. But

30. But by *Conſtr.* in 18°, ... □M = □L -|- □C.
31. Therefore out of 29° and 30°, *per* 1. *Ax. Chap.* 2. □NL = □M.
32. Therefore out of 31°, *per* 14. *prop.* 6. *Elem.* N . M :: M . L.

Which was to be Demonſtr. therefore the Problem propounded is ſatisfied.

Having by the preceding Examples of Reſolution and Compoſition given the Learner a taſt of the manner of arguing by Analogies, which is the beſt way when the nature of a Problem will admit the ſame, I ſhall now proceed to Examples of arguing partly by Equations, and partly by Analogies. But it muſt be remembred, that when in the Reſolution you paſs from one ſtep to another by Addition, in the Demonſtration of the Problem you muſt return by Subtraction: For Addition in the Reſolution requires Subtraction in the Compoſition, and Subtraction in the one, Addition in the other; alſo Compoſition of Reaſon in the one requires Diviſion of Reaſon in the other, as before hath been ſaid in the *Note* at the end of *Probl.* 1. of this Chapter.

Probl. VIII.

A right line equal to the ſumm of three proportional right lines being given, as alſo a right line whoſe Square is equal to the ſumm of the Squares of all the ſaid Proportionals, to find out the Proportionals ſeverally. But the firſt of thoſe lines given muſt be greater than the latter, yet not greater than the right line ariſing out of the Application of the triple Square of the ſaid latter line to the firſt.

L ———————— 8
M ———— 4
N —— 2
B ———————————— 14
C ———————— √84

Suppoſ.

1. L, M, N ∴, *viz.* L . M :: M . N.
2. *b* = L -|- M -|- N is given.
3. *c* = √: □L -|- □M -|- □N: is given; therefore,
4. *cc* = □L -|- □M -|- □N is given alſo.

Req. to find L, M, N.

Reſolution.

5. For the mean Proportional ſought put *a*, viz. ſuppoſe . . . . . *a* = M.
6. Therefore from 2° and 5°, the ſumm of the extremes ſhall be . . . . . *b* — *a* = L -|- N.
7. Therefore the Square of the ſumm of the extremes is . . . . . *bb* -|- *aa* — 2*ba* = □L -|- □N -|- 2□LN.
8. And the Square of the mean Proportional, or the Rectangle of the extremes is . . . *aa* = □M = □LN.
9. Which Square or Rectangle in 8°, being ſubtracted from the Square in 7°, leaves the ſumm of the Squares of all the three Proportionals, *viz.* *bb* — 2*ba* = □L -|- □M -|- □N.
10. But by *Suppoſition* in 4°, ... *cc* = □L -|- □M -|- □N.
11. Therefore from 9° and 10°, (*per Ax.* 1. *Chap.* 2.) . . . . . *bb* — 2*ba* = *cc*.
12. And by adding 2*ba* to each part of the Equation in 11°, this ariſeth, ... *bb* = *cc* -|- 2*ba*.
13. And by ſubtracting *cc* from each part of the Equation in 12°, ... *bb* — *cc* = 2*ba*.
14. And becauſe (*per Theor.* 8. *Chap.* 4.) . . *bb* — *cc* = $\overline{b+c} \times \overline{b-c}$.
15. Therefore from 13° and 14°, (*per Ax.* 1. *Chap.* 2.) . . . . . 2*ba* = $\overline{b+c} \times \overline{b-c}$.
16. Therefore by reſolving the laſt preceding Equation into Proportionals, it ſhall be . . 2*b* : *b* -|- *c* :: *b* — *c* . *a*.

From

From the ninth ſtep ariſeth

THEOREM 1.

17. If three right lines be Proportionals, the exceſs whereby the Square of their ſumm exceeds twice the Rectangle made of that ſumm and the mean proportional, ſhall be equal to the ſumm of the Squares of all the three proportionals.

From the laſt ſtep of the Reſolution ariſeth

THEOREM 2.

18. If three right lines be Proportionals, then this Analogy will attend them, *viz.* As the double ſumm of all the three Proportionals is to the ſimple ſumm increaſed with the ſide of a Square equal to the ſumm of the Squares of the three Proportionals; ſo is the exceſs whereby the ſumm of the three Proportionals exceeds the ſaid ſide, to the mean Proportional.

Therefore the ſumm of three Proportionals being given, as alſo the ſumm of their Squares, the mean Proportional ſhall be given alſo by the preceding *Theor.* 2. whence the ſumm of the extremes is conſequently given. And laſtly, the ſumm of the extremes being given, as alſo the mean, the extremes ſhall be given ſeverally, by *Probl.* 13. *Chap.* 5.

But to ſolve this *Probl.* 8. Arithmetically, the following Canon, (deducible from the 13th ſtep,) will be more ready than *Theor.* 2.

CANON.

19. From the Square of the given ſumm of three Proportionals, ſubtract the given ſumm of their Squares, and divide the remainder by the double of the firſt given ſumm; ſo ſhall the Quotient be the mean Proportional; which ſubtracted from the ſumm of all three, leaves the ſumm of the extremes. And laſtly, the ſumm of the extremes being given, as alſo the mean, the extremes ſhall be given ſeverally, by *Theor.* in 21° of *Probl.* 13. *Ch.* 5.

But for the greater evidence, I ſhall demonſtrate the truth of the preceding *Theor.* 1, and 2. and conſequently the Canon, by the ſteps of the foregoing Reſolution in a direct order, *viz.* by proceeding from the beginning to the end of the Reſolution.

Suppoſ.

20. L, M, N ÷÷, *viz.* L . M :: M . N.
21. $B = L + M + N.$
22. $C = \sqrt{:\square L + \square M + \square N:}$
23. $\square C = \square L + \square M + \square N.$

L ——— 8
M ——— 4
N —— 2
B ——— 14
C ——— $\sqrt{84}$

24. . . *Req. demonſtr.* . . . . { *Theor.* 1. $\square B - 2\square BM = \square C.$ *Theor.* 2. $2B . B + C :: B - C . M.$

Demonſtration.

25. By *Suppoſition* in 21°, ... } $B = L + M + N.$
26. Therefore by ſubtracting M from each part, ... } $B - M = L + N.$
27. And by ſquaring each part in 26°, this Equation will ariſe, (*per Theor.* 5, and 2. of *Chap.* 4.) . . . . . } $\square B + \square M - 2\square BM = \square L + \square N + 2\square LN,$
28. And from 20°, (*per prop.* 17. *Elem.* 6.) } $\square M = \square LN.$
29. Therefore by ſubtracting the Equation in 28° from that in 27°, this will manifeſt, (*per Ax.* 9, & 6. *Chap.* 2.) . . } $\square B - 2\square BM = \square L + \square M + \square N,$
30. But by *Suppoſ.* in 23°, ... } $\square C = \square L + \square M + \square N,$
31. Therefore from 29° and 30°, (*per Ax.* 1. *Chap.* 2.) . . . . . } $\square B - 2\square BM = \square C.$

Which was *Theor.* 1. to be demonſtr.

32. Again, by adding $2\square BM$ to each part of the Equation in 31°, this ariſeth, . } $\square B = \square C + 2\square BM.$

33. And

33. And by ſubtracting $\square C$ from each part of the Equation in 32°, ... } $\square B - \square C = 2\square BM.$
34. But by *Theor.* 8. *Chap.* 4. . . . . } $\square B - \square C = \square:\overline{B+C} \times \overline{B-C}:$
35. Therefore from 33° and 34°, (*per Ax.* 1.) . } $2\square BM = \square:\overline{B+C} \times \overline{B-C}:$
36. Therefore, (*per prop.* 14. *Elem.* 6.) . . . } $2B . B + C :: B - C . M.$

Which was *Theor.* 2. to be dem. Therefore the truth of both the preceding Theorems is made manifeſt; alſo the Canon in 19° is evident from the Equation in 33°, by Application of each part thereof unto 2 B.

37. I ſhall in the next place, in order to the Compoſition of *Probl.* 8. before propounded, demonſtrate the Determination annex'd to it for limiting the lines given, that they may be capable of effecting the Problem.

Determination. $\begin{cases} B \sqsubset C, \\ B \text{ not } \sqsubset \frac{3\square C}{B}. \end{cases}$

That is, the line given for the ſumm of three Proportionals muſt be greater than that given right line whoſe Square is equal to the ſumm of the Squares of all the three Proportionals, yet not greater than the right line ariſing out of the Application of the triple of the ſaid Square, to the right line given for the ſumm of the three Proportionals.

38. The Scope of the Determination is, to remove two Objections that may be brought againſt the Conſtruction of the Problem (in the following 76th and 87th ſteps,) unleſs the given lines be limited as the Determination preſcribes, whoſe firſt part, to wit, that $B \sqsubset C$ is diſcovered by the laſt ſtep of the Reſolution, and already demonſtrated. The latter part of the Determination, to wit, that the given line B ought not to be greater than $\frac{3\square C}{B}$, is neither apparent in the propoſition of the Problem, nor in either of the Theorems reſulting from the Reſolution; but that it is a property adherent to three Proportionals, I ſhall demonſtrate by the following *Lemma*, and afterwards ſhew that it is neceſſary to make the Problem poſſible.

LEMMA.

39. If three right lines be Proportionals, their ſumm ſhall ſometimes be equal to the right line ariſing out of the Application of the triple ſumm of all their Squares to their ſaid ſumm; ſometimes leſs, but never greater than the right line ariſing by the ſaid Application.

40. Two Caſes are to be demonſtrated to prove this *Lemma*, for three Proportionals are either equal between themſelves, or unequal. In the firſt Caſe, 'tis eaſie to perceive that the Square of the ſumm of three Proportionals is equal to the triple ſumm of their Squares; for ſuppoſing N, N, N to repreſent three Proportionals, their ſumm is 3N, the Square whereof is $9\square N$, which is manifeſtly equal to $3\square N + 3\square N + 3\square N$, to wit, the triple ſumm of the Squares of the ſaid three Proportionals; and therefore if $9\square N$, the triple ſumm of the Squares of the ſaid three equal Proportionals N, N, N, be applied to (or divided by) 3 N the ſumm of the ſame Proportionals, the line (or Quotient) ariſing by that Application ſhall neceſſarily be 3 N, (the ſumm of the ſaid Proportionals.) Therefore the firſt Caſe of the *Lemma* is manifeſt.

Suppoſ. in *Caſe* 2.

41. L, M, N ÷÷, *viz.* L . M :: M . N.
42. L $\sqsubset$ M; and conſequently M $\sqsubset$ N.

Prepar.

43. By *Probl.* 1. *Chap.* 5. make $B = L + M + N.$
44. By *Probl.* 2. *Chap.* 5. make $C = \sqrt{:\square L + \square M + \square N:}$
45. Thence it follows, that $\square C = \square L + \square M + \square N.$
46. By *Probl.* 8. *Chap.* 5. let it be made, B . C :: 3 C (to a fourth) T.
47. Thence it follows, that $T = \frac{3\square C}{B}.$

L ——————— 8
M ——— 4
N —— 2
B ——————————— 14
C ——————— √84
T ——————————————— 18

48. . . *Req. demonſtr.* . . . . . B ⊐ T, or $\frac{3\square C}{B}$.

Demonſtration.

49. Becauſe by *Suppoſ.* in 41°, ... L . M :: M . N.
50. Therefore, *(per prop.* 1. *Elem.* 6.) by reaſon of the common altitude L, . . . □L . □LM :: L . M :: M . N.
51. Likewiſe by reaſon of the common altitude N, ... □MN . □N :: M . N :: L . M.
52. Therefore out of 50° and 51°, *(per prop.* 11. *Elem.* 5.) . . . □MN . □N :: □L . □LM.
53. And out of 42° and 52°, by *Diviſion of Reaſon*, ... □MN−□N . □N :: □L−□LM . □LM.
54. And inverſly, ... □N . □MN−□N :: □LM . □L−□LM.
55. But from 41° and 42°, . . . □N ⊐ □LM.
56. Therefore from 54° and 55°, *(per prop.* 14. *Elem.* 5.) . . . . □MN − □N ⊐ □L − □LM.
57. And by adding □LM, to each part in 56°, ... □LM + □MN − □N ⊐ □L.
58. And by adding □N to each part in 57°, ... □LM + □MN ⊐ □L + □N.
59. And by adding □M to each part in 58°, ... □LM+□M+□MN ⊐ □L+□M+□N.
60. But by *Conſtr.* in 45°, . . . □C = □L + □M + □N.
61. Therefore from 59° and 60°, *(per Ax.* 3. *Chap.* 2.) . . . . □LM + □M + □MN ⊐ □C.
62. And becauſe by *Conſtr.* in 43°, L + M + N = B.
63. Therefore from 62°, by reaſon of the common altitude M, . . . □LM + □M + □MN = □BM.
64. Therefore from 61° and 63°, *(per Ax.* 4. *Chap.* 2.) . . . □BM ⊐ □C.
65. And conſequently, ... 2□BM ⊐ 2□C.
66. And by adding □C to each part in 65°, ... □C + 2□BM ⊐ 3□C.
67. But it hath been proved in 32°, (in the preceding Demonſtration of *Theor.* 1, and 2.) that . . □C + 2□BM = □B.
68. Therefore out of 66° and 67°, *(per Ax.* 4. *Chap.* 2.) . . . □B ⊐ 3□C.
69. Therefore by Application of each part in 68° to B, ... B ⊐ $\frac{3\square C}{B}$.
70. But by *Conſtr.* in 46°, . . . T = $\frac{3\square C}{B}$.

Therefore from 69° and 70°, *(per Ax.* 3. *Chap.* 2.) . . . . . B ⊐ T or $\frac{3\square C}{B}$.

Which was to be Demonſtr.

71. Now becauſe every three Proportionals whatever, are either equal or unequal between themſelves, and it hath been ſhewn, that when they be equal to one another, their ſumm is equal to the right line ariſing out of the Application of the triple ſumm of their Squares to their ſumm; but when unequal, the ſumm of the three Proportionals is leſs than the right line ariſing by the ſaid Application; it is manifeſt that the ſaid ſumm can never be greater than the ſaid right line: And therefore the truth of the preceding *Lemma*, and conſequently the reaſon of the latter part of the Determination annext to *Probl.* 8. are evident.

The

The Compoſition of the foregoing Probl. 8.

L ——————— 8
M ——— 4
N —— 2
B ——————————— 14
C ——————— √84
T ——————————————— 18
G ——————————————————————— 14 + √84
H ——————— 14 − √84
F ——————— 10

Suppoſ.

72. B = the ſumm of three proportional right lines is given.
73. C = a right line, whoſe Square is equal to the ſumm of the Squares of the ſaid three Proportionals, is given.
74. B ⊏ C; yet B not ⊏ $\frac{3\square C}{B}$. *(Determination.)*

Req. to find out the three Proportionals.

Conſtruction.

75. By *Probl.* 1. *Chap.* 5. find a right line, as G, equal to the ſumm of the given right lines B and C, therefore . . . . . G = B + C.
76. By *Probl.* 3. *Chap.* 5. find a right line, as H, equal to the exceſs of B above C; which effection is poſſible, for by *Suppoſ.* in 74°, B⊏C, therefore, H = B − C.
77. By *Probl.* 8. *Chap.* 5. let it be made as 2B to G; ſo H to a fourth proportional line, call it M, therefore 2B . G :: H . M.
78. Find a right line F = B − M, which is poſſible to be done if B exceeds M, but that B⊏M I prove thus,
79. By *Conſtruction* in 75°, ... B + C = G.
80. And by *Conſtr.* in 76°, ... B − C = H.
81. Therefore *(per Theor.* 8. *Chap.* 4.) by comparing the Rectangles made of the two laſt Equations, . . . □B − □C = □GH.
82. And by the firſt part of the laſt Equation 'tis manifeſt that . . . . . 2□B ⊏ □B − □C.
83. Therefore from 81° and 82°, *(per Ax.* 3. *Chap.* 2.) 2□B ⊏ □GH.
84. But from 77°, *(per prop.* 16. *Elem.* 6.) . . . 2□BM = □GH.
85. Therefore from 83° and 84°, *(per Ax.* 3. *Chap.* 2.) 2□B ⊏ 2□BM.
86. Therefore by Application of each part in 85° to 2B, ... B ⊏ M.

Which was to be ſhewn.

87. By *Probl.* 14. *Chap.* 5. divide the line F (before found in 78°) into two ſuch parts, that the line M may be a mean Proportional between the parts; which is poſſible to be done if M be not greater than ½F, but that M is not greater than ½F, I prove thus,
88. By *Suppoſition* in 74°, ... B not ⊏ $\frac{3\square C}{B}$.
89. Therefore by drawing B into each part, . . . □B not ⊏ 3□C.
90. And by adding 2□B to each part in 89°, . . 3□B not ⊏ 2□B + 3□C.
91. And becauſe by *Suppoſ.* in 74°, ... B ⊏ C.
92. And conſequently, ... 3□B ⊏ 3□C.
93. Therefore by ſubtracting 3□C from each part in 90°, ... 3□B − 3□C not ⊏ 2□B.
94. It hath been proved in 81°, that . . . . . □B − □C = □GH.
95. And in 84°, that . . . . . 2□BM = □GH.
96. Therefore from the two laſt preceding Equations, □B − □C = 2□BM.
97. And conſequently, ... 3□B − 3□C = 6□BM.
98. Therefore from 93° and 97°, *(per Ax.* 4. *Chap.* 2.) 6□BM not ⊏ 2□B.
99. And conſequently, ... 3□BM not ⊏ □B.

100. Therefore by Application of each part in 99° to B, ⊦ 3 M not ⊏ B.
101. It hath been proved in 86°, that . . . . . ⊦ B ⊏ M.
102. Therefore by ſubtracting M from each part in 100°, ⊦ 2 M not ⊏ B — M.
103. But by *Conſtr.* in 78°, ... ⊦ F = B — M.
104. Therefore from 102° and 103°, ... ⊦ 2 M not ⊏ F.
105. And conſequently, ... ⊦ M not ⊏ ½ F.
Which was to be Dem.

106. Therefore 'tis poſſible (by *Probl.* 14. *Chap.* 5.) to cut the line F into two ſuch parts, that the line M may be a mean Proportional between the parts; ſuppoſe it therefore done, and that the ſaid parts are found L and N, therefore,

L . M :: M . N.

107. I ſay the right lines L, M, N are the three Proportionals ſought: Now we muſt ſhew that they will ſatisfie the Problem propounded; Firſt then, by Conſtruction in the laſt preceding ſtep they are Proportionals. Secondly, that the ſumm of the ſaid L, M, N isequal to the given ſumm B, I prove thus,

108. Becauſe by *Conſtr.* in 106°, ... ⊦ L ⊦ N = F.
109. Therefore by adding M to each part, ... ⊦ L ⊦ M ⊦ N = F ⊦ M.
110. But by *Conſtr.* in 78°, ... ⊦ B — M = F.
111. And conſequently, ... ⊦ B = F ⊦ M.
112. Therefore from 109° and 111°, (*per Ax.* 1. *Chap.* 2.) ⊦ L ⊦ M ⊦ N = B.
Which was to be proved.

113. It remains only to ſhew, that the ſumm of the Squares of the ſaid right lines L, M and N is equal to the Square of the given line C; but that is made manifeſt by the following Demonſtration, which is formed out of the ſteps of the preceding Reſolution, in a retrograde (not in a direct) order.

114. . . *Req. demonſtr.* . . . . . □L ⊦ □M ⊦ □N = □C.

Demonſtration.

115. Becauſe by *Conſtr.* in 77°, ... ⊦ 2 B . G :: H . M.
That is, in 16°, (the laſt ſtep of the Reſolution,) ⊦ 2*b* . *b* ⊦ *c* :: *b* — *c* . *c*.
116. Therefore from 115°, (*per prop.* 16. *Elem.* 6.) ⊦ 2 □BM = □GH.
That is, in 15°, ... ⊦ $2ba = \overline{b+c} \times \overline{b-c}$.
117. But from 81°, ... ⊦ □B — □C = □GH.
That is, in 14°, ... ⊦ $bb - cc = \overline{b+c} \times \overline{b-c}$.
118. Therefore from 116° and 117°, (*per Ax.* 1. *Chap.* 2.) . . . . . ⊦ □B — □C = 2 □BM.
That is, in 13°, ... ⊦ $bb - cc = 2ba$.
119. And by addition of □C to each part in 118°, ⊦ □B = □C ⊦ 2 □BM.
That is, in 12°, ... ⊦ $bb = cc + 2ba$.
120. And by ſubtraction of 2 □BM from each part in 119°, ... ⊦ □B — 2 □BM = □C.
That is, in 11°, ... ⊦ $bb - 2ba = cc$.
121. By *Conſtr.* in 106°, ... ⊦ L . M :: M . N.
122. And it hath been proved in 112°, that . . ⊦ L ⊦ M ⊦ N = B.
123. Therefore out of 121° and 122° (*per Theor.* 1. in 17° of this *Probl.* 8.) . . . . ⊦ □B — 2 □BM = □L ⊦ □M ⊦ □N.
124. Therefore from 120° and 123°, (*per Ax.* 1. *Chap.* 2.) . . . . . ⊦ □L ⊦ □M ⊦ □N = □C.
Which was to be Dem.

Thus you have ſeen the preceding *Probl.* 8. effected and demonſtrated Synthetically, that is, by working and arguing with given Quantities only; but the ſubſtance of the Effection and Demonſtration was firſt found out Analytically.

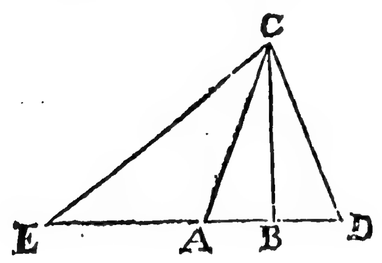
125. But me-thinks I hear the Learner object, That although the preceding Reſolution of *Probl.* 8. be clear and eaſie, yet the preſcribed Determination, eſpecially its latter part, and the reaſon thereof ſeems hard to be found out: For anſwer to this, I ſay, That the firſt part of the Determination is apparent in the laſt ſtep of the Reſolution, in which alſo the latter part of the Determination is tacitly implied, and may thence be eaſily inferr'd thus;

126. Sup-

126. Suppoſe (as before in the Reſolution,) *b* to repreſent the given ſumm of three Proportionals, and *cc* a given Square equal to the ſumm of the Squares of thoſe Proportionals, then the mean Proportional, (by the Canon in the 19th ſtep,) will be found equal to . . . . .

7. BA = BD.
8. AE = BE — BD (BA) the difference of the ſegments.
9. b = ED the Baſe is given.
10. c = EC the greater legg is given.
11. d = DC the leſſer legg is given.

Req. to find BE and BD.



ED = 63
EC = 60
DC = 39
EA = 33
BA = 15 = BD
BE = 48
BC = 36
△EDC = 1134

Reſolution.

12. Put a for the difference of the ſegments of the Baſe, *viz.* } a = AE.
13. Then (*per Theor.* 9. *Chap.* 4.) the greater ſegment of the Baſe ſhall be } $\frac{1}{2}b + \frac{1}{2}a$ = BE.
14. Likewiſe by the ſame *Theor.* the leſſer ſegment ſhall be } $\frac{1}{2}b - \frac{1}{2}a$ = BD.
15. Therefore the Square of the greater ſegment ſhall be } $\frac{1}{4}bb + \frac{1}{2}ba + \frac{1}{4}aa$ (= □BE.)
16. And the Square of the leſſer ſegment ſhall be } $\frac{1}{4}bb - \frac{1}{2}ba + \frac{1}{4}aa$ (= □BD.)
17. By 47. *prop.* 1. *Elem.* } □EC = □BE + □BC.
18. Therefore by equal ſubtraction of □BE, } □EC — □BE = □BC.
19. That is, in the letters belonging to the Reſolution, } $cc - \frac{1}{4}bb - \frac{1}{2}ba - \frac{1}{4}aa$ = □BC.
20. In like manner, (*per* 47. *prop.* 1. *Elem.*) } □DC — □DB = □BC.
21. That is, in the letters of the Reſolution, } $dd - \frac{1}{4}bb + \frac{1}{2}ba - \frac{1}{4}aa$ = □BC.
22. Therefore out of 19° and 21°, (*per* 1. *Axiom.* 2. *Chap.*)

$$cc - \tfrac{1}{4}bb - \tfrac{1}{2}ba - \tfrac{1}{4}aa = dd - \tfrac{1}{4}bb + \tfrac{1}{2}ba - \tfrac{1}{4}aa.$$

23. Therefore from 22°, by equal addition and ſubtraction, } $cc - dd = ba$.
24. Therefore from 23°, by Application of each part of that Equation to b, } $\frac{cc - dd}{b} = a$.
25. Moreover, becauſe by *Theor.* 8. *Chap.* 4. } $cc - dd = \overline{c + d} \times \overline{c - d}$.
26. Therefore out of 23° and 25°, } $c + d \times c - d = ba$.
27. Which laſt Equation may be reſolved into this Analogy, } $b \,.\, c + d :: c - d \,.\, a$.

Out of 24° ariſeth

THEOR. 1.

28. If the difference of the Squares of the leggs of a plain Triangle which hath unequal acute angles at the Baſe, be applied to, (or divided by) the Baſe; the line or number ariſing ſhall be equal to the difference of the ſegments of the Baſe which are made by the falling of the Perpendicular from the angle oppoſite to the Baſe.

Out of 27° ariſeth

THEOR. 2.

29. As the Baſe of a plane Triangle which hath unequal acute angles at the Baſe, is to the ſumm of the leggs; ſo is the difference of the leggs, to the difference of the ſegments of the Baſe which are made by the falling of the Perpendicular from the angle oppoſite to the Baſe.

30. Then the Baſe being given, as alſo the difference of the ſegments of the Baſe, the ſegments themſelves ſhall be given ſeverally; for (by *Theor.* 9. *Chap* 4.) half the Baſe, together with half the difference of the ſegments ſhall be equal to the greater ſegment; and half the Baſe leſs by half the difference of the ſegments ſhall be equal to the leſſer ſegment. Laſtly, the ſquare Root of the exceſs of the Square of the greater legg above the Square of the greater ſegment of the Baſe; or the ſquare Root of the exceſs of the Square of the leſſer legg above the Square of the leſſer ſegment, will give the Perpendicular, which multiplied into half the Baſe, gives the Area of the Triangle.

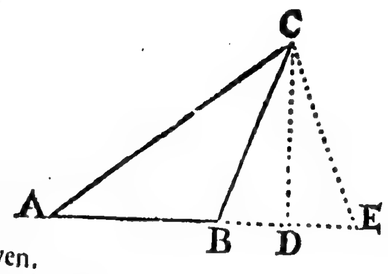
Probl. X.

Probl. X.

The Baſe and leggs of a plane Triangle obtuſangled at the Baſe being ſeverally given, to find out Arithmetically the diſtance from the foot of the Perpendicular falling without the Triangle, to each end of the Baſe: And conſequently the Perpendicular and Area of the Triangle.

Suppoſ.

1. ABC is a △ obtuſangled at B, therefore
2. AC ⌐ BC.
3. ABE is a right line.
4. CD ⊥ AE.
5. DE = DB.
6. AE = AB + 2BD, (2DE.)
7. b = AB the Baſe is given.
8. c = AC the greater legg is given.
9. d = BC the leſſer legg is given.



AB = 33
AC = 60
BC = 39
AE = 63
BD = 15 = DE
DC = 36
△ABC = 594

Req. to find DB, and DA.

Reſolution.

10. Suppoſe } a = AE.
11. Then becauſe } BE = AE — AB.
12. Therefore the halves of all in 11° being taken, } $\frac{1}{2}$BE (or BD) = $\frac{1}{2}$AE — $\frac{1}{2}$AB.
13. That is, in the letters belonging to the Reſolution, } $\frac{1}{2}a - \frac{1}{2}b$ = BD.
14. And becauſe } AB + BD = AD.
15. Therefore in the letters of the Reſolution, } $b + \frac{1}{2}a - \frac{1}{2}b$ = AD.
16. That is, } $\frac{1}{2}a + \frac{1}{2}b$ = AD.
17. The Square whereof is } $\frac{1}{4}aa + \frac{1}{2}ba + \frac{1}{4}bb$ = □AD.
18. And the Square of the Equation in 13° is } $\frac{1}{4}aa - \frac{1}{2}ba + \frac{1}{4}bb$ = □BD.
19. By 47. *prop.* 1. *Elem.* } □AC = □AD + □DC.
20. Therefore by equal ſubtraction of □AD, } □AC — □AD = □DC.
21. That is, in the letters of the Reſolution, } $cc - \frac{1}{4}aa - \frac{1}{2}ba - \frac{1}{4}bb$ = □DC.
22. In like manner, by 47. *prop.* 1. *Elem.* } □BC — □BD = □DC.
23. That is, in the letters of the Reſolut. } $dd - \frac{1}{4}aa + \frac{1}{2}ba - \frac{1}{4}bb$ = □DC.
24. Therefore out of 21° and 23°, (*per* 1. *Axiom.* 2. *Chap.*)

$$cc - \tfrac{1}{4}aa - \tfrac{1}{2}ba - \tfrac{1}{4}bb = dd - \tfrac{1}{4}aa + \tfrac{1}{2}ba - \tfrac{1}{4}bb.$$

25. Therefore out of 24°, by equal addition and ſubtraction, } $cc - dd = ba$.
26. Therefore each part of the laſt Equation being applied to b, } $\frac{cc - dd}{b} = a$.
27. Moreover, becauſe by *Theor.* 8. *Chap.* 4. } $cc - dd = \overline{b + c} \times \overline{b - c}$.
28. Therefore out of 25° and 27°, } $c + d \times c - d = ba$.
29. Which laſt Equation may be reſolved into this Analogy, } $b \,.\, c + d :: c - d \,.\, a$.

Out of 26° ariſeth

THEOR. 1.

30. If the difference of the Squares of the leggs of a plane Triangle obtuſangled at the Baſe, be applied to (or divided by) the Baſe, the line (or number) ariſing ſhall be equal to the ſumm of the Baſe and the double of the diſtance from the obtuſe angle to the foot of the Perpendicular, falling upon the Baſe produced without the Triangle.

Out of 29° ariſeth

THEOR. 2.

31. As the Baſe of a plane Triangle obtuſangled at the Baſe, is to the ſumm of the leggs; ſo is the difference of the leggs, to a fourth Proportional, which is compos'd of the Baſe and the double of the diſtance from the obtuſe angle to the foot of the Perpendicular, falling without the Triangle upon the Baſe prolonged.

32. Then

32. Then the Baſe, and the line compos'd of the Baſe and the double diſtance from the obtuſe angle to the foot of the Perpendicular being ſeverally given by either of the ſaid Theorems, the diſtance from the foot of the Perpendicular to each end of the Baſe ſhall be given alſo; for out of 13° and 16° of the preceding Reſolution it is manifeſt, that half the ſaid compos'd line together with half the Baſe, is equal to the diſtance from the foot of the Perpendicular to the remoter end of the Baſe; and half the ſaid compos'd line leſs by half the Baſe, is equal to the diſtance from the foot of the Perpendicular to the nearer end of the Baſe. Laſtly, the ſquare Root of the exceſs of the Square of the leſſer legg above the Square of the diſtance from the foot of the Perpendicular to the obtuſe angle, (or nearer end of the Baſe,) will give the Perpendicular, which multiplied into half the Baſe, gives the Area of the Triangle.

Here it will not be improper to add the following

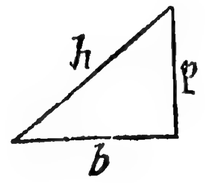
THEOR. 3.

33. If the difference of the Squares of the Hypothenuſal and Perpendicular of a right-angled Triangle be applied to the Baſe, the line ariſing ſhall be equal to the Baſe. Or, As the Baſe is to the ſumm of the Hypothenuſal and Perpendicular, ſo is their difference to the Baſe it ſelf.

This may be demonſtrated in manner following.

Suppoſ.

34. h = the Hypothenuſal of a right-angled Triangle.
35. p = the Perpendicular.
36. b = the Baſe.



Req. demonſtr.

37. $\frac{hh - pp}{b} = b$, that is,
38. $b \;.\; h + p \;::\; h - p \;.\; b$.

Demonſtration.

39. By 47. *prop.* 1. *Elem.* . . . . . $\} \; hh = bb + pp.$
40. Therefore, ... $\} \; hh - pp = bb.$
41. Therefore by Application of each part of the laſt Equation to b, ... $\} \; \frac{hh - pp}{b} = b.$

Which was to be dem.

42. Moreover, becauſe by *Theor.* 8. *Chap.* 4. . . $\} \; hh - pp = \square \begin{cases} h+p, \\ h-p. \end{cases}$
43. Therefore out of 40° and 42°, (*per Ax.* 1. *Chap.* 2.) . . . . . $\} \; bb = \square \begin{cases} h+p, \\ h-p. \end{cases}$
44. Therefore, from the laſt Equation this Analogy ariſeth, ... $\} \; b \;.\; h+p \;::\; h-p \;.\; b.$

Which was to be dem.

From *Theor.* 1. in 28° of the foregoing *Probl.* 9. and from *Theor.* 1, and 3. in 30° and 33° of the preceding *Probl.* 10. we may deduce the following

COROLLARY.

45. If the three ſides of any plane Triangle be ſeverally given in numbers, the kind of every one of the three angles is given alſo: For ſuppoſing any one of the three ſides to be called the Baſe, and the other two the leggs: Firſt, if the leggs be equal to one another, the angles at the Baſe are equal acute angles; ſecondly, if the leggs be unequal, and the Quotient that ariſeth by dividing the difference of the Squares of the leggs by the Baſe be leſs than the Baſe, then there will be unequal acute angles at the Baſe; thirdly, if the ſaid Quotient be greater than the Baſe, then that angle at the Baſe which is oppoſite to the greater legg ſhall be obtuſe, and conſequently the other two angles acute; and laſtly, if the ſaid Quotient be equal to the Baſe, then the angle at the Baſe oppoſite to the greater legg is a right angle.

Or more eaſily, thus;

If the Square of one of the three ſides of a plain Triangle be equal to the ſumm of the Squares of the other two, then the angle oppoſite to that ſide is a right angle; if greater, obtuſe; if leſſer, acute; as is manifeſt by *prop.* 48. *Elem.* 1. and by *prop.* 12, and 13. *Elem.* 2.

A Lemma,

A Lemma, *uſeful in the following* Probl. 11, *and* 12.

If within a plane Triangle a right line be drawn parallel to the Baſe, the Perpendicular falling upon the Baſe from the oppoſite angle, ſhall be to the Baſe, as that ſegment of the Perpendicular which is intercepted between the ſaid parallel line and the ſaid angle, *is* to the ſaid parallel line.

Suppoſ.

1. CEG is a △.
2. NM ∥ CE.
3. GL ⊥ CE.
4. . . *Req. demonſtr.* . . . . . LG . CE :: OG . NM.

Demonſtration.

5. By *Suppoſition* in 2°, ... } NM ∥ CE.
6. Therefore (*per* 29. *prop.* 1. *Elem.*) . . } ∠GNM = ∠GCE.
7. Likewiſe . . . . . } ∠GMN = ∠GEC.
8. Therefore, (*per Coroll. prop.* 32. *Elem.* 1.) } △CLG and △NOG are equiangular.
9. Likewiſe . . . . . } △ELG and △MOG are equiangular.
10. Therefore (*per prop.* 4. *Elem.* 6.) . . } LC . LG :: ON . OG.
11. Therefore alternately, ... } LC . ON :: LG . OG.
12. And by the like argumentation, . . . } LE . OM :: LG . OG.
13. Therefore out of 11° and 12°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } LC . ON :: LE . OM.
14. Therefore (*per* 12. *prop.* 5. *Elem.*) . . } LC . ON :: LC+LE . ON+OM.
15. But it hath above been proved in 11°, that } LC . ON :: LG . OG.
16. Therefore out of 14° and 15°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } LC+LE . ON+OM :: LG . OG.
17. That is, (viewing the *Fig.*) . . . . } CE : NM :: LG . OG.
18. Therefore alternly, ... } CE . LG :: NM . OG.
19. Therefore inverſly, ... } LG : CE :: OG . NM.

Which was to be Dem.

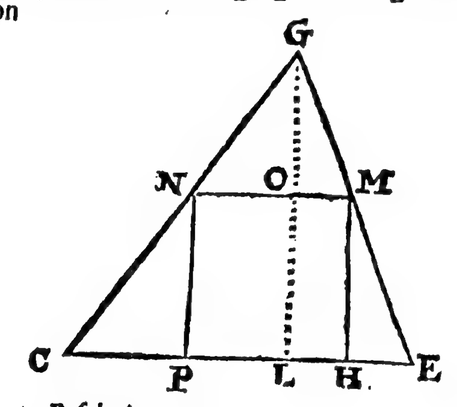
Probl. XI.

To inſcribe a Square in a Triangle given.

Note. If the given Triangle be obtuſangled, one of the ſides of the Square to be inſcribed muſt neceſſarily be a ſegment of that ſide of the Triangle which is oppoſite to the obtuſe angle, for otherwiſe, (as may eaſily be perceived) all the angular points of the Square cannot lye in the ſides of the Triangle; but if the Triangle given be right-angled or acute-angled, the Square may ſtand upon any one of the three ſides as a Baſe.

Suppoſ.

1. CEG is a △ given.
2. ∠GCE is not obtuſe.
3. ∠GEC is not obtuſe.
4. GL ⊥ CE.
5. b = CE the Baſe is given.
6. p = LG the Perpendic. is given.
7. . . *Req.* to inſcribe □ PNMH in the △ CEG.



CE = 182
CG = 195
EG = 169
GL = 156
CL = 117
LE = 65
NP = 84
NM = 84

Reſolution.

8. Suppoſing PNMH to be the Square required, put a to repreſent its ſide, *viz.* . . . . . } a = LO, or NM, or NP.
9. It is manifeſt by the foregoing *Lemma*, that . } LG . CE :: OG . NM.
10. That is, in the letters belonging to the *Reſolut.* } $p \;.\; b \;::\; p-a \;.\; a.$
11. Therefore by *Compoſition of Reaſon*, . . . } $p+b \;.\; b \;::\; p \;.\; a.$

Hence this

CANON.

12. As the ſumm of the Perpendicular and Baſe is to the Baſe, ſo is the Perpendicular to the ſide of the inſcribed Square ſought.

The Compoſition of Probl. 11.

Suppoſ.

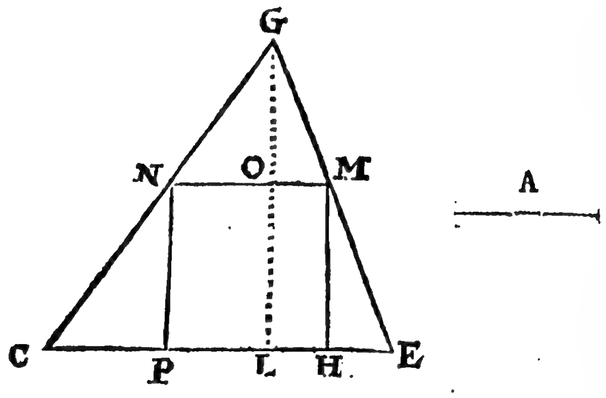
13. CEG is a △ given.

14. ∠ GCE is not obtuſe.

15. ∠ GEC is not obtuſe.

Req.

To inſcribe □ PNMH in △ CEG.



A ——

Conſtruction.

16. Firſt, make GL ⊥ CE.

17. Then making LG + CE the firſt of four Proportionals, CE the ſecond, and LG the third; find (*per Probl.* 8. *Chap.* 5.) a fourth Proportional, which ſuppoſe to be the line A, therefore,

LG + CE . CE :: LG . A.

18. From LG cut off LO = A, which is poſſible to be done, for in the Analogy in 17° the firſt Proportional is manifeſtly greater than the ſecond, therefore the third Proportional ſhall be greater than the fourth, (*per Schol. prop.* 14. *Elem.* 5.) that is, LG ⊏ A; therefore 'tis poſſible to cut off from LG a ſegment equal to A, as LO.

19. By the point O draw NOM ∥ CE.

20. Alſo draw NP and MH ∥ OL.

21. I ſay PNMH is the Square required.

22. . . *Req. demonſtr.* . . . . . : that PNMH is a □.

Demonſtration.

23. By *Conſtr.* in 17°, . . .	LG + CE . CE :: LG . A.
24. And by *Conſtr.* in 18°, . .	. . . . . LO = A.
25. Therefore out of 23° and 24°,	LG + CE . CE :: LG . LO.
That is, in 11°, ...	. *p* + *b* . *b* :: *p* . *a*.
26. Therefore from 25°, by *Diviſion of Reaſon*, ...	LG . CE :: LG − LO : LO.
That is, in 10°, ...	*p* . *b* :: *p* − *a* . *a*.
27. It is manifeſt by the Figure that	LG − LO = OG.
28. Therefore out of 26° and 27°,	LG . CE :: OG : LO.
29. By the *Lemma* prefixt to this Problem, ...	LG . CE :: OG : NM.
30. Therefore out of 28° and 29°, (*per* 11. *prop.* 5. *Elem.*) . .	OG . LO :: OG : NM.
31. Therefore (*per* 14. *prop.* 5. *El.*)	. . . LO = NM.
32. By *Conſtr.* in 19° and 20°, and by *deſin.* 35. *Elem.* 1. .	PNMH, PNOL and LOMH are Parallelograms.
33. Therefore (*per prop.* 34. *El.* 1.)	PN = MH = LO. Alſo, NM = PH.
34. And from 31° and 33°, . .	PN = NM = MH = PH.
35. And becauſe by *Conſtr.* in 16°,	∠GLC = ⌟ = ∠GLE.
36. Therefore from 32° and 35°, (*per prop.* 29, & 34. *Elem.* 1.)	∠PNM = ⌟ = ∠NMH = ∠MHP = ∠HPN.
37. Therefore from 34° and 36°, (*per deſin.* 29. *Elem.* 1.) . .	PNMH is a □.

Which was to be Dem.

Probl. XII.

Within a given Triangle to inſcribe a long Square, whoſe ſides ſhall be in a given Reaſon.

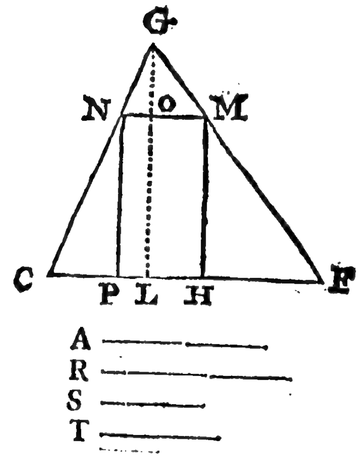
Suppoſ.

1. CEG is a Triangle given.
2. ∠ GCE is not obtuſe.
3. ∠ GEC is not obtuſe.
4. GL ⊥ CE.
5. *b* = CE the Baſe is given.
6. *p* = LG the Perpendicular is given.
7. *r* and *s* are the Terms of a given Reaſon.

Req. to inſcribe

8. PNMH a □ within the △ CEG, and ſuch, that

9. PN . NM :: R . S.



A ——
R ——
S ——
T ——

CE = 140
CG = 130
EG = 150
GL = 120
LE = 90
LC = 50
PN = 84
NM = 42
R = 2
S = 1
T = 60
A = 84

Reſolution.

10. Suppoſing PNMH to be the Rectangle ſought, put	*a* = PN, or LO.
11. Therefore out of 6°, and 10°, ...	*p* − *a* = OG.
12. It is manifeſt by the *Lemma* prefixt to the foregoing *Probl.* 11. that	LG . CE :: OG : NM.
13. That is, in the letters belonging to the Reſolution, ...	*p* . *b* :: *p* − *a* . $\frac{bp-ba}{p}$.
14. It is required in 9°, that	R : S :: PN . NM.
15. That is, ...	*r* . *s* :: *a* : $\frac{bp-ba}{p}$.
16. Therefore the two latter terms of the laſt Analogy being drawn into *p*, this Analogy will ariſe, . .	*r* . *s* :: *pa* . *bp* − *ba*.
17. Now to avoid an Equation between Solids, (for 'tis improper to introduce Solids in arguing about a plane Problem,) let it be made as *r* to *s*, ſo *p* to a fourth Proportional, which ſuppoſe to be *t*, therefore . .	*r* : *s* :: *p* : *t*.
18. Therefore from the Analogies in 16° and 17°, this will ariſe, (*per* 11. *prop.* 5. *Elem.*) . . .	*p* . *t* :: *pa* : *bp* − *ba*.
19. And by drawing *a* as a common altitude into *p* and *t* ſeverally, this Analogy will be manifeſt, (*per* 1. *prop.* 6. *Elem.*)	*p* . *t* :: *pa* : *ta*.
20. Therefore from 18° and 19° this Analogy will ariſe, (*per* 11. *prop.* 5. *Elem.*)	*pa* . *bp* − *ba* :: *pa* : *ta*.
21. Therefore out of 20°, (*per* 14. *prop.* 5. *Elem.*)	*bp* − *ba* = *ta*.
22. Therefore out of 21°, by addition of *ba* to each part,	*bp* = *ba* + *ta*.
23. Which laſt Equation gives this following Analogy, (*per* 14. *prop.* 6. *Elem.*)	*b* + *t* . *b* :: *p* : *a*.

Out of 17° and 23° ariſeth this

CANON.

24. Let it be made as R to S, ſo the Perpendicular GL to a fourth Proportional T; alſo let it be made as the ſumm of the Baſe CE and the fourth Proportional T to the Baſe CE; ſo the Perpendicular GL to a fourth Proportional A, which ſhall be equal to the altitude PN of the Rectangle required to be inſcribed. Laſtly, as R to S, ſo A to NM, to wit, that ſide of the Rectangle which muſt be parallel to the Baſe CE.

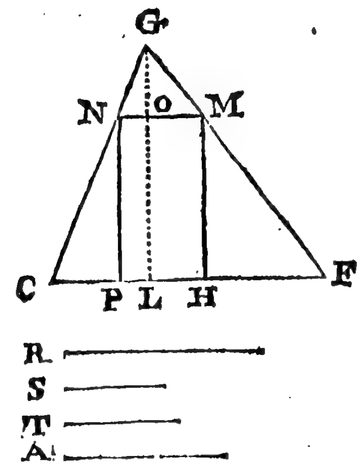
The Composition of Probl. 12.

Suppos.

25. CEG is a △ given.
26. < GCE is not obtuse.
27. < GEC is not obtuse.
28. R and S are two right lines given.

Req. to inscribe

29. PNMH a □ within the △ CEG, and such, that
30. PN . NM :: R . S.



CE = 140
CG = 130
EG = 150
GL = 120
LE = 90
LC = 50
PN = 84
NM = 42
R = 2
S = 1
T = 60
A = 84

Construction.

31. First, make GL ⊥ CE.
32. Then making R the first of four Proportionals, S the second, and GL the third, find a fourth Proportional, which suppose to be T, therefore
R . S :: GL . T.
33. Again, making CE + T the first of four Proportionals, CE the second, and GL the third, find a fourth Proportional, let it be A, therefore
CE + T . CE :: GL . A.
34. From GL cut off LO = A, which is possible to be done; for in the Analogy in 33° the first Proportional is manifestly greater than the second, therefore (*per Schol.* of 14. *prop.* 5. *Elem.*) the third shall be greater than the fourth, therefore GL ⊐ A.
35. By the point O draw NOM ‖ CE, also by the points N and M draw NP and MH ‖ OL.
36. I say PNMH is the right-angled Parallelogram required to be inscribed. Now we must shew that PN is to NM as R to S; also that the angles NPH, MHP, NMH and PNM are right angles. First, that the side PN is to the side NM, as the line R to the line S, I shall make manifest by a retrograde repetition of the steps of the foregoing Resolution of *Probl.* 12.
37. . . *Req. demonstr.* . . . . . R . S :: PN . NM.

Demonstration.

38. Because by *Construction* in 33°, . . . } CE + T . CE :: GL . A.
That is, in 23°, (the last step of the *Resolut.*) } . *b* + *t* . *b* :: *p* . *a*.
39. And by *Constr.* in 34°, ... } . . . . LO = A.
40. Therefore out of 38° and 39°, . . . } CE + T . CE :: GL . LO.
41. Therefore (*per* 16. *prop.* 6. *Elem.*) . } □GE,GL = □CE,LO + □T,LO.
That is, in 22°, ... } . . *bp* = *ba* + *ta*.
42. Therefore out of 41°, by equal subtraction of □ CE, LO, ... } □CE,GL − □CE,LO = □T,LO.
That is, in 21°, ... } . . *bp* − *ba* = *ta*.
43. From 42°, (*per* 7. *prop.* 5. *Elem.*) this subsequent Analogy will be manifest,
□GL,LO . □CE,GL − □CE,LO :: □GL,LO . □T,LO.
That is, in 20°, *pa* . *bp* − *ba* :: *pa* . *ta*.
44. And by reason of the common altitude LO this Analogy is manifest, (*per* 1. *prop.* 6. *El.*)
GL . T :: □GL,LO . □T,LO.
That is, in 19°, *p* . *t* :: *pa* . *ta*.
45. Therefore from the Analogies in 43° and 44°, (*per* 11. *prop.* 5. *Elem.*)
GL . T :: □GL,LO . □CE,GL − □CE,LO.
That is, in 18°, *p* . *t* :: *pa* . *bp* − *ba*.
46. By *Constr.* in 32°, ... GL . T :: R . S.
That is, in 17°, ... *p* . *t* :: *r* . *s*.

47. There-

47. Therefore from 45° and 46°, (*per* 11. *prop.* 5. *Elem.*)
R . S :: □GL,LO . □CE,GL − □CE,LO.
That is, in 16°, *r* . *s* :: *pa* . *bp* − *ba*.
48. It is manifest by the Figure, that . . } GL − LO = OG.
49. Therefore by taking CE as a common altitude, ... } □CE,GL − □CE,LO = □CE,OG.
50. Therefore out of 47° and 49°, by exchanging equal Rectangles, this Analogy is manifest, ... } R . S :: □GL,LO . □CE,OG.
51. By the *Lemma* placed next before *Probl.* 11. it's manifest that . . . . } GL . CE :: OG . NM.
52. And consequently, (*per* 16 *prop.* 6. *El.*) } □GL,NM = □CE,OG.
53. Therefore out of 50° and 52, by exchanging equal Rectangles, ... } R . S :: □GL;LO . □GL,NM.
54. Therefore, the common altitude GL being cast away, ... } R . S :: LO . NM.
55. And because by *Constr.* in 35°, (and by 35. *defin.* 1. *Elem.*) PNOL is a Parallelogram, therefore (*per* 34. *prop.* 1. *Elem.*) } PN = LO.
56. Therefore out of 54° and 55°, by setting PN in the place of LO, ... } R . S :: PN . NM.
Which was to be Demonstr.

But that the angles of the inscribed Parallelogram PNMH are right angles, is easie to prove out of the Construction, after the same manner as in the preceding *Probl.* 11. from 31° to 37°.

Probl. XIII.

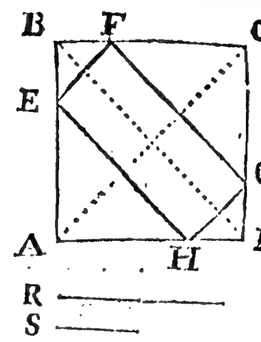
Within a given Square to inscribe a long Square, whose sides shall be in a given Reason, and parallel to the Diameters of the Square.

Suppos.

1. ABCD is a □ given.
2. *b* = BC (or BA) is given.
3. *r* and *s* are the Terms of a given Reason.

Req. to inscribe

4. EFGH a □ within the □ ABCD, so, as that the points E, F, G and H may lye in the sides of the □ABCD; also, that



R = 2
S = 5
AB = 28 = BC
BF = 8 = BE
DG = 8 = DH
FC = 20 = CG
AE = 20 = AH
EF = 4√8 = HG
FG = 10√8 = EH

5. EF ‖ AC ‖ HG; likewise, FG ‖ BD ‖ EH. Also;
6. EF . FG :: *r* . *s*.

Preparat.

7. Suppose EFGH to be the □ required, then according to the tenour of the Problem, . . } . . . EF ‖ AC.
8. Therefore (*per* 29. *prop.* 1. *Elem.*) . . . } < BEF = < BAC.
9. Likewise, ... } < BFE = < BCA.
10. Therefore equiangular are . . . . . } △ BEF and △ BAC.
11. Therefore (*per* 4. *prop.* 6. *Elem.*) . . . } BA . BC :: BE . BF.
12. Therefore alternately, ... } BA . BE :: BC . BF.
13. But by *Supposition* in 1°, ... } . . . BA = BC.
14. Therefore (*per* 14. *prop.* 5. *Elem.*) . . . } . . . BE = BF.
15. And because by *Suppos.* in 1°, ... } < EBF = ⌟.
16. Therefore out of 14° and 15°, (*per* 47. *prop.* 1. *Elem.*) . . . . . } . . 2□ BF = □ EF.
17. And by the like argumentation, ... } . . 2□ FC = □ FG.

Resolu-

Reſolution.

18. Suppoſe . . . . . ⟩ a = BF, or BE.
19. Therefore out of 2° and 18°, ... ⟩ $b - a$ = FC, or CG.
20. The Square of the ſegment in 18° is . . . ⟩ aa = □ BF.
21. The Square of the ſegment in 19° is . . . ⟩ $bb - 2ba + aa$ = □ FC.
22. Therefore out of 20° and 16°, ... ⟩ $2aa$ = □ EF.
23. Alſo out of 21° and 17°, ... ⟩ $2bb - 4ba + 2aa$ = □ FG.
24. Therefore out of 22°, ... ⟩ $\sqrt{2aa}$ = EF.
25. And out of 23°, ... ⟩ $\sqrt{2bb - 4ba + 2aa}$ = FG.
26. Therefore according to the condition in the Problem,
$r . s :: \sqrt{2aa} . \sqrt{2bb - 4ba + 2aa}.$
27. But their Squares alſo are Proportionals, therefore
$rr . ss :: 2aa . 2bb - 4ba + 2aa.$
28. And becauſe the two latter Terms of the laſt Analogy are in the ſame Reaſon as their halves, therefore
$rr . ss :: aa . bb - 2ba + aa.$
29. But the Roots of the four laſt Proportionals are alſo Proportionals, therefore . . . . . ⟩ $r . s :: a . b - a.$
30. Therefore inverſly, ... ⟩ $s . r :: b - a . a.$
31. Therefore by *Compoſition of Reaſon*, . . . ⟩ $s + r . r :: b . a$ (BF.)
32. Alſo out of 29°, by *Compoſ. of Reaſon*, . . ⟩ $r + s . s :: b . FC.$

From the premiſſes ariſeth this

CANON.

33. By *Probl.* 1. of this *Chapt.* divide the ſide of the given Square into two ſegments that may be one to the other in the given Reaſon; then find out the ſide of a Square equal to the double Square of the greater ſegment, alſo the ſide of a Square equal to the double Square of the leſſer ſegment; ſo ſhall thoſe ſides be the length and bredth of the deſired Rectangle, (or long Square,) which may be inſcribed within the given Square by this following

Conſtruction.

34. Firſt divide BC into two parts in F, that may be one to the other as R to S, (*per Probl.* 1. of this *Chapt.*) *viz.* ſuppoſe it be made as . . } R . S :: BF . FC.
35. Then let BE = BF; alſo DG = DH = BF, and draw the right lines EF, FG, GH and HE, ſo ſhall EFGH be the long Square given to be inſcribed: Now we muſt ſhew that it will ſatisfie the Problem propounded; firſt, that EF is to FG as R to S, I ſhall demonſtrate by a retrograde repetition of the ſteps of the preceding Reſolution.
36. . . *Req. demonſtr.* . . . . . R . S :: EF . FG.

Demonſtration.

37. Becauſe by *Conſtr.* in 34°, ... ⟩ R . S :: BF . FC.
38. Therefore (*per prop.* 22. *Elem.* 6.) . . ⟩ □R . □S :: □ BF . □ FC.
39. And by taking the doubles of the two latter Terms of the laſt preceding Analogy, . . } □R . □S :: 2□ BF . 2□ FC.
40. By *Conſtr.* in 35°, ... ⟩ . . BE = BF.
41. And by *Suppoſ.* . . . . . ⟩ ∠EBF = ⌟.
42. Therefore from 40° and 41°, (*per prop.* 47. *Elem.* 1.) . . . . . } . □ EF = 2□ BF.
43. Likewiſe, ... ⟩ . □ FG = 2□ FC.
44. Therefore out of 39°, 42° and 43°, by exchange of equal quantities, ... } □R . □S :: □ EF : □ FG.
45. Therefore from 44°, (*per prop.* 22. *Elem.* 6.) ⟩ R . S :: EF : FG.

Which was to be Dem.

It remains to demonſtrate that the quadrilateral Figure EFGH is right-angled at E, F, G and H.

46. . . *Req. demonſtr.* . ∠EFG = ⌟ = ∠FGH = ∠GHE = ∠HEF.

Demon-

Demonſtration.

47. Becauſe by *Conſtr.* in 35°, ... ⟩ . . BF = BE.
48. And by *Suppoſ.* . . . . . ⟩ ∠EBF = ⌟.
49. Therefore (*per Coroll. prop.* 32. *Elem.* 1.) . . ⟩ ∠BFE = ½ ⌟.
50. Likewiſe . . . . . ⟩ ∠CFG = ½ ⌟.
51. Therefore from 49° and 50°, (*per Coroll. prop.* 13. *Elem.* 1.) . . . . . } ∠EFG = ⌟.
52. And by the like argumentation, ... ⟩ ∠FGH, ∠GHE, ∠HEF are ⌟.

Which was to be Dem. Therefore the Problem is ſatisfied.

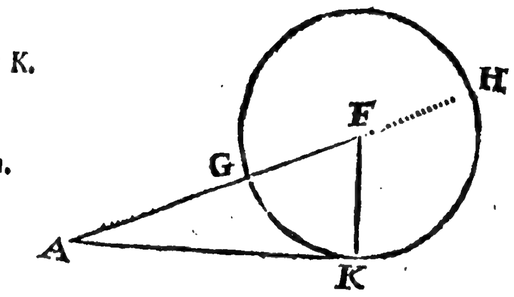
Probl. XIV.

In a right-angled Triangle, one of the ſides about the right angle being given, as alſo the ſumm of the other ſide about the right angle and the Hypothenuſal, to find out the Triangle. But the ſaid ſumm muſt be greater than the given ſide, as is manifeſt by *prop.* 22. *Elem.* 1.

Suppoſ.

1. △ AKF is right-angled at K.
2. b = AK is given.
3. c = AF + FK is given.
4. $c \sqsubset b$.

Req. to find △ AKF.



AK = 12
KF = 5
FA = 13

Reſolution.

5. For the difference between the Hypothenuſal AF and the Perpendicular FK, put . . . . . } a.
6. Then from 3° and 5°, (*per Theor.* 9. *Chap.* 4.) the Hypothenuſal ſhall be . . . . . } $\frac{1}{2}c + \frac{1}{2}a$.
7. And by the ſame Theorem, the Perpendicular KF ſhall be . ⟩ $\frac{1}{2}c - \frac{1}{2}a$.
8. Therefore from 6° the Square of the Hypothenuſal ſhall be . ⟩ $\frac{1}{4}cc + \frac{1}{4}aa + \frac{1}{2}ca$.
9. And from 7° the Square of the Perpendicular is . . . . ⟩ $\frac{1}{4}cc + \frac{1}{4}aa - \frac{1}{2}ca$.
10. The difference of thoſe Squares is . . . . . ⟩ ca.
11. But (*per prop.* 47. *Elem.* 1.) the difference of the Square of the Hypothenuſal and the Square of one of the ſides about the right angle is equal to the Square of the other ſide about the right angle, therefore from 10° and 2°, ... } $ca = bb$.
12. Therefore, by reſolving the laſt preceding Equation into Proportionals, ... } $c . b :: b . a$.
13. Therefore out of 12° and 6°, the Hypothenuſal is now given, *viz.* . . . . . } $\frac{1}{2}c + \frac{bb}{2c} = AF$.
14. And from 12° and 7°, the deſired ſide about the right angle is alſo given, *viz.* . . . . . } $\frac{1}{2}c - \frac{bb}{2c} = FK$.

Out of 12° ariſeth this

THEOREM.

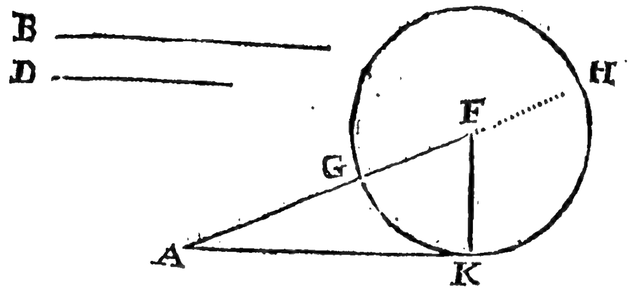
15. One of the ſides about the right angle of a right-angled Triangle is a mean Proportional between the ſumm and difference of the Hypothenuſal and the other ſide about the right-angle. And conſequently (*per prop.* 17. *Elem.* 1.) the Rectangle of the ſaid ſumm and difference is equal to the Square of the firſt mentioned ſide, or mean Proportional.
16. Therefore, the Baſe of a right-angled Triangle, (that is, which of the two ſides about the right angle you pleaſe,) being given, as alſo the ſumm of the Hypothenuſal and Perpendicular, the difference of the Hypothenuſal and Perpendicular ſhall be given alſo. For by the Theorem above expreſt, (which is the ſame with *Prop.* 36. *Elem.* 3. when

the

the line there mentioned to cut the Circle paſſeth through the Center,) the ſaid difference is a third Proportional proceeding from the ſaid Baſe and Summ.

Therefore alſo, the ſaid Hypothenuſal and Perpendicular ſhall be given ſeverally, (*per Theor. 9. Chap. 4.*) for they ſhall be equal to the given Quantities before expreſt in 13° and 14° of this Problem.

The Compoſition of Probl. 14.



AK = 12 = B
KF = 5 = FH
FA = 13
AH = 18
D = 8 = AG

Suppoſ.

17. B = the Baſe of a right-angled Triangle is given.
18. AH = the ſumm of the Hypothenuſal and Perpendicular is given.
19. AH ⊏ B.

Req. to find out the △.

Conſtruction.

20. Let it be made as AH to B, ſo B to a third Proportional, call it D, therefore . . . } AH . B :: B . D.
21. From AH cut off AG = D, which is poſſible to be done if AH ⊏ D; but by *Suppoſition* AH ⊏ B, therefore from the Analogy in 20°, AH ⊏ D.
22. Divide GH into two equal parts in F, therefore FG = FH.
23. From F as a Center, at the diſtance of FG or FH deſcribe the Circle FGKH, and (*per prop. 17. Elem. 3.*) draw the right line AK to touch the ſaid Circle in K, draw alſo the Semidiameter FK; then ſhall AKF be the Triangle ſought. Now we muſt ſhew that it will ſatisfie the Problem.
24. Firſt then, by *Conſtruction* in 23°, FH = FK, therefore AF + FK = AH the given ſumm of the Hypothenuſal and Perpendicular. It remains to ſhew that AK is equal to the given Baſe B, and that the angle AKF is a right angle; therefore,
25. . . *Req. demonſtr.* . . . . . AK = B. Alſo, ∠ AKF = ⊥.

Demonſtration.

26. Becauſe by *Conſtr.* in 20° and 21°, . . } AH . B :: B . D (AG.)
27. Therefore, (*per prop. 17. Elem. 6.*) . . } □ AH, AG = □ B.
28. And becauſe by *Conſtr.* in 23°, AK toucheth ⊙FGKH in K, therefore (*per prop. 36. Elem. 3.*) } □ AH, AG = □ AK.
29. Therefore out of 27° and 28°, (*per Ax. 1. Chap. 2.*) . . . . . } □ AK = □ B.
30. Therefore, (*per Schol. prop. 48. Elem. 1.*) } . AK = B. Which was to be dem.
31. And becauſe by *Conſtr.* in 23°, AK toucheth ⊙ FGKH in K, and FK is Radius, therefore, (*per prop. 18. Elem. 3.*) . . . . . } FK ⊥ KA. Therefore, ∠AKF = ⊥.

Which was alſo to be Dem. Therefore the Problem is ſatisfied.

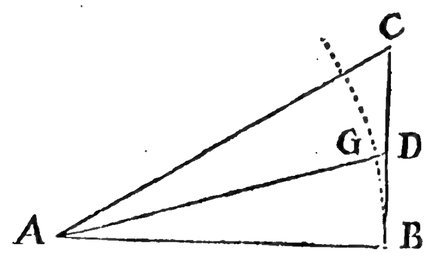
Probl. XV.

The Radius (or Semidiameter) of a Circle being given, as alſo the Tangent of an arch, to find the Tangent of the double of that arch. But the given Tangent muſt be leſs than the Radius.

Suppoſ.

Suppoſ.

1. △ ABC is right-angled at B.
2. ∠ BAD = ∠ DAC = ½ ∠ CAB.
3. *r* = AB the Radius of a Circle, whoſe Center is A, is given.
4. *t* = BD the Tangent of the arch BG, or angle DAB, is given.
5. BD ⊐ AB.



Req. to find BC the Tangent of the angle CAB, that is, the Tangent of 2 ∠DAB.

Reſolution.

6. Put . . . . . } *a* = BC.
7. Therefore out of 4° and 6°, ... } *a* − *t* = DC.
8. And out of 1°, 3° and 6°, (*per 47. prop. 1. Elem.*) . . . . . } $\sqrt{:rr+aa:}$ = AC.
9. And becauſe by *Suppoſition* ∠ BAD = ∠ DAC, therefore (*per 3. prop. 6. Elem.*) } DC . BD :: AC . AB.
10. That is, in the letters of the Reſolution, } . *a* − *t* . *t* :: $\sqrt{:rr+aa:}$. *r*.
11. But their Squares alſo ſhall be Proportionals, (*per 22. prop. 6. Elem.*) *viz.* . . } *aa* − 2*ta* + *tt* . *tt* :: *rr* + *aa* . *rr*.
12. Therefore from the Analogy in the 11th ſtep, by *Diviſion of Reaſon*, ... } *aa* − 2*ta* . *tt* :: *aa* . *rr*.
13. Therefore alternly, ... } *aa* − 2*ta* . *aa* :: *tt* . *rr*.
14. And by reaſon of the common altitude *a*, in the two firſt Terms of this Analogy, it is manifeſt (*per 1. prop. 6. Elem.*) that . . } *aa* − 2*ta* . *aa* :: *a* − 2*t* . *a*.
15. Therefore from the two laſt preceding Analogies, (*per 11. prop. 5. Elem.*) . . } . *a* − 2*t* . *a* :: *tt* . *rr*.
16. Therefore inverſly, ... } . . *rr* . *tt* :: *a* . *a* − 2*t*.
17. And converſly, ... } . . *rr* . *rr* − *tt* :: *a* . 2*t*.
18. And by doubling the Antecedents, (*per Schol. Clavii in prop. 22. Elem. 5.*) . . } . 2*rr* . *rr* − *tt* :: 2*a* . 2*t*.
19. And inverſly, ... } *rr* − *tt* . 2*rr* :: 2*t* . 2*a*.
20. But (*per 15. prop. 5. Elem.*) . . . . } . *t* . *a* :: 2*t* . 2*a*.
21. Therefore out of 19° and 20°, (*per 11. prop. 5. Elem.*) . . . . . } *rr* − *tt* . 2*rr* :: *t* . *a*.

Hence this

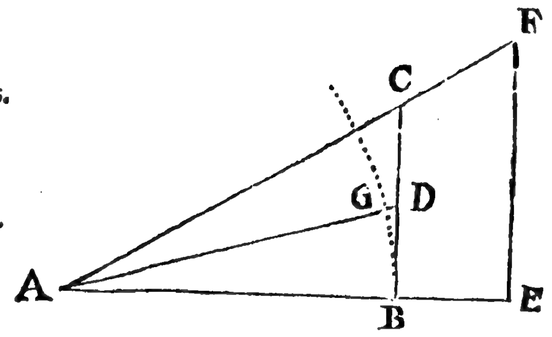
THEOREM.

22. As the exceſs whereby the ſquare of the Radius exceeds the ſquare of the Tangent of an arch leſs than half a Quadrant, is to the double ſquare of the Radius; ſo is the ſaid Tangent, to the Tangent of the double of that arch.

Therefore the Radius being given, as alſo the Tangent of an arch leſs than 45. degrees, the Tangent of the double of that arch is given alſo, without extracting any Root.

This Theorem was found out by the Learned Dr. *John Pell*, who thereby hath clearly ſhew'd the falſity of *Longomontanus*'s pretended quadrature of the Circle, as appears by the Controverſie between them, (printed at *Amſterdam* in 1647.) where the Demonſtrations of divers eminent Mathematicians to manifeſt the truth of the Theorem are inſerted, which may alſo be eaſily demonſtrated by the ſteps of the preceding Reſolution, by proceeding in a direct order from the beginning to the end thereof; but leaving that Demonſtration to the Learner's practice, I ſhall give another which here follows.

Suppoſ.



23. ∠ABC and ∠AEF are right angles.
24. ACF and ABE are right lines.
25. ∠ DAB leſs than half a right angle.
26. ∠ DAB = ∠ DAC = ½ ∠ FAE.
27. . . *Req. demonſtr.* . . . . □AB − □BD . 2□AB :: BD . BC.

Demonſtration.

28. By *Suppoſition*, ... } ∠ ABC = ⌟ = ∠ AEF.
29. By *Suppoſ.* alſo, ... } ∠ FAE = 2 ∠ DAB.
30. Therefore, (*per Theor.* 2. *angular. ſect.* Viet.) . . . . . } AE . EF :: □AB − □BD . 2□AB,BD.
31. And becauſe from 23° and 24° △ AEF and △ ABC are like, it's manifeſt (*per* 4. *prop.* 6. *Elem.*) that } AE . EF :: AB . BC.
32. Therefore from the two laſt preceding Analogies, (*per* 11. *prop.* 5. *Elem.*) } □AB − □BD . 2□AB,BD :: AB . BC.
33. But by reaſon of the common altitude 2 AB, this Analogy is manifeſt, (*per* 1. *prop.* 6. *Elem.*) *viz.* . . . . } 2□AB,BD . 2□AB :: BD . AB.
34. Wherefore from the two laſt preceding Analogies, (*per* 23. *prop.* 5. *Elem.* agreeable to *Definit.* 8. of Inordinate proportion in the foregoing *Chap.* 3.) } □AB − □BD . 2□AB :: BD . BC.

Which was to be dem.

Illuſtration by Numbers.

35. Let the Radius AB be . . . . . } . . . . . 100000.
36. Then ſuppoſing the arch BG or angle DAB to be 15. degrees, its Tangent BD (by the Trigonometrical Canon) is } . . . . . 26795.
37. Now to find out BC, the Tangent of 30. degrees, (the double of the given angle,) according to the preceding Theorem, multiply the given Tangent 26795, by the double Square of the Radius, *viz.* by . . . . . } . . 20000000000.
38. The Product is to be reſerved for a Dividend, to wit, . } 535900000000000.
39. Then from the Square of the Radius, *viz.* from . . . } . . . 10000000000.
40. Subtract the Square of the given Tangent 26795, *viz.* . } . . . 717972025.
41. The remainder is a Diviſor, to wit, ... } . . . 9282027975.
42. Laſtly, divide the Dividend in the 38th ſtep, by the Diviſor in the 41th, and the Quotient, neglecting the remainder, is } . . . . . 57735.

Which 57735 is the deſired Tangent of 30. degrees, the double of the given angle or arch 15. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being aſſumed 100000.

Probl. XVI.

The Radius of a Circle, and the Tangents of two arches being given ſeverally, to find the Tangents of the ſumm and difference of thoſe arches. But when the Tangent of the ſumm is required, the Rectangle of the given Tangents muſt be leſs than the Square of the Radius.

In finding out the Demonſtration of the Theorem in the 22th ſtep of the foregoing *Probl.* 15. I diſcovered the two following Theorems, to ſolve this *Probl.* 16. which I have not met with in any Author.

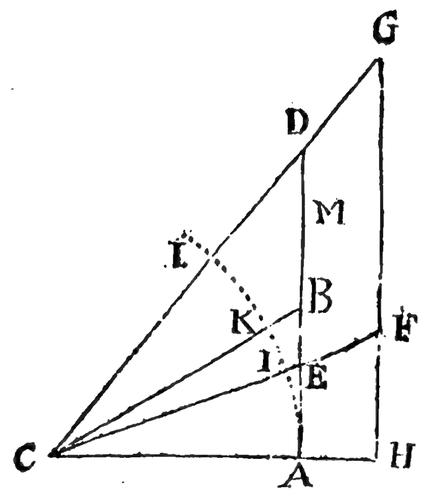
THEOR. 1.

THEOR. 1.

As the exceſs whereby the Square of the Radius exceeds the Rectangle of the Tangents of two arches, is to the Square of the Radius; ſo is the ſumm of thoſe Tangents, to the Tangent of the ſumm of the ſaid arches.

Therefore the Radius of a Circle, and the Tangents of two arches whoſe ſumm is leſs than a Quadrant, being given ſeverally, the Tangent of the ſumm of thoſe arches ſhall be given alſo.

Suppoſ.



1. CA is the Radius of a Circle whoſe Center is C.
2. AL an arch leſs than a Quadrant.
3. Arch LK = arch AI, therefore
4. Arch AL = arch AK + arch AI.
5. AD ⊥ CA.
6. AB the Tangent of the arch AK, or ∠ BCA.
7. AE the Tangent of the arch AI, or ∠ ECA.
8. AD the Tangent of the arch AL, or of ∠ BCA + ∠ ECA.
9. CAH and CDG are right lines.
10. HG ⊥ CAH.
11. BM = AE, therefore AM = BA + AE.

Req. demonſtr.

12. □CA − □BA,AE . □CA :: AM . AD. (BA + AE)

Demonſtration.

13. Becauſe by *Suppoſition*, ... } ∠ CAD = ⌟ = ∠ CHG.
14. Alſo by *Suppoſ.* in the right-angled Triangles GCH, BCA and ECA, ... } ∠ GCH = ∠ BCA + ∠ ECA.
15. Therefore, (*per Theor.* 2. *Angular. ſection.* Vi.) CH . HG :: □CA − □BA,AE . □AM,CA.
16. And becauſe by *Suppoſ.* in 5°, 9° and 10°, △ CAD and △ CHG are equiangular, therefore, (*per* 4. *prop.* 6. *Elem.*) CH . HG :: CA . AD.
17. Therefore from the two laſt preceding Analogies, (*per* 11. *prop.* 5. *Elem.*) □CA − □BA,AE . □AM,CA :: CA . AD.
18. But by reaſon of the common altitude CA this following Analogy is manifeſt, (*per* 1 *prop.* 6. *Elem.*) □AM,CA . □CA :: AM . CA.
19. Therefore from the two laſt preceding Analogies, (*per* 23. *prop.* 5. *Elem.* agreeable to *Defin.* 8. of Inordinate proportion, in the foregoing *Chap.* 3.) □CA − □BA,AE . □CA :: AM . AD.

Which was to be dem.

Illuſtration by Numbers.

20. Let the Radius CA be . . . . . } . . . . . 100000.
21. Then ſuppoſing the arch AK to be 30. degrees, its Tangent AB by the Trigonometrical Canon is . . . . . } . . . . . 57735.
22. Likewiſe ſuppoſing the arch AI to be 20. degrees, its Tangent AE is . . . . . } . . . . . 36397.

Now to find out AD the Tangent of the arch AL, 50. degrees, *viz.* the Tangent of the ſumm of the given arches AK, 30. degrees, and AI, (or KL,) 20. degrees, the Operation, according to the preceding *Theor.* 1. will be as it here follows, *viz.*

23. Multiply the ſumm of the given Tangents 57735 and 36397, to wit, ... } . . . . . 94132.
24. By the Square of the Radius, *viz.* by . . . . . } . . 10000000000.
25. The Product ſhall be a Dividend, to wit, ... } 941320000000000.
26. Then from the ſaid Square of the Radius in the 24th ſtep, ſubtract the Product of the given Tangents, to wit, . . } . . . 210138 0795.

27. The remainder is a Diviſor, to wit, ... } . . . 7898619205.
28. Laſtly, divide the Dividend in the 25th ſtep, by the Diviſor in the 27th and the Quotient, (neglecting the remainder,) is } . . . . . 119175.

Which 119175 is the deſired Tangent of 50. degrees, the ſumm of the given arches 30, and 20. degrees, and exactly agrees with the Trigonometrical Canon, the Radius being aſſumed 100000.

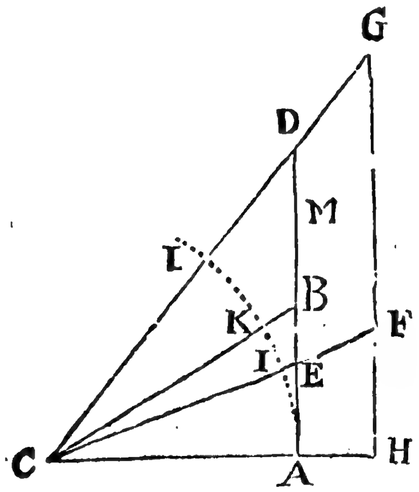
THEOR. 2.

29. As the ſumm of the Square of the Radius and the Rectangle of the Tangents of two arches is to the Square of the Radius; ſo is the difference of thoſe Tangents, to the Tangent of the difference of the ſaid arches.

Therefore, the Radius and the Tangents of two arches being ſeverally given, the Tangent of the difference of thoſe arches ſhall be given alſo.

Suppoſ.

30. CA is the Radius of a Circle whoſe Center is C
31. AL is an arch leſs than a Quadrant.
32. Arch AI = arch LK, therefore
33. Arch AI = arch AL — arch AK.
34. AD ⊥ CA.
35. AD the Tangent of the arch AL or ∠DCA.
36. AB the Tangent of the arch AK or ∠BCA.
37. AE the Tangent of the arch AI, *viz.* of ∠DCA — ∠BCA.
38. CAH and CEF are right lines.
39. HF ⊥ CAH.



Req. demonſtr.

40. $\square CA + \square AD, AB \;.\; \square CA :: BD \;.\; \frac{DA \cdot AB}{AE}$.

Demonſtration.

41. Becauſe by *Suppoſition* . . . . . } ∠CAD = ⊥ = ∠CHG.
42. Alſo by *Suppoſ.* in the right-angled Triangles FCH, DCA, BCA, ... } ∠FCH = ∠DCA − ∠BCA.
43. Therefore, (*per Theor.* 1. *Angular. ſection.* Viet.)
CH . HF :: □CA + □AD,AB . □BD,CA; (that is, $\overline{DA - AB} \times CA$.)
44. And becauſe by *Suppoſ.* in 34°, 38° and 39°, △ CAE and △ CHF are equiangular, therefore, (*per* 4. *prop.* 6. *Elem.*)
CH . HF :: CA . AE.
45. Therefore from the two laſt preceding Analogies, (*per* 11. *prop.* 5. *Elem.*)
□CA + □AD, AB . □BD, CA :: CA . AE.
46. But by reaſon of the common altitude CA, this following Analogy is manifeſt, (*per* 1. *prop.* 6. *Elem.*)
□BD, CA . □CA :: BD . CA.
47. Therefore from the two laſt preceding Analogies, (*per* 23. *prop.* 5. *Elem.* agreeable to *Defin.* 8. of Inordinate proportion, in the foregoing *Chap.* 3.)
$\square CA + \square AD, AB \;.\; \square CA :: BD \;.\; \frac{DA \cdot AB}{AE}$.

Which was to be dem.

Illuſtration by Numbers.

48. Let the Radius CA be . . . . . } . . . . . 100000.
49. Then ſuppoſing the arch AL to be 50. degrees, its Tangent AD by the Trigonometrical Canon is . . . . . } . . . . . 119175.
50. Likewiſe, ſuppoſing the arch AK to be 30. degrees, its Tangent is . . . . . } . . . . . 57735.

Now to find out AE the Tangent of the arch AI, 20. degrees, *viz.* the Tangent of the difference of the given arches AL, 50. degrees, and AK, 30. degrees, the Operation, according to the preceding *Theor.* 2. will be as followeth, *viz.*

51. Multiply the difference of the given Tangents 119175 and 57735, to wit, ... } . . . . . 61440.

52. By

52. By the Square of the Radius, *viz.* by . . . . . } . . . 10000000000.
53. The Product ſhall be a Dividend, to wit, ... } 614400000000000.
54. Then to the ſaid Square of the Radius, add the Product of the given Tangents 119175 and 57735, *viz.* . . . } . . . 6880568625.
55. The ſumm is a Diviſor, to wit, ... } . . . 16880568625.
56. Laſtly, divide the Dividend in the 53th ſtep by the Diviſor in the 55th, and the Quotient, neglecting the remainder, will be } . . . . . 36396.

Which 36396 is the deſired Tangent of 20. degrees, the difference of the given arches 50, and 30. degrees, and nearly agrees with the Trigonometrical Canon, the Radius being aſſumed 100000. But for more exactneſs, the Radius and Tangents given ought to conſiſt of more places than thoſe before expreſt in 48°, 49° and 50°.

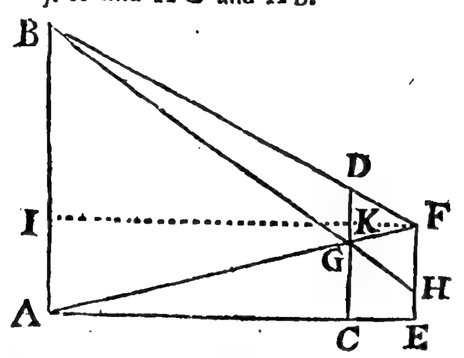
Probl. XVII.

To find the altitude of a Tower, when the foot thereof is inacceſſible.

Suppoſ.

1. EA = AC + CE is an horizontal line.
2. CA part of EA is acceſſible only at C.
3. AB the height of a Tower.
4. CD and EF two ſtaves perpendicular to EA, and at ſuch a diſtance one from the other, that FDB is a right line.
5. FA a right line cutting CD in G.
6. BH a right line cutting CD in G.
7. FI = and ∥ EA, and cuts CD in K.
8. b = DG,
9. d = CE, } are given ſeverally.
10. f = FH;

Req. to find AC and AB.



DG = 24
CE = 30
FH = 25
AC = 720
AB = 600

Preparat.

11. Becauſe in △ BFH, DG ∥ FH, therefore, △ BFH and △ BDG are like, therefore (*per prop.* 4. *Elem.* 6.) } BF . BD :: FH . DG.
12. Likewiſe becauſe in △ FBI, DK ∥ BI, therefore } BF . BD :: IF . IK.
13. Therefore out of 11° and 12°, (*per pro.* 11. *Ele* 5.) } FH . DG :: IF . IK.
14. That is, (becauſe AE = IF, and AC = IK,) } FH . DG :: AE . AC.

Reſolution.

15. Put, ... } a = AC.
16. Therefore from 9° and 15°, ... } $a + d$ = AE.
17. Then from the 8th, 10th, 14th, 15th and 16th ſteps this Analogy ariſeth, *viz.* . . . . . } $f \;.\; b :: a + d \;.\; a$.
18. Therefore by *Diviſion of Reaſon*, ... } $f - b \;.\; b :: d \;.\; a$.
19. Again, in order to find AB, add d, that is, CE, to $\frac{db}{f-b}$, (which by the 18th ſtep is manifeſtly equal to a,) that is, AC, and it makes . . } $\frac{fd}{f-b}$ = AE.
20. Then becauſe △ FBA and △ FDG are like, (for DG ∥ BA,) this Analogy will be manifeſt (by the *Lemma* prefixt to *Probl.* 11. of this *Chapt.*) viz. } FK . GD :: FI . AB.

21. That

21. That is, by exchange of equal right lines, } CE . GD :: AE . AB.
22. That is, in the letters belonging to the Resolution, ... } $d . b :: \frac{fd}{f-b} . \frac{fb}{f-b}$ (AB.)

From the 18th, 21th and 22th steps ariseth this

CANON.

23. As the excess whereby FH exceeds DG, is to DG, so is CE to AC; and so is FH to AB.

The Demonstration of this Canon being very easie to be deduced from the premisses, I shall only illustrate it by

An Example in Numbers.

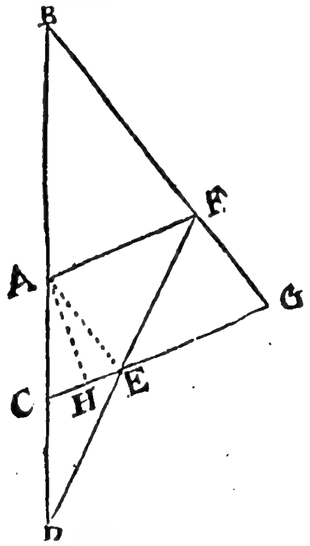
Suppose . . . { DG = b = 24, CE = d = 30, FH = f = 25 } Inches.

Then by the Canon,

I.	FH—DG	.	DG	::	CE	.	AC.
	1	.	24	::	30	:	720.
II.	FH—DG	.	DG	::	FH	.	AB.
	1	.	24	::	25	.	600.

Probl. XVIII.

To find the length of a right line AB, when there is access to one of its ends, suppose to A.



GC = 51
CA = 26
AF = 34 = GE
FG = 25 = AE
AH = 24
AB = 52
FB = 50
CE = 17
HE = 7
HC = 10

Preparat.

1. From the given point A measure two equal distances, as AC = CD, and so, as DCAB may be a right (or straight) line; measure likewise DE and EF, so as DE may be equal to EF and DEF a straght line, making an angle ADF at pleasure: then measure FA, which shall be parallel to a line drawn from C to E, (*per* 2. *prop.* 6. *Elem.*) for DC . CA :: DE . EF; then measure CEG, that is, the parallel CE produced to such a Station G, that GFB may be a straight line, measure also GF. Now of the Trapezium CAFG the two sides AF and CG being parallels, and every one of the four sides CA, AF, FG, GC, given by measure as above, in Yards or what equal parts you please, the length of AB, also of FB may be found out in the same kind of parts, by the following Resolution.

Suppos.

2. . . . . { CAB and GFB are right lines. AF ∥ CG. b = CG, c = CA, d = AF, f = FG } given severally.

Req. to find AB and FB.

Resolu-

Resolution.

3. Put . . . . . } a = AB.
4. Then because △ BCG and △ BAF are equiangular, for AF ∥ CG, this Analogy is manifest, (*per* 4. *prop.* 6. *Elem.*) . . . . . } CG . CB :: AF . AB.
5. That is, in the letters of the Resolution, . . } . b . c + a :: d . a.
6. Therefore alternately, ... } . b . d :: c + a . a.
7. Therefore by *Division of Reason*, ... } b − d . d :: c . a.
8. And by the like argumentation this Analogy will arise, ... } b − d . d :: f . FB.

The Analogies in the seventh and eighth steps afford this

THEOR. 1.

9. In a Trapezium CAFG having two parallel sides AF, CG; As the difference of those two parallel sides is to the lesser of them; so is each of the two other sides not parallels, to its continuation to the point of their concurrence, *viz.* As, CE . AF :: CA . AB :: FG . FB.

10. The Demonstration whereof is manifest; for drawing AE ∥ FG, the Triangles CAE and ABF shall be like; and because AE ∥ FG and AF ∥ EG, it follows (*per prop.* 34. *Elem.* 1.) that AE = FG, also AF = EG, and CE = CG — AF. Wherefore CE . AF :: CA . AB :: FG . FB, as the Theorem affirms.

An Example in Numbers.

11. Suppose . . . . . { AF ∥ CG in the Trapezium CAFG; also, CG = 51. CA = 26. AF = 34 = GE. FG = 25 = EA. CE = 17 = CG — AF.

12. Then by the Theorem before expresst in 9°,

$$\underset{17}{CE} . (CG - \underset{34}{AF} .) \underset{26}{AF} :: \begin{cases} CA . AB. \\ 26 \quad 52 \\ GF . FB. \\ 25 \quad 50 \end{cases}$$

Here by the way, the Learner may observe, that the four sides of a Trapezium having two parallel sides being severally given, the Area of that Trapezium shall be given also by this following

THEOR. 2.

13. If a Trapezium hath two parallel sides, the Rectangle or Product made by the multiplication of half the summ of the parallel sides, into the shortest distance between the said Parallels, shall be equal to the Area of the Trapezium.

14. . . *Suppos.* . . . . { AF ∥ CG in the Trapezium CAFG. CG ⊏ AF. AC not ∥ FG.

Prepar.

15. Make AH ⊥ CG; also GE = AF, and draw AE.

16. . . *Req. demonstr.* . . { $AH \times \overline{\frac{1}{2}CG + \frac{1}{2}AF} = $ Trapez. CAFG. That is, $AH \times \overline{EG + \frac{1}{2}CE} = $ Trapez. CAFG.

Demonstration.

17. By *Supposition* in 14°, ... } AF ∥ CG.
18. And by *Constr.* in 15°, ... } GE = AF and AH ⊥ HG.
19. Therefore out of 17° and 18°, (*per* 33, & 35. *prop.* 1. *Elem.*) . . . . . } AH × EG = Parallelogr. EAFG.
20. And (*per Schol.* 41. *prop.* 1. *Elem.*) . . . } $AH \times \frac{1}{2}CE = \triangle CAE$.
21. Therefore from the two last Equations, $AH \times \overline{EG + \frac{1}{2}CE} = EAFG + \triangle CAE =$ Trapez. CAFG.

Which was to be Demonstr.

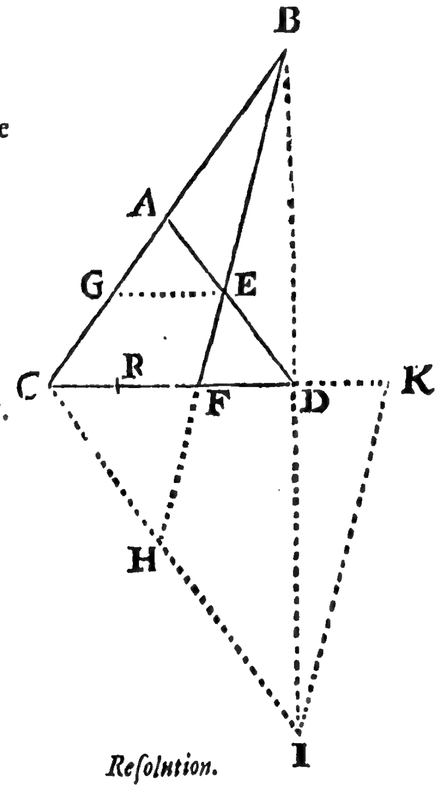
Ar-

An Example in Numbers.

Let the ſame things be given before as in the 11th ſtep, whence the three ſides of the Triangle CEA will be 17, 26, 25, and by the help of theſe, the Perpendicular AH will be found 24, (by *Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.) which multiplied by $42\frac{1}{2}$, that is, half the ſumm of the parallel ſides AF, CG, gives 1020 for the Area of the Trapezium CAFG.

Probl. XIX.

Another way to meaſure a diſtance, when we can come to one end thereof.



CA = 210
AD = 210
CD = 252
AE = 90
CF = 152
CK = $354\frac{2}{3}$
AB = $204\frac{6}{11}$

Suppoſ.

1. AB a diſtance acceſſible only at A.
   CAB a right line.
   b=CA, c=AD, d=DC, f=AE, g=CF — given ſeverally.
   GE ‖ CD.
   BEF a right line.

Req. to find AB.

Reſolution.

2. Put . . . . . a = AB.
3. Then becauſe △ ADC and △ AEG are equiangular, (for GE ‖ CD,) it ſhall be, (*per prop.* 4. *Elem.* 6.) . . . . . AD . AE :: DC . GE.
4. That is, in the letters of the Reſolution, . . c . f :: d . $\frac{df}{c}$.
5. Likewiſe in the ſame △ ADC and △ AEG, AD . AE :: CA . GA.
6. That is, in the letters of the Reſolution, c . f :: b . $\frac{bf}{c}$.
7. Again, in the equiangular Triangles CBF and GBE, ... CF . CA + AB :: GE . GA + AB.
8. That is, ... g . $b+a$ :: $\frac{df}{c}$. $\frac{bf}{c}+a$.
9. Which Analogy gives this Equation, *viz.* $\frac{bfg}{c}+ga = \frac{bdf}{c}+\frac{dfa}{c}$.
10. Therefore by ſubtracting $\frac{dfa}{c}$ from each part, (for g, or CF, is greater than $\frac{df}{c}$ or EG,) . . . . . $\frac{bfg}{c}+ga-\frac{dfa}{c} = \frac{bdf}{c}$.
11. And by ſubtracting $\frac{bfg}{c}$ from each part of the laſt Equation, this ariſeth, . . . $\overline{g-\frac{df}{c}}$ into $a = \frac{bdf-bfg}{c}$.
12. That is, ... $\overline{g-\frac{af}{c}}$ into $a = \frac{df-fg}{c}$ into b.

13. There-

13. Therefore by reſolving the laſt preceding Equation into Proportionals, this Analogy ariſeth, *viz.* . . . $g-\frac{df}{c}$. $\frac{df-fg}{c}$ :: b . a.
14. And by multiplying each of the two firſt Terms of that Analogy by c, this ariſeth, *viz.* . . . . $gc-df$. $df-fg$ :: b . a.
15. Let fm be made equal to gc, *viz.* As f to c, ſo g to a fourth, which may be called m, then ſetting fm in the place of gc in the firſt Term of the Analogy in the 14th ſtep, this ariſeth, *viz.* . . . . . $fm-df$. $df-fg$ :: b . a.
16. Therefore by dividing the firſt and ſecond Terms of the laſt preceding Analogy by f, this ariſeth, *viz.* $m-d$. $d-g$ :: b . a.
    Which Analogy laſt before expreſt affords this

CANON.

17. Let it be made as f to c, ſo g to a fourth, which may be called m; (that is, in the Diagram, as AE to AD, or as CH to CI, ſo CF to CK;) then let it be made, as $m-d$ to $d-g$, ſo b to a the diſtance ſought; (that is, in the Diagram, as KD to DF, or as ID to DB, ſo CA to AB.)

The Demonſtration whereof, (in regard CI is parallel to AD, and IK to HFB,) is manifeſt (*per prop.* 2, & 4. *Elem.* 6.) the Proportionals in the Canon being compared to the reſpective lines in the Diagram belonging to this Problem. Which Canon may be demonſtrated alſo by proceeding according to the ſteps of the Reſolution, in a direct order, *viz.* from the beginning to the end; and although the Demonſtration in this way be prolix, yet ſince 'tis certain, and may ſerve to improve the younger Analyſt's skill in the Reſolution of Plane Problems, I ſhall here form it at large.

Prepar.

18. Let it be made as . . . . . AD . AE :: CF . CR.
    That is, ... c . f :: g . $\frac{fg}{c}$.

Then the ſame things being ſuppoſed and made as in the firſt and ſeventeenth ſteps,

19. . . *Req. demonſtr.* . . . . KD . DF :: CA . AB.

Demonſtration.

20. Foraſmuch as △ ADC and △ AEG are like, it ſhall be (*per prop.* 4. *Ele.* 6.) AD . AE :: DC . GE.
    That is, in the fourth ſtep, . . . c . f :: d . $\frac{df}{c}$.
21. Again, in the ſame △ ADC and △ AEG, ... AD . AE :: CA . GA.
    That is, in 6°, ... c . f :: b . $\frac{bf}{c}$.
22. And becauſe △ CBF and △ GBE are like, it ſhall be . . . . . CF . CA + AB :: GE . GA + AB.
    That is, in 8°, ... g . $b+a$ :: $\frac{df}{c}$. $\frac{bf}{c}+a$.
23. Therefore from the 22th ſtep, this Analogy is manifeſt, (*per* 16. *prop. Elem.* 6.) *viz.* □GA,CF + □CF,AB = □CA,GE + □GE,AB.
    That is, in 9°, $\frac{bfg}{c}+ga = \frac{bdf}{c}+\frac{dfa}{c}$.
24. And by ſubtracting □ GE, AB from each part of the Equation in the 23th ſtep; this remains, *viz.* □GA,CF + □CF,AB − □GE,AB = □CA,GE.
    That is, in 10°, $\frac{bfg}{c}+ga-\frac{dfa}{c} = \frac{bdf}{c}$.
25. And by ſubtracting □ GA, CF from each part of the Equation in the 24th ſtep, this remains, *viz.* □CF,AB − □GE,AB = □CA,GE − □GA,CF.
    That is, in 11°, $\overline{g-\frac{df}{c}}$ into $a = \frac{bdf-bfg}{c}$.
    That is, in 12°, $\overline{g-\frac{df}{c}}$ into $a = \frac{df-fg}{c}$ into b.

Now that □CR,CA may be ſet in the place of □GA,CF in the latter part of the Equation in the 25th ſtep, I ſhall by the four following ſteps prove that □CR, CA = □GA, CF. Firſt then,

26. By the 21th ſtep, ... } AD . AE :: CA . GA.
27. And by the 18th ſtep, ... } AD . AE :: CF . CR.
28. Therefore from the 26th and 27th ſteps, (*per* 11. *prop.* 5. *Elem.*) . . . . . } CA . GA :: CF . CR.
29. Therefore from the 28th ſtep, (*per* 16. *prop. El.* 6.) } □CR, CA = □GA, CF.
30. Therefore by ſetting □CR, CA in the place of □GA, CF in the latter part of the Equation in the 25th ſtep, that Equation will be converted into this, *viz.*

□CF,AB — □GE,AB = □CA,GE — □CR,CA.

That is, in 12°, } . . $g - \frac{df}{c}$ into $a = \frac{df - fg}{c}$ into b.

31. Therefore by reſolving the Equation in the 30th ſtep into Proportionals, this Analogy ariſeth, *viz.*

CF — GE . GE — CR :: CA . AB.

That is, in 13°, } . . $g - \frac{df}{c}$. $\frac{df - fg}{c}$ :: b . a.

32. And by drawing AD into the firſt and ſecond Terms of the Analogy in 31°, this will be manifeſt, (*per* 1. *prop.* 6. *Elem.*)

□CF,AD — □GE,AD . □GE,AD — □CR,AD :: CF — GE . GE — CR.

33. Therefore out of 31° and 32°, (*per* 11. *prop.* 5. *Elem.*)

□CF,AD — □GE,AD . □GE,AD — □CR,AD :: CA . AB.

That is, in 14°, $gc - df$. $df - fg$ :: b . a.

34. And becauſe from the *Conſtruction* in 17°, . . } □CK,AE = □CF,AD.
35. And out of 20°, ... } □DC,AE = □GE,AD.
36. And out of 18°, ... } □CF,AE = □CR,AD.
37. Therefore out of 33°, 34°, 35° 36°, by exchange of equal Rectangles, this Analogy ariſeth, *viz.*

□CK,AE — □DC,AE . □DC,AE — □CF,AE :: CA . AB.

That is, in 15°, $fm - df$. $df - gf$ :: b . a.

38. And this following Analogy, by reaſon of the common altitude AE, will be manifeſt by *prop.* 1. *Elem.* 6.

□CK,AE — □DC,AE . □DC,AE — □CF,AE :: CK — DC . DC — CF.

39. Therefore out of 37° and 38°, (*per* 11. *prop.* 5. *Elem.*)

CK — DC . DC — CF :: CA . AB.

That is, in 16°, $m - d$. $d - g$ :: b . a.

40. Therefore from 39°, reſpect being had to the Diagram,

KD . DF :: CA . AB.

Which was to be Dem.

Divers other Canons might be deduced from the premiſſes, which for brevity ſake I ſhall paſs over; but to ſatisfie ſuch Artiſts as delight in Problems of this kind, becauſe they may be practically applied to meaſure diſtances in the field without any Mathematical Inſtrument to obſerve angles, I ſhall illuſtrate the preceding Canon (in 17°,) by

An Example in Numbers.

Suppoſe . . . . { CA = *b* = 210; AD = *c* = 210; CD = *d* = 252; AE = *f* = 90; CF = *g* = 152 } Feet.

Then by the Canon in 17° of this *Probl.*

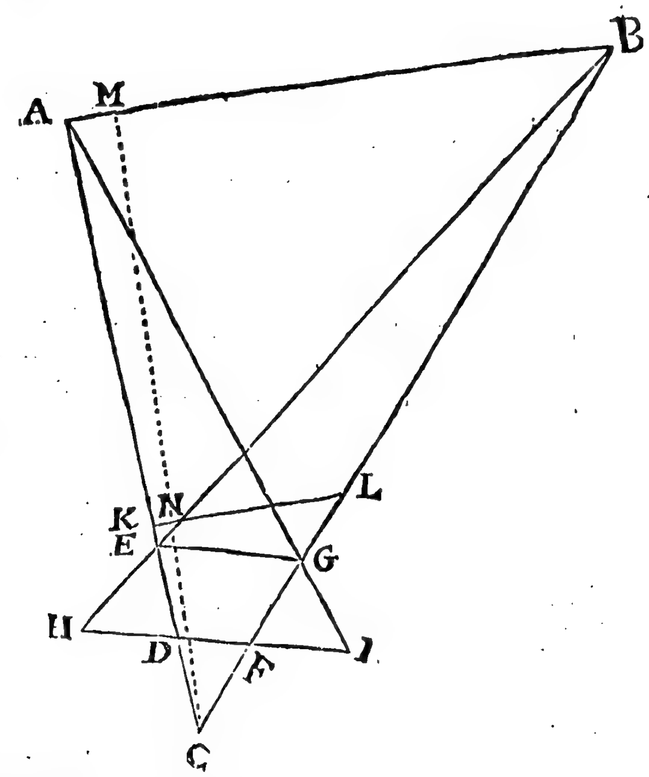
I. { AE . AD :: CF . CK.
  { 90 . 210 :: 152 . $354\frac{2}{3}$.

II. { CK — CD . DC — CF :: CA . AB.
   { or, KD . DF :: CA . AB.
   { $102\frac{2}{3}$. 100 :: 210 . $204\frac{4}{11}$.

Probl. XX.

Probl. XX.

To find the length of a right line AB, when neither of its ends A or B is acceſſible.



Make choice of ſome ſtation C, where you may both ſee, and have liberty to meaſure towards A and B; then meaſure two equal lines CD and DE, in Yards or Feet, *&c.* ſo as CDEA may be a right (or ſtraight) line; likewiſe meaſure CF and FG equal to one another, and ſo as CFGB may be a ſtraight line. Meaſure alſo EG, which ſhall be parallel to a line drawn from D to F, (*per* 2. *prop.* 6. *Elem.*) for CD . DE :: CF . FG. Then ſuppoſing DF to be produced to H and I, to wit, ſuch ſtations that HEB and IGA may be right lines, meaſure HE and GI; alſo HF and DI. Now there are given ſeverally the four ſides of the Trapezium HEGF, having two parallel ſides EG and HF, therefore the length of GB ſhall be given alſo by the preceding *Probl.* 18. In like manner, the four ſides of the Trapezium DEGI having two parallel ſides EG, DI, are given ſeverally, therefore the line EA ſhall be given alſo by the ſaid *Probl.* 18. Then EA and GB, alſo CE and CG being given (as before,) the lines CA and CB ſhall be given alſo, by Addition. Then take ſome Aliquot part of CA, ſuppoſe one third part, and meaſure it from C to K; meaſure likewiſe one third part of CB from C to L; ſo ſhall KL be parallel to the inacceſſible line AB ſought, (*per prop.* 2. *Elem.* 6.) Then meaſure KL. Laſtly, as CK to KL, ſo is CA to AB; or as CL to LK, ſo is CB to BA; but CK, KL, CA, CL, CB are given in Yards or Feet, *&c.* (as before,) and therefore AB the diſtance ſought is given alſo in the ſame kind of meaſure.

If you deſire to find the ſhorteſt diſtance from C to AB, to wit, the Perpendicular CM, it may be found by the help of the three ſides of the Triangle CAB, whoſe meaſures were before diſcovered. Or if you would draw a line in the field from C towards the inacceſſible line AB, that ſhall fall perpendicularly upon the ſaid AB produced infinitely, you may by the three ſides of the Triangle CKL, and by the *Coroll.* in 45° of *Probl.* 10. of this *Chapt.* diſcover whether a Perpendicular from C will fall within or without the Triangle CKL, and alſo find the diſtance from either end of the Baſe KL to the foot of the Perpendicular, which diſtance being meaſured in the field accordingly, the Perpendicular CN may be drawn, which being produced, ſhall be alſo perpendicular to the inacceſſible line AB. But how this Problem may be ſolved by meaſuring only five right lines, I ſhall hereafter ſhew in *Probl.* 20. *Chap.* 9.

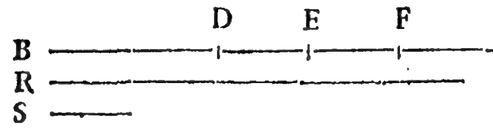
CHAP. VIII.

The ſecond Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems.*

IN which Examples, the Reſolution ends either in an Equation between the Square of the right line ſought, and one or more known Planes; or elſe in an Analogy whoſe three firſt Terms are known Planes, and the fourth gives the Square of the right line ſought.

Probl. I.

To cut a given right line into two ſuch parts, that the ſumm of the Squares of the parts may be to the Square of the difference of the parts in a given Reaſon. But the firſt Term of the Reaſon muſt exceed the latter.



BC = 18	DF = 6
R = 5	BF = 12
S = 1	FC = 6

Suppoſ.

1. b = BC a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = R \\ s = S \end{matrix}\right\}$ the Terms of a given Reaſon.
3. $r \sqsubset s$.

Req. to find

4. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
5. □ BF + □ FC . □ : BF — FC : : R . S.

Reſolution.

6. Put a for the difference of the parts ſought, *viz.* . } a = BF — FC.
7. Therefore the Square of the difference of the parts is } aa.
8. And from 1° and 6°, the ſumm of the Squares of the parts (*per Theor.* 6. *Chap.* 4.) is . . . . . } $\frac{1}{2}bb + \frac{1}{2}aa$.
9. Therefore according to the tenour of the Problem theſe muſt be Proportionals, *viz.* . . . . . } $r \cdot s :: \frac{1}{2}bb + \frac{1}{2}aa \cdot aa$.
10. Whence, by doubling the Antecedents, this Analogy ariſeth, ... } $2r \cdot s :: bb + aa \cdot aa$.
11. Therefore by *Diviſion of Reaſon*, ... } $2r - s \cdot s :: bb \cdot aa$.

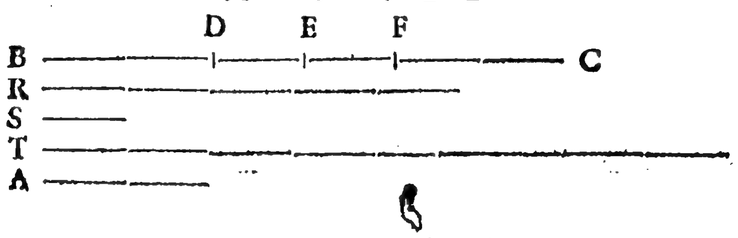
Which laſt Analogy gives this

CANON.

12. As the exceſs whereby the double of the firſt Term of the given Reaſon exceeds the latter Term, is to the latter Term; ſo is the Square of the line given to be cut into two parts, to the Square of the difference of the parts. Therefore the difference of the parts is given, and conſequently the parts are given ſeverally by *Theor.* 9. *Chap.* 4.

The reaſon of the Determination annex'd to the Problem is evident by *Theor.* 5. *Chap.* 4. which ſhews, that if a right line be divided into two unequal parts, the ſumm of the Squares of the parts is greater than the Square of the difference of the parts, by the double Rectangle of the parts.

The Compoſition of the foregoing Probl. 1.



Suppoſ.

Suppoſ.

13. BC is a right line given to be cut into two parts.
14. R and S are the Terms of a given Reaſon.
15. R ⊏ S.

Req. to find

16. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
17. □ BF + □ FC . □ : BF — FC : : R . S.

Conſtruction.

18. Find a right line T that may be equal to the exceſs of 2R above S, which is poſſible to be done, for by *Suppoſition* R ⊏ S; ſuppoſe therefore T = 2R — S.
19. By *Probl.* 11. *Chap.* 5. let it be made as T to S, ſo the Square of BC to another Square, whoſe ſide ſuppoſe to be A, therefore,

T . S :: □ BC . □ A.

20. Divide BC into two equal parts in E, therefore EB = EC.
21. From EC and EB cut off EF and ED, ſuch parts, that each may be equal to $\frac{1}{2}$ A, which is poſſible to be done, if EB (= EC) be greater than $\frac{1}{2}$ A. But that EB or EC is greater than $\frac{1}{2}$ A, I prove thus;

By *Suppoſ.* in 15°, ...	R ⊏ S.
And conſequently, ...	2R ⊏ 2S.
Therefore by ſubtracting S from each part, ...	2R — S ⊏ S.
But by *Conſtr.* in 18°, ...	T = 2R — S.
Therefore from the two laſt preceding ſteps, (*per Ax.* 4. *Chap* 2.)	T ⊏ S.
Therefore from the Analogy in 19°, and from the laſt preceding ſtep,	BC ⊏ A.
And conſequently, ...	$\frac{1}{2}$ BC ⊏ $\frac{1}{2}$ A.
But by *Conſtr.* in 20°, ...	$\frac{1}{2}$ BC = EB = EC.
Therefore from the two laſt preceding ſteps, ...	EB or EC ⊏ $\frac{1}{2}$ A.

Which was to be Dem.

22. I ſay BF and FC are the deſired parts of BC. For firſt, their ſumm is manifeſtly equal to BC; and by *Conſtr.* in 20° and 21° the difference between the ſaid parts BF and FC, that is, BF — FC (BD) is equal to DF. But that the ſumm of the Squares of the parts BF and FC, is to the Square of their difference DF, as R to S, I ſhall demonſtrate by a repetition of the ſteps of the foregoing Reſolution in a backward order.

23. . . *Req. demonſtr.* . . . . . R . S :: □ BF + □ FC . □ DF.

Demonſtration.

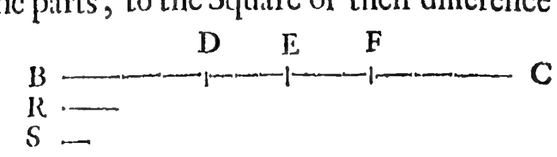
24. By *Conſtr.* in 19°, ... }	T . S :: □ BC . □ A.
25. And by *Conſtr.* in 18° and 21°, . }	2R — S = T. And DF = A.
26. Therefore from 24° and 25°, by exchange of equal quantities, . . . }	2R — S . S :: □ BC . □ DF.
That is, in 11°, ... }	2r — s . s :: bb . aa.
27. Therefore from 26°, by *Compoſition of Reaſon*, ... }	2R . S :: □ BC + □ DF . □ DF.
That is, in 10°, ... }	2r . s :: bb + aa . aa.
28. Therefore from 27°, by halving the Antecedents, ... }	R . S :: $\frac{1}{2}$ □ BC + $\frac{1}{2}$ □ DF . □ DF.
That is, in 9°, ... }	r . s :: $\frac{1}{2}bb + \frac{1}{2}aa$. aa.
29. By *Conſtr.* in 20° and 21°, BC is the ſumm, and DF the difference of the parts BF and FC, therefore (*per Theor.* 6. *Chap.* 4.) . . . . . }	□ BF + □ FC = $\frac{1}{2}$ □ BC + $\frac{1}{2}$ □ DF.
30 Therefore from 28° and 29°, by exchanging equal quantities, . . . }	R . S :: □ BF + □ FC . □ DF.

Which was to be Demonſtr. Therefore that is done which was required by the Problem.

Probl. II.

Probl. II.

To cut a given right line into two ſuch parts, that the Rectangle of the parts, to the Square of their difference, may have a given Reaſon.



BC = 15 | DF = 5
R = 2 | BF = 10
S = 1 | FC = 5

Suppoſ.

1. b = BC a right line given to be cut into two parts.
2. r = R, s = S the Terms of a given Reaſon.

Req. to find

3. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
4. □ BF, FC . □ : BF — FC : : R . S.

Reſolution.

5. Put a for the difference of the parts ſought, *viz.* ⊦ a = BF — FC.
6. Therefore from 1° and 5°, (*per Theor. 9. Chap. 4.*) the greater part ſhall be ⊦ $\frac{1}{2}b + \frac{1}{2}a$ (= BF.)
7. And by the ſame Theorem, the leſſer part ſhall be ⊦ $\frac{1}{2}b - \frac{1}{2}a$ (= FC.)
8. Therefore from 6° and 7°, the Rectangle (or Product) of the parts is ⊦ $\frac{1}{4}bb - \frac{1}{4}aa$.
9. And from 5°, the Square of the difference of the parts is ⊦ aa.
10. Therefore from 4°, 8° and 9°, according to the tenor of the Problem, this Analogy ariſeth, *viz.* ⊦ $r . s :: \frac{1}{4}bb - \frac{1}{4}aa . aa$.
11. Whence, by quadrupling the Antecedents, ⊦ $4r . s :: bb - aa . aa$.
11. Therefore by *Compoſition of Reaſon*, ⊦ $4r + s . s :: bb . aa$.

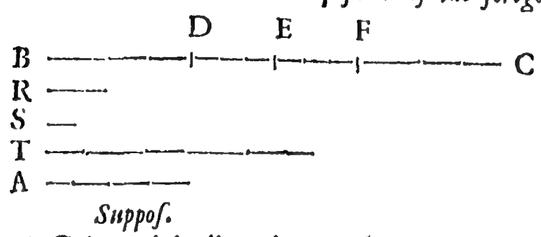
Which laſt Analogy affords this

CANON.

13. As the ſumm of the ſecond Term of the given Reaſon and the quadruple of the firſt is to the ſecond Term; ſo is the Square of the line given to be divided into two parts, to the Square of the difference of the parts.

Therefore the difference of the parts is given, and conſequently the parts ſeverally, by *Theor. 9. Chap. 4.*

The Compoſition of the foregoing Probl. 2.



BC = 15 | A = 5
R = 2 | BF = 10
S = 1 | FC = 5
T = 9 | DF = A

Suppoſ.

14. BC is a right line given to be cut into two parts.
15. R and S are the Terms of a given Reaſon.

Req. to find

16. BF and FC ſuch parts of BC, that BF + FC = BC. Alſo,
17. □ BF, FC . □ : BF — FC : : R . S.

Conſtruction.

18. Find a right line T = 4R + S.
19. By *Probl. 11. Chap. 5.* let it be made as T to S, ſo the Square of BC to another Square, whoſe ſide ſuppoſe to be A, therefore,

T . S :: □ BC . □ A.

20. Divide BC into two equal parts in E, therefore EC = EB.
21. From EC and EB cut off EF and ED, ſuch parts, that each may be equal to $\frac{1}{2}$A, which may be done, if EC (= EB) be greater than $\frac{1}{2}$A; but that EC or EB is greater than $\frac{1}{2}$A, I prove thus;

By

By *Conſtruction* in 18°, ... T ⊏ S.
Therefore from 19°, ... BC ⊏ A.
And conſequently, ... $\frac{1}{2}$BC ⊏ $\frac{1}{2}$A.
But by *Conſtr.* in 20°, ... $\frac{1}{2}$BC = EC = EB.
Therefore from the two laſt preceding ſteps, ... EC or EB ⊏ $\frac{1}{2}$A.

Which was to be Dem.

22. I ſay BF and FC are ſuch parts of BC as will ſatisfie the Problem. For firſt, BF + FC = BC; and by *Conſtruction* in 20° and 21°, the difference between the ſaid parts BF and FC, that is, BF — FC (BD) is equal to DF. But that the Rectangle of the ſaid parts BF and FC is to the Square of their difference DF as R to S, the following Demonſtration, form'd out of the preceding Reſolution by a repetition of its ſteps in a backward order will make manifeſt.

23. . . *Req. demonſtr.* . . . . . R . S :: □ BF, FC . □ DF.

Demonſtration.

24. By *Conſtr.* in 19°, ... ⊦ T . S :: □ BC . □ A.
25. And by *Conſtr.* in 18° and 21°, . . ⊦ 4R + S = T. And DF = A.
26. Therefore from 24° and 25°, by exchange of equal quantities, ... ⊦ 4R + S . S :: □ BC . □ DF.
27. Therefore by *Diviſion of Reaſon*, . . ⊦ 4R . S :: □ BC — □ DF . □ DF.
28. And by taking $\frac{1}{4}$ of the Antecedents in 27°, ⊦ R . S :: $\frac{1}{4}$□ BC — $\frac{1}{4}$□ DF . □ DF.
29. By *Conſtr.* in 20° and 21°, BC is the ſumm, and DF the difference of the parts BF and FC; therefore by *Theor. 7. Chap. 4.* ⊦ □ BF, FC = $\frac{1}{4}$□ BC — $\frac{1}{4}$□ DF.
30. Therefore from 28° and 29°, by exchange of equal quantities, ... ⊦ R . S :: □ BF, FC . □ DF.

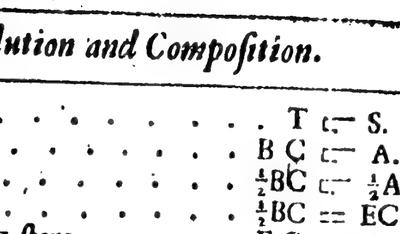
Which was to be Dem. Therefore that is done which was required by the Problem.

A LEMMA, *leading to the following* Probl. 3.

If a Square, or long Square be inſcribed in a Semicircle, the Center of the Semicircle is in the middle of the Baſe of the Square, or long Square.

Suppoſ.

1. CBFGD a Semicircle, whoſe Center is C.
2. EFGH is a Square.
3. . . *Req. demonſtr.* . CE = CH.



Prepar.

4. Draw the right lines CF and CG.

Demonſtration.

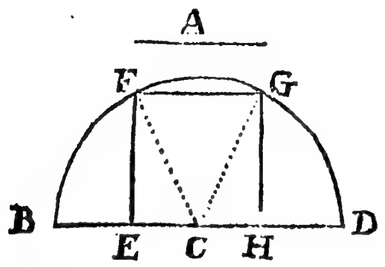
5. By *Deſin. 15. Elem. 1.* . . . . . ⊦ . . CF = CG.
6. And by *Suppoſ.* in 2°, ... ⊦ < FEC = ⌟ = < GHC.
7. Therefore (*per prop. 47. Elem. 1.*) . . ⊦ □ EF + □ CE = □ CF = □ CG.
8. Likewiſe, ... ⊦ □ HG + □ CH = □ CG = □ CF.
9. Therefore from 7° and 8°, (*per Ax. 1. Chap. 2.*) . . . . . ⊦ □ EF + □ CE = □ HG + □ CH.
10. But by *Suppoſ.* . . . . . ⊦ EF = HG.
11. And conſequently, ... ⊦ □ EF = □ HG.
12. Therefore by ſubtracting □ EF or □ HG from each part of the Equation in 9°, the remainders will be equal, *viz.* . . . □ CE = □ CH.
13. Therefore . . . . . ⊦ . CE = CH.

Which was to be Dem. The ſame Demonſtration may be made when a long Square is inſcribed in a Semicircle.

Probl. III.

Probl. III.

To inſcribe a Square in a given Semicircle.



$CB = CD = 10$
$EF = EH = \sqrt{80}$
$CE = CH = \sqrt{20}$
$EB = HD = 10 - \sqrt{20}$

Suppoſ.

1. CBFGD is a Semicircle, whoſe Center is C.
2. $r = CB = CD$ is given.

Req. to inſcribe □ EFGH.

Reſolution.

3. Put a for the ſide of the Square required, *viz.* . . $a = EF = EH$.
4. Therefore by the preceding *Lemma*, ... $\frac{1}{2}a = CE = CH$.
5. And becauſe (*per prop.* 47. *Elem.* 1.) . . . . . $\square EF + \square CE = \square CF$.
6. Therefore in the letters of the Reſolution, . . . $aa + \frac{1}{4}aa = rr$.
7. Therefore, by multiplying the laſt Equation by 4, . . $4aa + aa = 4rr$.
8. That is, ... $5aa = 4rr$.
9. Therefore by taking $\frac{1}{5}$ of the laſt Equation, . . . $aa = \frac{4}{5}rr$.
10. Therefore, by extracting the ſquare Root out of each part of the laſt preceding Equation, ... $a = \sqrt{:\frac{4}{5}rr}$.

Hence this

CANON.

11. The ſquare Root of $\frac{4}{5}$ parts of the Square of the Semidiameter is equal to the ſide of the Square inſcribed in the Semicircle.

The Compoſition of the foregoing Probl. 3.

Suppoſ.

12. CBFGD is a Semicircle, whoſe Center is C.
13. CB = CD the Radius is given.

Req. to inſcribe □ EFGH.

Conſtruction.

14. By *Probl.* 9. *Chap.* 5. find a mean proportional line A between the given Radius CB and $\frac{4}{5}$ CB, therefore . . . CB . A :: A . $\frac{4}{5}$CB.
15. From CB and CD cut off CE and CH, ſuch ſegments, that as well CE as CH may be equal to $\frac{1}{2}$ A, and conſequently EH = A; which Effection is poſſible, for by *Conſtruction* in 14° CB is the greateſt of three Proportionals, whereof A is the mean, therefore CB ⊏ A, and becauſe CD = CB, therefore alſo CD ⊏ A; and conſequently a ſegment equal to $\frac{1}{4}$A, as CE or CH may be cut off from CB or CD.
16. Make EF and HG ⊥ BD, and draw FG, ſo is EFGH the Square required to be inſcribed, as will be evident by the following Demonſtration, form'd out of the preceding Reſolution, by a repetition of its ſteps in a backward (not direct) order.
17. . . *Req. demonſtr.* . . . . . EFGH is a □.

Preparat.

18. Draw the right lines CF and CG.

Demonſtration.

19. By *Conſtr.* in 14°, ... CB . A :: A . $\frac{4}{5}$CB.
20. And by *Conſtr.* in 15°, ... EH = A.
21. Therefore from 19° and 20°, by exchanging equal right lines, ... CB . EH :: EH . $\frac{4}{5}$CB.
22. And from 21°, (*per prop.* 17. *Elem.* 6.) . . . □ EH = $\frac{4}{5}$ □ CB.
23. But the quintuples of equal quantities are alſo equal, therefore from 22°, ... 5□ EH = 4 □ CB.

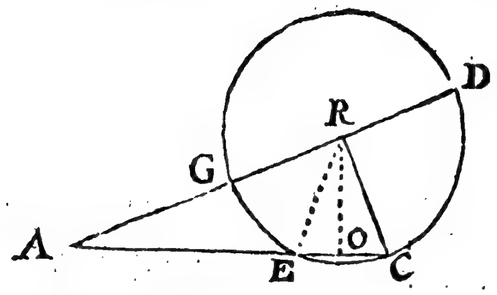
24. That is . . . . . 4 □ EH + □ EH = 4 □ CB.
25. And by taking $\frac{1}{4}$ of all in 24°, ... □EH + $\frac{1}{4}$□EH = □CB = □CF.
26. And becauſe by *Conſtr.* in 15°, ... CE = $\frac{1}{2}$EH = CH.
27. And conſequently, (*per Theor.* 3. *Chap.* 4.) □ CE = $\frac{1}{4}$ □ EH.
28. Therefore from 25° and 27°, by exchanging equal quantities, ... □ EH + □ CE = □ CF.
29. And becauſe by *Conſtr.* in 16°, ... < FEH = ⊿ = < GHE.
30. And conſequently, (*per prop.* 47. *Elem.* 1.) □ EF + □ CE = □ CF.
31. Therefore from 28° and 30°, (*per Ax.* 1. *Ch.* 2.) □EH + □ CE = □EF + □CE.
32. Therefore from 31°, by ſubtracting □ CE from from each part, ... □EH = □ EF.
33. But the ſides of equal Squares are alſo equal, therefore from 32°, ... EH = EF.
34. And by arguing in like manner as in the ſix laſt preceding ſteps, it will be manifeſt that . . . EH = HG.
35. And becauſe from 33°, 34° and 29°, (*per Ax.* 1. & *prop.* 28. *Elem.* 1.) . . . . EF = and ‖ HG.
36. Therefore from 35°, (*per prop.* 33. *Elem* 1.) EH = and ‖ FG.
37. Therefore from 33°, 34° and 36°, . . . EFGH is equilateral.
38. And from 29°, and *Coroll. prop.* 29. *Elem* 1. EFGH is right-angled.
39. Therefore from 37° and 38°, (*per def.* 29. *El.* 1.) EFGH is a □.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. IV.

The Hypothenuſal of a right-angled Triangle being given, as alſo the ſumm of the leggs containing the right angle, to find the Triangle.

But the ſumm of the leggs muſt be greater than the Hypothenuſal, yet not greater than the right line ariſing by application of the double Square of the Hypothenuſal to the ſumm of the leggs.



AC = 169	AE = 119
AR = 156	EC = 50
RC = 65	OE = 25
AD = 221	OC = 25
AG = 91	RO = 60

Suppoſ.

1. ARG is a △ right-angled at R.
2. b = AC the Hypothenuſal is given.
3. b = AR + RC, the ſumm of the leggs is given.

Req. to find △ ARC.

Reſolution.

4. Suppoſing the leggs about the right angle to be unequal, to wit, AR ⊏ RC, put a for their difference, *viz.* . . $a = AR - RG = AG$.
5. Therefore from 3° and 4°, the ſumm of the Squares of the leggs (*per Theor.* 6. *Chap.* 4.) is . . . . . $\frac{1}{2}bb + \frac{1}{2}aa$.
6. Therefore from 5° and 2°, (*per prop.* 47. *Elem.* 1.) this Equation ariſeth, ... $\frac{1}{2}bb + \frac{1}{2}aa = hh$.
7. And by doubling each part of that Equation, . . . $bb + aa = 2hh$.
8. And by ſubtracting bb from each part of the laſt Equation, $aa = 2hh - bb$.
9. Therefore by extracting the ſquare Root out of each part of the laſt preceding Equation, ... $a = \sqrt{:2hh - bb}$.

Hence this

CANON.

10. The difference of the leggs about the right angle is equal to the ſquare Root of the exceſs whereby the double Square of the Hypothenuſal exceeds the Square of the ſumm of the leggs.

Therefore the difference of the leggs is given, and conſequently by the given ſumm and difference of the leggs, the leggs ſhall be given ſeverally, *per Theor. 9. Chap. 4.*

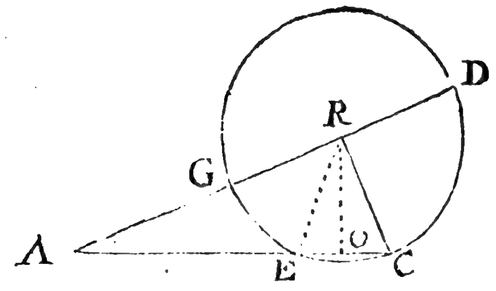
11. But in order to the Geometrical Effection of the Problem propounded, the truth and reaſon of the Determination annex'd to it muſt be made manifeſt. First then, the reaſon of the firſt part of the Determination, to wit, that the right line given for the ſumm of the leggs about the right angle muſt be longer than the line given for the Hypothenuſal, is evident by *prop. 22. Elem. 1.* which ſhews that the ſumm of every two ſides of a plain Triangle is greater (or longer) than the third. The latter part of the Determination is diſcovered by the Canon, which requires that bb be ſubtracted from $2hh$, and therefore bb muſt not exceed $2hh$, and conſequently, by dividing as well bb as $2hh$ by b, 'tis manifeſt that b muſt not be greater than $\frac{2hh}{b}$; that is, the right line given for the ſumm of the leggs about the right angle, muſt not be longer than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſumm of the leggs. The truth of this Determination will more fully appear by the following

THEOREM.

12. In a right-angled plain Triangle, the ſumm of the leggs about the right angle is ſometimes leſs than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſumm of the leggs, and ſometimes equal to, but never greater than the ſaid right line.

The leggs about the right angle are either unequal; or elſe equal between themſelves; I ſhall begin with the firſt Caſe.

Suppoſ. in *Caſe* 1.



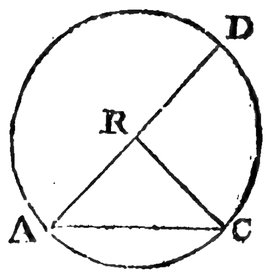
13. ARC is a △.
14. ∠ ARC is ⊿.
15. AR ⊏ RC.
16. RCGD is a ○, whence
17. AD = AR + RC (RD.)
18. AG = AR − RC (RG.)

19. . . *Req. demonſtr.* . . . . . : AD ⊐ $\frac{2\square AC}{AD}$.

Demonſtration.

20. By *Suppoſition* in 17°, ... } AD = AR + RC.
21. And by *Suppoſ.* in 18°, ... } AG = AR − RC.
22. Therefore from 20° and 21, (*per Theor. 6. Ch. 4.*) } $\frac{1}{2}\square$AD + $\frac{1}{2}\square$AG = □AR + □RC.
23. And becauſe by *Suppoſ.* in 14°, ∠ ARC is ⊿, therefore (*per prop. 47. Elem. 1.*) . . . . } . . . □AC = □AR + □RC.
24. Therefore from 22° and 23°, (*per Ax. 1. Ch. 2.*) } $\frac{1}{2}\square$AD + $\frac{1}{2}\square$AG = □AC.
25. And by doubling the laſt Equation, . . . } □AD + □AG = 2□AC.
26. Therefore from 25°, ... } □AD ⊐ 2□AC.
27. Therefore by Application of each part to AD, } . AD ⊐ $\frac{2\square AC}{AD}$.
Which was *Caſe* 1. to be Dem.

Suppoſ. in *Caſe* 2.



28. ARC is a △.
29. ∠ ARC is ⊿.
30. AR = RC = RD.
31. AD = AR + RC.

32. . . *Req. demonſtr.* . AD = $\frac{2\square AC}{AD}$.

Demon-

Demonſtration.

33. Becauſe by *Suppoſ.* in 30° and 31°, ... } AR = RC = $\frac{1}{2}$AD.
34. Therefore their Squares are alſo equal, *viz.* . . } □AR = $\frac{1}{4}$□AD.
35. Likewiſe from 33°, ... } □RC = $\frac{1}{4}$□AD.
36. The ſumm of the Equations in 34° and 35°, gives, } □AR + □RC = $\frac{1}{2}$□AD.
37. And becauſe by *Suppoſ.* ∠ ARC is ⊿, therefore (*per prop. 47. Elem. 1.*) . . . . . } □AR + □RC = □AC.
38. Therefore from 36° and 37°, (*per Ax. 1. Chap. 2.*) } $\frac{1}{2}\square$AD = □AC.
39. And by doubling the laſt Equation, ... } □AD = 2□AC.
40. Therefore by Application of each part to AD, . . } AD = $\frac{2\square AC}{AD}$.
Which was *Caſe* 2. to be Demonſtr.

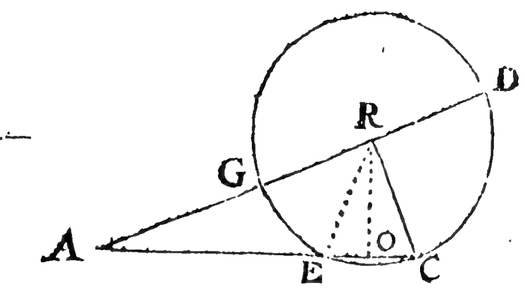
41. Now becauſe in every right-angled Triangle, the ſides about the right angle are either unequal or equal between themſelves, and it hath been demonſtrated, that when the ſaid ſides are unequal, their ſumm is leſs than the right line ariſing by the Application of the double Square of the Hypothenuſal to the ſaid ſumm; but when the ſaid ſides are equal to one another, their ſumm is equal to the ſaid right line; it is evident that the ſumm of the ſides about the right angle can never be greater than the right line ariſing by the ſaid Application. Therefore the truth of the Theorem is manifeſt, and conſequently the Hypothenuſal and the ſumm of the leggs about the right angle muſt be given with due Caution, according to the import of the Determination annex'd to the Problem, that its Solution may be poſſible.

The Compoſition of the foregoing Probl. 4.

H ——————

B ——————————

F ——————



Suppoſ.

42. H = the Hypothenuſal of a right-angled Triangle is given.
43. B = AD the ſumm of the leggs about the right angle is given.
44. AD ⊏ H, but AD not ⊏ $\frac{2\square H}{AD}$. (*Determination.*)

Req. to find the Triangle.

Conſtruction.

45. By the *Determination* in 44° AD is not greater than $\frac{2\square H}{AD}$, ſuppoſe then it be granted, or diſcovered by H and B given in numbers, that AD is leſs than $\frac{2\square H}{AD}$, and conſequently, (by multiplying each part by AD,) that □AD ⊐ 2□H; then it evidently follows, that 'tis poſſible (*per Probl. 4. Chap. 5.*) to find out a right line F, ſuch, that its Square ſhall be equal to 2□H − □AD; ſuppoſe therefore

F = √: 2□H − □AD :

46. From AD cut off AG = F, which may be done, for that AD is greater than F, I prove thus;

By *Suppoſ.* in 44°, ... AD ⊏ H.
Therefore . . . . . □AD ⊏ □H.
And by doubling each part, ... 2□AD ⊏ 2□H.
And by ſubtracting □AD from each part, ... □AD ⊏ 2□H − □AD.
But by *Conſtr.* in 45°, ... □F = 2□H − □AD.
Therefore from the two laſt preceding ſteps, (*per Ax. 3. Ch. 2.*) □AD ⊏ □F.
Therefore . . . . . AD ⊏ F.
Which was to be Dem.

47. Divide GD into two equal parts in R, therefore RG = RD.

48. Make

48. Make R C ⊥ A R; alſo R C = R D or R G, and draw A C.
49. I ſay A R C is the Triangle ſought. Now we muſt ſhew that it will ſatisfie the Problem; firſt then by *Conſtruction* in 48°, R C ⊥ A R, therefore the angle A R C is a right angle; ſecondly, the ſumm of the leggs A R and R C about the right angle, is equal to A D (= B) the given ſumm of the leggs. It remains only to prove that A C is equal to the given Hypothenuſal H; but that will be made manifeſt by the following Demonſtration, form'd out of the preceding Reſolution by a repetition of its ſteps in a backward (not direct) order.
50. . . *Req. demonſtr.* . . . . . A C = H.

Demonſtration.

51. By *Conſtr.* in 45°, ... } $\sqrt{:2\square H - \square AD:} = F.$
52. And by *Conſtr.* in 46°, ... } . . . . . A G = F.
53. Therefore from 51° and 52°, *(per Ax. 1. Chap. 2.)* } A G = $\sqrt{:2\square H - \square AD:}$
54. But the Squares of equal right lines are alſo equal, therefore from 53°, ... } $\square AG = 2\square H - \square AD.$
55. Therefore from 54°, by adding □ AD to each part, } $\square AD + \square AG = 2\square H.$
56. And by taking the halves of all in 55°, . . . } $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square H.$
57. And becauſe by *Conſtr.* in 47° and 48°, A D is the ſumm, and A G the difference of the parts A R and R C, (or R D,) therefore *(per Theor. 6. Chap. 4.)* } $\frac{1}{2}\square AD + \frac{1}{2}\square AG = \square AR + \square RC.$
58. Therefore from 56° and 57°, *(per Ax 1. Cha. 2.)* } $\square H = \square AR + \square RC.$
59. But by *Conſtr.* in 48°, ∠ A R C is ⊥, and conſequently, *(per prop. 47. Elem. 1.)* . . . . } $\square AC = \square AR + \square RC.$
60. Therefore from 58° and 59°, *(per Ax. 1. Chap. 2.)* } $\square AC = \square H.$
61. But the ſides of equal Squares are alſo equal, therefore . . . . . } AC = H.

Which was to be Dem. Therefore the Problem is ſatisfied.

Probl. V.

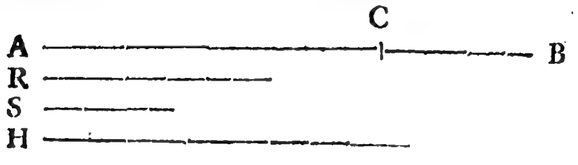
To cut a given right line into two ſuch parts, that the Square of the whole line, to the ſumm of the Squares of the parts may have a given Reaſon.

Or thus,

In a right-angled Triangle, the ſumm of the leggs about the right angle being given, as alſo the Proportion which the Square of the ſaid ſumm hath to the Square of the Hypothenuſal, to find the Triangle. But the given quantities muſt be liable to this

Determination.

The firſt Term of the given Reaſon muſt be greater than the latter Term, yet not greater than the double of the latter Term. For in a right-angled Triangle, the Square of the ſumm of the leggs about the right angle is always greater than the Square of the Hypothenuſal, but never greater than the double Square of the Hypothenuſal, as hath been demonſtrated in the preceding *Probl.* 4.



A B = 34 | H = 26
R = 289 | A C = 24
S = 169 | C B = 10

Suppoſ.

1. A B is a right line given to be cut into two parts.
2. R and S are the Terms of a given Reaſon.
3. R ⊏ S; but R not ⊏ 2 S.

Req. to find

4. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,
5. □ A B . □ A C + □ C B :: R . S.

Con-

Conſtruction.

6. By *Probl.* 11. *Chap.* 5. let it be made, as R to S, ſo the Square of A B to another Square, whoſe ſide ſuppoſe to be H, therefore

R . S :: □ A B . □ H.

7. Then ſuppoſing H to be the Hypothenuſal of a right-angled Triangle, and A B the ſumm of the ſides about the right-angle, find out the ſaid ſides and Triangle by the foregoing *Probl.* 4. For if R be greater than S, but not greater than 2S, according to the import of the Determination added to that Problem, 'tis poſſible to find out ſuch a right-angled Triangle, and then the ſides about the right-angle ſhall be equal to A C and C B, the parts ſought by this *Probl.* 5. Therefore,
8. . . *Req. demonſtr.* . . . . . R . S :: □ A B . □ A C + □ C B.

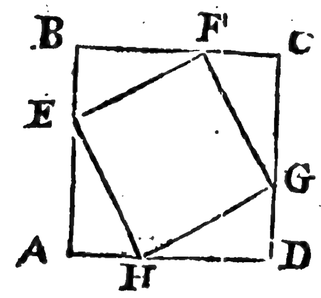
Demonſtration.

9. Becauſe by *Conſtr.* in 6°, ... } R . S :: □ A B . □ H.
10. And by *Conſtr.* in 7°, ... } □ A C + □ C B = □ H.
11. Therefore from 9° and 10°, by exchanging equal quantities, ... } R . S :: □ A B . □ A C + □ C B.

Which was to be Dem. Therefore the Problem is ſatisfied.

A LEMMA, *leading to the following* Probl. 6.

Suppoſ.



1. A B C D is a Square.
2. B F and F C are parts of B C.
3. C F = B E = A H = D G; therefore,
4. B F = A E = D H = C G.
5. E F, F G, G H, H E are right lines.
6. . . *Reg. demonſtr.* that E F G H is a Square.

Demonſtration.

7. Becauſe by *Suppoſ.* in 3° and 4°, ... } C F = B E; and B F = C G.
8. And by *Suppoſ.* in 1°, ... } ∠ E B F = ∠ F C G.
9. Therefore from 7° and 8°, *(per prop. 4. El. 1.)* } E F = F G; and ∠ B E F = ∠ G F C.
10. And in like manner, ... } E F = E H = H G = F G.
11. Again, becauſe by *Suppoſ.* in 1°, . . . } ∠ E B F is ⊥.
12. Therefore *(per Coroll. prop. 32. Elem. 1.)* } ∠ B F E + ∠ B E F = ⊥.
13. But it hath been proved in 9°, that . . . } ∠ G F C = ∠ B E F.
14. Therefore from 12° and 13°, *(per Ax. 6. Chap. 2.)* . . . . . } ∠ B F E + ∠ G F C = ⊥.
15. And from 14°, *(per Coroll. prop. 13. Elem. 1.)* } ∠ E F G is ⊥.
16. And by arguing in like manner as in the five laſt preceding ſteps, it will be evident that . } ∠ F G H = ⊥ = ∠ G H E = ∠ G E F.
17. Therefore from 10°, 15° and 16°, *(per Defin. 29. Elem. 1.)* . . . . . } E F G H is a Square.

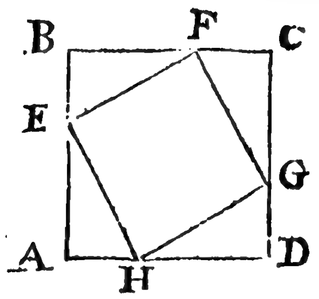
Which was to be Dem.

Probl. VI.

In a given Square to inſcribe another Square whoſe angular points may lye in the ſides of the given Square; and that the Square given to the Square inſcribed may be in a given Reaſon, ſuppoſe as R to S. But R muſt be greater than S, yet not greater than 2 S; as may eaſily be inferr'd from the preceding *Probl.* 5.

R ——

R ——————
S ——————



BC = 34 = BA
R = 289
S = 169
BF = 24
FC = 10 = BE
EF = 26 = FG

Suppoſ.

1. ABCD is a Square, whoſe ſide AB or BC is given.
2. R and S are the Terms of a given Reaſon.
3. R ⊏ S; but R not ⊏ 2 S.

Req. to inſcribe

4. □ EFGH in the □ ABCD. Alſo, that
5. □ ABCD . □ EFGH :: R . S.

Conſtruction.

6. By the foregoing *Probl.* 5. of this *Chapt.* cut BC the ſide of the given Square into two ſuch parts in F, that the Square of BC, that is, □ ABCD, may be in ſuch proportion to the ſumm of the Squares of the parts BF, FC, as R to S; which may be done, becauſe by *Suppoſition* R ⊏ S, yet R not ⊏ 2S; ſuppoſe therefore
R . S :: □ BC . □ BF + □ FC.
7. Make BE = CF = AH = DG, and draw the right lines EF, FG, GH and HE; ſo is EFGH the inſcribed Square required. Now we muſt ſhew that it will ſatisfie the Problem, to which end, two things are to be proved, *viz.* Firſt, that EFGH is a Square, and then that the Square ABCD hath ſuch proportion to the Square EFGH as R to S: Therefore,
8. . . *Req. demonſtr.* . . . . . { EFGH is a □. Alſo, that R . S :: □ABCD . □EFGH.

Demonſtration.

9. By *Suppoſ.* in 1°, ... ABCD is a □.
10. Therefore . . . . . BC = BA = AD = DC.
11. By *Conſtr.* in 7°, ... CF = BE = AH = DG.
12. Therefore by ſubtracting the Equations in 11° from thoſe in 10°, there will remain . . . BF = AE = DH = CG.
13. Therefore out of 9°, 10°, 11° and 12°, by the *Lemma* prefixt before this Problem,) . . EFGH is a □.
14. Again by *Conſtr.* in 6°, ... R . S :: □BC . □BF + □FC.
15. And becauſe by *Suppoſ.* in 1°, ... < EBF is ⌟.
16. And conſequently, (*per prop.* 47. *Elem.* 1.) □EF = □BF + □BE (□FC.)
17. Therefore from 14° and 16°, by exchanging equal quantities,
R . S :: □BC (or □ABCD) . □EF (or □ EFGH.)

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

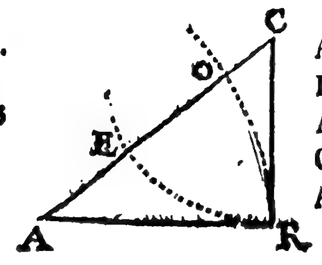
Probl. VII.

In a right-angled Triangle, the difference between the Hypothenuſal and each of the ſides about the right angle being given, to find the Triangle.

Suppoſ.

1. ARC is a △ right-angled at R.
2. b = OC = AC − AR is given.
3. d = AE = CA − CR is given; whence,
4. $g = d + b$ is given. And,
5. $kk = dd + bb$ is given.

Req. to find △ ARC.



AR=4
RC=3
AC=5
OC=1=AC−AR(AO)
AE=1=CA−CR(CE)

Reſolu-

Reſolution.

6. Put a for the exceſs of the Hypothenuſal above the ſumm of the given differences, *viz.* . . . . . a = EO.
7. Therefore from the premiſſes the Hypothenuſal ſhall be $a + g$ (= AC.)
8. And the Baſe . . . . . $a + d$ (= AR.)
9. And the Perpendicular . . . . . $a + b$ (= CR)
10. Therefore from 7° the Square of the Hypothenuſal is $aa + 2ga + gg$.
11. And from 8° the Square of the Baſe is . . . . $aa + 2da + dd$.
12. And from 9° the Square of the Perpendicular is . $aa + 2ba + bb$.
13. Therefore the ſumm of all in 11° and 12°, gives the ſumm of the Squares of the Baſe and Perpendicular, *viz.* $2aa + 2da + 2ba + dd + bb$.
14. That is, (becauſe by *Suppoſ.* in 5°, $kk = dd + bb$, and from 4°, $2g = 2d + 2b$,) . . . . . $2aa + 2ga + kk$.
15. Therefore from 10° and 14°, this Equation ariſeth, (*per prop.* 47. *Elem.* 1.) . . . . . $2aa + 2ga + kk = aa + 2ga + gg$.
16. Therefore, by ſubtracting $aa + 2ga$ from each part of that Equation, this ariſeth, ... $aa + kk = gg$.
17. And by ſubtracting kk from each part of the laſt Equation, ... $aa = gg - kk$.
18. But from 4°, ... $dd + bb + 2db = gg$.
19. And from 5°, ... $dd + bb = kk$.
20. And by ſubtracting the Equation in 19° from that in 18°, this remains, ... $2db = gg - kk$.
21. Therefore from 17° and 20°, (*per Ax.* 1. *Chap.* 2.) $aa = 2db$.
22. Therefore by extracting the ſquare Root out of each part of the laſt Equation, it gives . . . . . $a = \sqrt{2db}$.
23. Therefore out of 22°, 6°, 7° and 4°, the Hypothenuſal is given, to wit, ... $d + b + \sqrt{2db}$ = AC.
24. And from 22°, 6°, 8° and 3°, the Baſe is alſo given, *viz.* $d + \sqrt{2db}$ = AR.
25. And from 22°, 6°, 9° and 3°, the Perpendicular is alſo diſcovered, *viz.* . . . . . $b + \sqrt{2db}$ = CR.

Which three laſt preceding ſteps give this

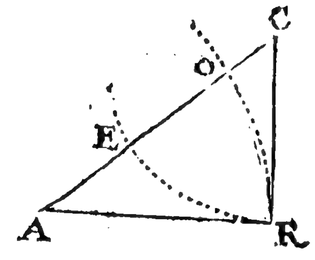
CANON.

26. To the ſumm of the given differences add the ſquare Root of their double Rectangle, ſo ſhall the ſumm of that Addition be the Hypothenuſal ſought. Then add that ſquare Root to the given differences ſeverally, and theſe two ſumms ſhall be the deſired ſides about the right angle.

The Compoſition of Probl. 7.

B —— 1
D ———— 2
M ———— 2
F ————— 3

AR = 4
RC = 3
AC = 5



Suppoſ.

27. B = the exceſs whereby the Hypothenuſal of a right-angled Triangle exceeds one of the ſides about the right angle is given.
28. D = the exceſs of the Hypothenuſal above the other ſide about the right angle is given alſo.

Req. to find out the Triangle.

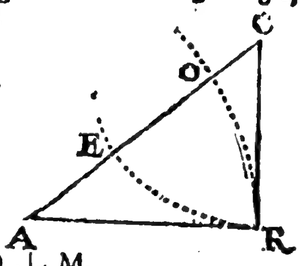
Conſtruction.

29. By *Probl.* 9. *Chap.* 5. find a mean Proportional line, as M, between 2 D and B, therefore, ... 2D . M :: M . B.

30. Then

30. Then let a Triangle, as A R C, be made of three right lines equal to theſe given, to wit, D-|-B-|-M, | D-|-M, | B-|-M; which may be done, (*per probl.* 22. *Elem.* 1.) for the ſumm of every two of thoſe three lines is manifeſtly greater than the third; ſuppoſe therefore . . . . . } AC = D-|-B-|-M, AR = D-|-M, RC = B-|-M.

31. I ſay A R C is the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. Firſt then, 'tis manifeſt that the difference between AC, that is, D-|-B-|-M, and A R, that is, D-|-M, is equal to the given difference B; alſo the difference between A C, that is, D-|-B-|-M, and R C, that is, B-|-M, is manifeſtly equal to the given difference D. So it remains only to prove that the angle A R C is a right angle, which will be made manifeſt by the following Demonſtration.



Prepar.

32. Make F = B-|-M, therefore
33. From 30° and 32°, A C = D-|-F. Alſo R C = F.
34. . . *Req. demonſtr.* . . . . < A R C = ⌟.

Demonſtration.

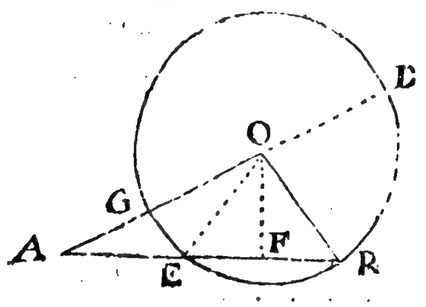
35. Becauſe by *Conſtr.* in 30°, ... } A R = D-|-M.
36. Therefore, (*per Theor.* 2. *Chap.* 4.) . . } □A R = □D-|-□M-|-2□DM.
37. By *Conſtr.* in 32°, ... } F = B-|-M.
38. Therefore from 37°, by drawing 2D into each part, (*per prop.* 1. *Elem.* 6.) . . . } 2□DF = 2□DB-|-2□DM.
39. But from the *Conſtr.* in 29°, it follows (*per prop.* 17. *Elem.* 6.) that . . . . } □M = 2□DB.
40. Therefore from 38° and 39°, (*per Ax.* 6. *Chap.* 2.) . . . . . } 2□DF = □M-|-2□DM.
41. Likewiſe from 36° and 40°, ... } □AR = □D-|-2□DF.
42. By *Conſtr.* in 33°, ... } RC = F.
43. And conſequently, ... } □RC = □F.
44. Therefore the ſumm of the Equations in 41° and 43°, gives (*per Ax.* 8. *Chap.* 2.) . . } □AR-|-□RC = □D-|-2□DF-|-□F.
45. By *Conſtr.* in 33°, ... } AC = D-|-F.
46. And conſequently, (*per Theor.* 2. *Chap.* 4.) } □AC = □D-|-2□DF-|-□F.
47. Therefore from 44° and 46°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □AC = □AR-|-□RC.
48. Therefore, (*per prop.* 48. *Elem.* 1.) . . } < A R C = ⌟.

Which was to be Dem. Therefore the Problem is ſatisfied.

Probl. VIII.

The Baſe, Perpendicular and ſumm of the leggs of a plain Triangle being given ſeverally, to find the Triangle. But the lines given muſt be ſubject to the Determination hereafter declared.

Note. There is more than enough given in this Problem, unleſs it requires a Triangle that hath either unequal acute angles at the Baſe, in which Caſe the Perpendicular falls within the Triangle; or elſe is obtuſangled at the Baſe, in which latter Caſe the Perpendicular falls without. The following Reſolution handles the firſt Caſe, but with a little alteration it may be applied to the latter, as will hereafter appear.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

Preparat.

1. Suppoſe A R O to be the Triangle ſought, having unequal acute angles A and R at the ends of the Baſe A R; then from the Center O, at the diſtance of the leſſer legg O R, deſcribe the Circle O R G D cutting the greater legg O A in G; ſo ſhall A G be the difference of the leggs O A and O R, for O G = O R.
2. Produce A O to the Circumference in D, then is A D equal to the ſumm of the leggs A O and O R; for O D = O R.
3. Draw the Semidiameter O E, and let fall O F ⊥ E R, ſo will O F cut E R into two equal parts in F, (*per prop.* 3. *Elem.* 3.) Theſe things premiſed, the Reſolution of the Problem propos'd may be formed in manner following.

Suppoſ.

4. *b* = A R, the Baſe of △ A R O is given.
5. *p* = O F the Perpendicular is given.
6. *c* = A D = A O -|- O R, the ſumm of the leggs is given.

Req. to find the Triangle.

Reſolution.

7. Put *a* for the unknown difference of the leggs, *viz.* aſſume . . . . } *a* = AG (= AO − OR.)
8. Then the difference of the ſegments of the Baſe made by the falling of the Perpendicular ſhall be $\frac{ca}{b}$, for, (by *Theor.* 2. *Probl.* 9. *Chap.* 7.) . . } *b* . *c* :: *a* . $\frac{ca}{b}$ (AE.)
9. Therefore from 4° and 8°, (by *Theor.* 9. *Chap.* 4.) the greater ſegment of the Baſe ſhall be . . . } $\frac{1}{2}b + \frac{ca}{2b}$ (= AF.)
10. And (by the Theorem laſt mentioned,) the leſſer ſegment ſhall be } $\frac{1}{2}b - \frac{ca}{2b}$ (= FR.)
11. Therefore the Square of the greater ſegment (in 9°) ſhall be . . . } $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca.$
12. And the Square of the leſſer ſegment (in 10°) ſhall be . . . } $\frac{1}{4}bb + \frac{ccaa}{4bb} = \frac{1}{2}ca$
13. By *prop.* 47. *Elem.* 1. . . . } □ AF -|- □ FO = □ AO.
14. Therefore out of 11°, 13° and 5°, the Square of the greater legg A O ſhall be . . . . . } $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp.$
15. And out of 5° and 12°, the Square of the leſſer legg O R ſhall be . . } $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp.$
16. Therefore from 14° and 15°, the ſumm of the Squares of the leggs ſhall be . . . . . } $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp.$

Now becauſe by *Theor.* 2. *Chapt.* 4.
□ AO -|- □ RO -|- 2□ AO, RO = □ : $\overline{AO + OR}$: = □ AD;
Therefore to the end the quantities in 16° may be made a compleat Square of A D, let the double Rectangle of the leggs A O and R O be found out in this manner, *viz.*

17. From 6° and 7°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . } $\frac{1}{2}c + \frac{1}{2}a$ (= AO.)
18. And (by the Theorem laſt mentioned,) the leſſer legg is . . . } $\frac{1}{2}c - \frac{1}{2}a$ (= RO.)
19. Therefore the Rectang. of the legs is } $\frac{1}{4}cc - \frac{1}{4}aa$ (= □ AO, RO.)
20. And the double Rectangle of the leggs is . . . . . } $\frac{1}{2}cc - \frac{1}{2}aa$ (= 2□ AO, RO.)
21. Therefore the ſumm of all in 16° and 20°, gives the Square of the ſumm of the leggs, *viz.* . . . } $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa$ (= □ AD.)

22. Which quantities in 21° muſt be equal to *cc* the Square of the given ſumm of the leggs; hence this Equation,

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \tfrac{1}{2}cc - \tfrac{1}{2}aa = cc.$$

23. Therefore by ſubtracting $\frac{1}{2}cc$ from each part of that Equation, this will ariſe;

$$\tfrac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \tfrac{1}{2}aa = \tfrac{1}{2}cc.$$

24. And by doubling all in 23°, $bb + \frac{ccaa}{bb} + 4pp - aa = cc.$

25. Now to the end that known quantities may be ſeparated from unknown, enquiry muſt be made, whether $\frac{ccaa}{bb}$ be greater or leſs than *aa*. But becauſe (as appears in 8°,) theſe are Proportionals, *viz.* $b \,.\, a :: c \,.\, \frac{ca}{b}$, and $c \sqsubset b$, for the ſumm of the leggs of any plain Triangle is longer than the Baſe, therefore (*per prop.* 14. *Elem.* 5.) $\frac{ca}{b}$ ſhall be greater than *a*, and conſequently the Square of the former greater than the Square of the latter, *viz.* $\frac{ccaa}{bb} \sqsubset aa$, therefore *aa* may be ſubtracted from $\frac{ccaa}{bb}$, and there will remain a quantity greater than nothing. From the premiſſes therefore it is manifeſt that $bb + 4pp$ may be ſubtracted from each part of the Equation above expreſt in 24°, and the quantity remaining on each part will be greater than nothing, and the Equation ariſing by that ſubtraction will be this,

$$\frac{ccaa}{bb} - aa = cc - bb - 4pp.$$

26. That is, by reducing $\frac{ccaa}{bb} - aa$ into the form of a Fraction;

$$\frac{ccaa - bbaa}{bb} = cc - bb - 4pp.$$

27. Which laſt Equation may be reſolved into this Analogy,

$$bb \,.\, cc - bb :: aa \,.\, cc - bb - 4pp.$$

28. Therefore by Inverſe and Altern Reaſon,

$$cc - bb \,.\, cc - bb - 4pp :: bb \,.\, aa.$$

29. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt preceding Analogy,

$$\sqrt{:} cc - bb \,.\, \sqrt{:} cc - bb - 4pp :: b \,.\, a.$$

Of which laſt Analogy the three firſt Terms are given, therefore the fourth Term, which is the difference of the leggs ſought, is given alſo. Moreover, by the ſame quantities firſt given, the line A E, which is the difference of the ſegments of the Baſe made by the falling of the Perpendicular, ſhall be given alſo; for,

30. By *Theor.* 2. *Probl.* 9. *Chap.* 7. $c \,.\, AE :: b \,.\, a.$

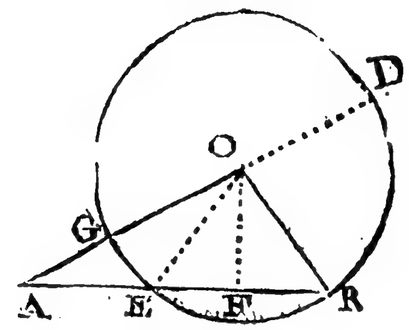
31. Therefore out of 29° and 30°, (by *prop.* 11. *Elem.* 5.)

$$\sqrt{:} cc - bb \,.\, \sqrt{:} cc - bb - 4pp :: c \,.\, AE.$$

32. *Note.* In this Reſolution, the Perpendicular is ſuppoſed to fall within the Triangle ſought, and $\frac{ca}{b}$ (the fourth Proportional of the Analogy in 8°,) repreſents A E the difference of the ſegments of the Baſe made by the falling of the Perpendicular; but when the Perpendicular falls without, as in △ A E O, where A E is the Baſe, the ſaid $\frac{ca}{b}$ repreſents the line A R, which is compos'd of the Baſe A E, and E R the double of the diſtance F E from F the foot of the Perpendicular to the obtuſe angle at E. But whether the Perpendicular falls within or without the Triangle, the Reſolution runs into that Equation before expreſt in 22°. Which things being well obſerved, the Theorems hereafter expreſt, (which will be very uſeful in the following Problems,) will clearly ariſe out of the preceding Reſolution; *viz.*

The Equation in 25° gives

THEOR. 1.



THEOR. 1.

33. In a plain Triangle whoſe leggs are unequal, if the Perpendicular falls within, the Square of the difference of the ſegments of the Baſe made by the falling of the Perpendicular, is greater than the Square of the difference of the leggs, by the exceſs whereby the Square of the ſumm of the leggs exceeds the ſumm of the Square of the Baſe and the Square of the double Perpendicular. But when the Perpendicular falls without the Triangle upon the Baſe increaſed, then the Square of the line compos'd of the Baſe and double diſtance between the foot of the Perpendicular and the obtuſe angle, is greater than the Square of the difference of the leggs, by the exceſs above-mentioned.

The Analogy in 29° gives

THEOR. 2.

34. As the right line whoſe Square is equal to the exceſs whereby the Square of the ſumm of the leggs of a plain Triangle exceeds the Square of the Baſe, is to the right line whoſe Square is equal to the exceſs of the ſaid Square of the ſumm of the leggs above the ſumm of the Square of the Baſe and the Square of the double Perpendicular; ſo is the Baſe to the difference of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the exceſs of the Square of the ſumm of the leggs above the Square of the Baſe, is to the exceſs of the Square of the ſumm of the leggs above the ſumm of the Square of the Baſe and the Square of the double Perpendicular; ſo is the Square of the Baſe to the Square of the difference of the leggs.

Therefore, if the Baſe, Perpendicular, and ſumm of the leggs of a plain Triangle whoſe leggs are unequal, be given ſeverally, the difference of the leggs ſhall be given alſo; and conſequently the leggs ſeverally, by *Theor.* 9. *Chap.* 4.

The Analogy in 31° gives

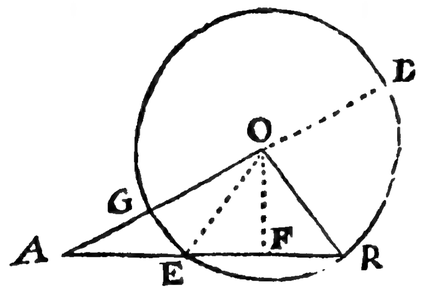
THEOR. 3.

35. As the right line whoſe Square is equal to the exceſs whereby the Square of the ſumm of the unequal leggs of a plain Triangle exceeds the Square of the Baſe, is to the right line whoſe Square is equal to the exceſs of the Square of the ſumm of the leggs above the ſumm of the Square of the Baſe and the Square of the double Perpendicular; ſo is the ſumm of the leggs to a fourth Proportional, which is leſs than the Baſe, when the Perpendicular falls within, for then it is the difference of the ſegments of the Baſe made by the Perpendicular. But when the Perpendicular falls without, the ſaid fourth Proportional exceeds the Baſe, and is compos'd of the Baſe and double diſtance between the foot of the Perpendicular and the obtuſe angle at the nearer end of the Baſe. Laſtly, when the ſaid fourth Proportional is equal to the Baſe, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 8. be expreſt by numbers, we may diſcover by *Theor.* 3. above expreſt, whether the Triangle ſought be acute-angled, or obtuſe-angled, or right-angled at the Baſe, *viz.* of what kind the angles at the Baſe are.

36. The truth of the three preceding Theorems, when the angles at the Baſe are acute and unequal, may be demonſtrated by the ſteps of the foregoing Reſolution, by proceeding in a direct order from the beginning to the end of the Reſolution, in manner following.

Let there be a Triangle propos'd, as A R O, having unequal acute angles at the ends of the Baſe A R, and let the ſame things be ſuppoſed as before in 1°, 2° and 3° of this Problem.



The three Theorems *to be demonſtrated are theſe following*, viz.

37. . □AE — □AG = □AD — □AR — 4□OF.

38. √:□AD — □AR: . √:□AD — □AR — 4□FO: :: AR . AG.

39. √:□AD — □AR: . √:□AD — □AR — 4□FO: :: AD . AE.

Demonſtration.

40. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. this Analogy is manifeſt, ... AR . AD :: AG . AE.
That is, in 8°, ... $b \cdot c :: a \cdot \frac{ca}{b}$.

41. And becauſe AR is divided into two unequal parts in F, and AE is the difference of thoſe parts, therefore (*per Theor.* 9. *Chap.* 4.) . . . . . $\frac{1}{2}$AR + $\frac{1}{2}$AE = AF.
That is, in 9°, ... $\frac{1}{2}b + \frac{ca}{2b}$ = AF.

42. And by the ſame Theorem, ... $\frac{1}{2}$AR — $\frac{1}{2}$AE = FR.
That is, in 10°, ... $\frac{1}{2}b - \frac{ca}{2b}$ (= FR)

43. And (by *Theor.* 2. *Chap.* 4.) the Square of the Equation in 41° gives
$\frac{1}{4}$□AR + $\frac{1}{4}$□AE + $\frac{1}{2}$□AR, AE = □AF.
That is, in 11°, . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca$ (= □AF.)

44. And (by *Theor.* 5. *Chap.* 4.) the Square of the Equation in 42° gives
$\frac{1}{4}$□AR + $\frac{1}{4}$□AE — $\frac{1}{2}$□AR, AE = □FR.
That is, in 12°, . . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca$ (= □FR.)

45. And by adding □OF to each part of the Equation in 43°, it makes (*per prop.* 47. *Ele.* 1.)
$\frac{1}{4}$□AR + $\frac{1}{4}$□AE + $\frac{1}{2}$□AR, AE + □OF = □AO.
That is, in 14°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} + \frac{1}{2}ca + pp$ (= □AO.)

46. And by adding □OF to each part of the Equation in 44°, this ariſeth, (*per pro.* 47. *El.* 1.)
$\frac{1}{4}$□AR + $\frac{1}{4}$□AE — $\frac{1}{2}$□AR, AE + □OF = □RO.
That is, in 15°, . . $\frac{1}{4}bb + \frac{ccaa}{4bb} - \frac{1}{2}ca + pp$ (= □RO.)

47. And the ſumm of the Equations in 45° and 46° gives
$\frac{1}{2}$□AR + $\frac{1}{2}$□AE + 2□OF = □AO + □RO.
That is, in 16°, . . . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp$ (= □AO + □RO.)

It is manifeſt (*per Theor.* 2. *Chap.* 4.) that the Equation in 47° wants only 2□AO, RO to compleat the Square of AO + RO, that is, the Square of AD; therefore in order to fill up that Square, I proceed as in the five ſteps next following.

48. By *Theor.* 9. *Chap.* 4. . . . . $\frac{1}{2}$AD + $\frac{1}{2}$AG = AO.
That is, in 17°, ... $\frac{1}{2}c + \frac{1}{2}a$ = AO.

49. And by the ſame Theorem, ... $\frac{1}{2}$AD — $\frac{1}{2}$AG = RO.
That is, in 18°, ... $\frac{1}{2}c - \frac{1}{2}a$ = RO.

50. And by *Theor.* 8. *Chap.* 4. the Rectangle of the Equations in 48° and 49° is . . . . $\frac{1}{4}$□AD — $\frac{1}{4}$□AG = □AO,RO.
That is, in 19°, ... $\frac{1}{4}cc - \frac{1}{4}aa$ = □AO,RO.

51. And

51. And by doubling the Equation in 50°, . $\frac{1}{2}$□AD — $\frac{1}{2}$□AG = 2□AO,RO.
That is, in 20°, ... $\frac{1}{2}cc - \frac{1}{2}aa$ = 2□AO,RO.

52. The ſumm of the Equations in 47° and 51° makes the Square of the ſumm of the leggs AO and RO, *viz.* the Square of AD,
$\frac{1}{2}$□AR + $\frac{1}{2}$□AE + 2□OF + $\frac{1}{2}$□AD — $\frac{1}{2}$□AG = □AD.
That is, in 22°, . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp + \frac{1}{2}cc - \frac{1}{2}aa = cc$.

53. And by ſubtracting $\frac{1}{2}$□AD from each part of the Equation in 52°, this remains,
$\frac{1}{2}$□AR + $\frac{1}{2}$□AE + 2□OF — $\frac{1}{2}$□AG = $\frac{1}{2}$□AD.
That is, in 23°, . $\frac{1}{2}bb + \frac{ccaa}{2bb} + 2pp - \frac{1}{2}aa = \frac{1}{2}cc$.

54. And by doubling the Equation in 53°,
□AR + □AE + 4□OF — □AG = □AD.
That is, in 24°, . . $bb + \frac{ccaa}{bb} + 4pp - aa = cc$.

55. And becauſe (by *prop.* 8. *Elem.* 3.) AE ⊏ AG, and conſequently □AE ⊏ □AG, therefore □AE — □AG ⊏ O; whence it is manifeſt, that if □AR + 4□OF be ſubtracted from each part of the Equation in 54°, there will remain on each part a quantity greater than nothing, and the Equation ariſing by that ſubtraction will be this that follows, *viz*
□AE — □AG = □AD — □AR — 4□OF.
That is, in 25°, . . $\frac{ccaa}{bb} - aa = cc - bb - 4pp$.
And in 26°, . . $\frac{ccaa - bbaa}{bb} = cc - bb - 4pp$.
Which was *Theor.* 1. to be demonſtr.

Now to paſs from the 26th ſtep to the 27th of the preceding Algebraical Reſolution, by the lines of the Diagram, ſome Analogies, not expreſt in the Reſolution, muſt be introduced; and in order to their diſcovery, the Learner may obſerve, that the Algebraical Fraction $\frac{ccaa - bbaa}{bb}$ in 26° denotes a Plane, which is the fourth Term of an Analogy whoſe three firſt Terms are theſe three Planes, to wit, *bb*, *cc* — *bb* and *aa*, which anſwer to theſe three Planes, (in the lines of the Diagram,) to wit, □AR, □AD — □AR and □AG; therefore, □AE — □AG which is correſpondent to the ſaid Algebraical Fraction $\frac{ccaa - bbaa}{bb}$, muſt likewiſe be the fourth Term of an Analogy whoſe three firſt Terms are the ſaid Planes □AR, □AD — □AR and □AG; but how the ſaid Analogy is brought to light, the four ſteps next following will ſhew.

56. Becauſe, (as hath been ſhewn in 40°,)
AD . AR :: AE . AG.

57. Therefore, (*per prop.* 22. *Elem.* 6.)
□AD . □AR :: □AE . □AG.

58. Therefore by *Diviſion of Reaſon*,
□AD — □AR . □AR :: □AE — □AG . □AG.

59. Therefore inverſly,
□AR . □AD — □AR :: □AG . □AE — □AG.

60. But it hath been ſhewn in 55°, that
□AE — □AG = □AD — □AR — 4□OF.

61. Therefore from 59° and 60° (by exchanging equal quantities) this Analogy ariſeth,
□AR . □AD — □AR :: □AG . □AD — □AR — 4□OF.
That is, in 27°,
bb . *cc* — *bb* :: *aa* . *cc* — *bb* — 4*pp*.

62. Therefore from 61°, by *inverſe* and *altern Reaſon*,
□AD — □AR . □AD — □AR — 4□OF :: □AR . □AG.
That is, in 28°,
cc — *bb* . *cc* — *bb* — 4*pp* :: *bb* . *aa*.

63. But

63. But the ſides of proportional Squares are alſo Proportionals, therefore from 62° this Analogy ariſeth,

$\sqrt{:\square AD - \square AR} \cdot \sqrt{:\square AD - \square AR - 4\square OF} :: AR \cdot AG.$

That is, in 29°,

$\sqrt{:cc - bb} \cdot \sqrt{:cc - bb - 4pp} :: b \cdot a.$

Which was *Theor.* 2. to be Demonſtr.

64. Again, becauſe by *Theor.* 2. in 29° of *Probl.* 9. *Chapt.* 7.

$AD \cdot AE :: AR \cdot AG.$

65. Therefore from 63° and 64°, (*per prop.* 11. *Elem.* 5.)

$\sqrt{:\square AD - \square AR} \cdot \sqrt{:\square AD - \square AR - 4\square OF} :: AD \cdot AE.$

Which was *Theor.* 3. to be Demonſtr.

66. Again, it hath been ſhewn in 62°, that

$\square AR \cdot \square AG :: \square AD - \square AR \cdot \square AD - \square AR - 4\square OF.$

67. Therefore by *converſe Reaſon*,

$\square AR \cdot \square AR - \square AG :: \square AD - \square AR \cdot 4\square OF.$

Which laſt Analogy affords

THEOR. 4.

68. As the Square of the Baſe of any plain Triangle whoſe leggs are unequal, is to the exceſs whereby the Square of the Baſe exceeds the Square of the difference of the leggs; ſo is the exceſs whereby the Square of the ſumm of the leggs exceeds the Square of the Baſe, to the Square of the double Perpendicular.

Therefore, the Baſe and leggs of any plain Triangle whoſe leggs are unequal, being ſeverally given in numbers, the Perpendicular falling upon that Baſe within the Triangle, or without upon the Baſe increaſed, ſhall be given alſo in numbers.

From the ſaid *Theor.* 4. and *prop.* 41. *Elem.* 1. 'tis eaſie to deduce this following

THEOR. 5.

69. The Rectangle made of theſe two right lines, to wit, the right line whoſe Square is equal to the exceſs whereby a quarter of the Square of the Baſe of a plain Triangle exceeds a quarter of the Square of the difference of the leggs; and the right line whoſe Square is equal to the exceſs of a quarter of the Square of the ſumm of the leggs above a quarter of the Square of the Baſe, ſhall be equal to the Triangle.

To make this manifeſt, let the $\triangle ARO$ be taken as before in the Reſolution, then

70. . . *Req. demonſtr.* $\square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:} = \triangle ARO.$

Demonſtration.

71. By *Theor.* 4. in 68° of this Problem,

$\square AR \cdot \square AR - \square AG :: \square AD - \square AR \cdot 4\square OF.$

72. And by taking $\frac{1}{4}$ of every Term of that Analogy,

$\frac{1}{4}\square AR \cdot \frac{1}{4}\square AR - \frac{1}{4}\square AG :: \frac{1}{4}\square AD - \frac{1}{4}\square AR \cdot \square OF.$

73. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt Analogy,

$\frac{1}{2}AR \cdot \sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG} :: \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR} \cdot OF.$

74. Therefore, (*per prop.* 16. *Elem.* 6.)

$\frac{1}{2}\square AR,OF = \square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR.}$

75. But (*per prop.* 41. *Elem.* 1.)

$\frac{1}{2}\square AR,OF = \triangle ARO.$

76. Therefore from 74° and 75°, (*per Ax.* 1. *Chapt.* 2.)

$\square$ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR} = \triangle ARO.$

Which was to be Dem.

Hence the following Canons are deducible, to find out the Area of a plain Triangle Arithmetically, without the help of the Perpendicular, the Baſe and leggs being ſeverally given in numbers, and the leggs unequal between themſelves.

CANON 1.

77. From a quarter of the Square of the Baſe ſubtract a quarter of the Square of the difference of the leggs, and reſerve the remainder; then from a quarter of the Square of the ſumm of the leggs ſubtract a quarter of the Square of the Baſe, and reſerve the remainder;

remainder; that done, multiply the firſt remainder by the ſecond, and extract the Square Root of the Product, ſo ſhall that ſquare Root be the Area of the Triangle.

78. Again, becauſe by *Theor.* 8. *Chapt.* 4.

$\frac{1}{4}\square AR - \frac{1}{4}\square AG = \square$ of $\frac{1}{2}AR + \frac{1}{2}AG \times \frac{1}{2}AR - \frac{1}{2}AG.$

79. Likewiſe by the ſame Theorem,

$\frac{1}{4}\square AD - \frac{1}{4}\square AR = \square$ of $\frac{1}{2}AD + \frac{1}{2}AR \times \frac{1}{2}AD - \frac{1}{2}AR.$

Therefore from 77°, 78° and 79°, by exchanging equal Factors, there will ariſe

CANON 2.

80. Multiply theſe four numbers one into another, to wit,

1. The ſumm of half the Baſe, and half the difference of the leggs.
2. The exceſs of half the Baſe above half the difference of the leggs.
3. The ſumm of half the ſumm of the leggs, and half the Baſe.
4. The exceſs of half the ſumm of the leggs above half the Baſe.

Then extract the ſquare Root of the Product made by the continual multiplication of thoſe four numbers, ſo ſhall that ſquare Root be the Area of the Triangle.

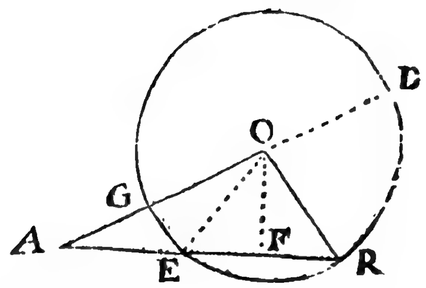
81. Again, if we ſuppoſe $\left\{\begin{array}{l} B = \text{the Baſe,} \\ A = \text{the greater legg,} \\ E = \text{the leſſer legg} \end{array}\right\}$ of a plain Triangle.

Then the four numbers above mentioned in *Canon* 2. may be expreſt thus, *viz.*

1. $\frac{1}{2}B + \frac{1}{2}A - \frac{1}{2}E,$
2. $\frac{1}{2}B - \frac{1}{2}A + \frac{1}{2}E,$
3. $\frac{1}{2}A + \frac{1}{2}E + \frac{1}{2}B,$
4. $\frac{1}{2}A + \frac{1}{2}E - \frac{1}{2}B.$

Or thus,

$\frac{1}{2}B + \frac{1}{2}A + \frac{1}{2}E - E,$
$\frac{1}{2}B + \frac{1}{2}A + \frac{1}{2}E - A,$
$\frac{1}{2}A + \frac{1}{2}E + \frac{1}{2}B,$
$\frac{1}{2}A + \frac{1}{2}E + \frac{1}{2}B - B.$



82. Therefore, if the four numbers laſt before expreſt, (to wit, thoſe ſtanding on the right hand,) be multiplied one into another continually, the Product ſhall be equal to the Square of the Area of the Triangle whoſe three ſides are repreſented by *B, A, E.* But if thoſe four numbers be well obſerved, it will be evident that the number third in order is the half ſumm of the three ſides of the Triangle, and the other three numbers are the Remainders ariſing by the ſubtraction of the three ſides ſeverally from their half ſumm. Hence therefore ariſeth the vulgar Canon, to find out the Area of any plain Triangle whoſe three ſides are ſeverally given in numbers, *viz.*

CANON 3.

83. From half the ſumm of the three ſides of any plain Triangle ſubtract the three ſides ſeverally; then multiply the ſaid half ſumm and the three remainders one into another, according to the Rule of Continual Multiplication, and extract the ſquare Root of the laſt Product, ſo ſhall that ſquare Root be the Area of the Triangle.

Divers other Canons might be raiſed from the premiſſes, to find out the Area of a plain Triangle; but 'tis now time to proceed to the Compoſition of the Problem in hand, and that its Conſtruction may be poſſible, the lines given muſt be ſubject to this

Determination.

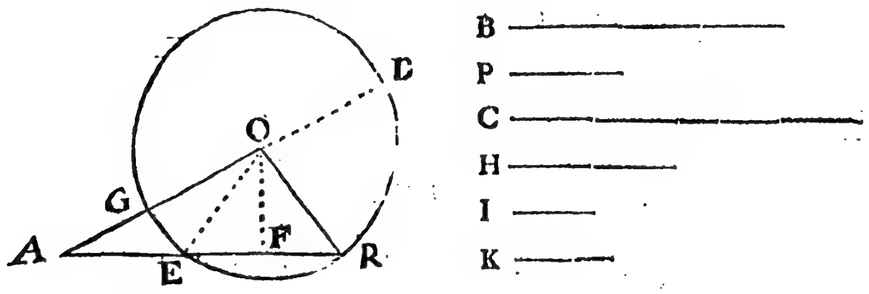
84. . . . . $c \sqsubset \sqrt{:bb + 4pp}$: that is, in words,

The given ſumm of the leggs muſt be longer than that right line whoſe Square is equal to the ſumm of the Square of the Baſe and the Square of the double of the Perpendicular.

This Determination doth openly ſhew it ſelf in *Theor.* 2. in the 34th ſtep of this Problem, and therefore that Theorem having already been demonſtrated, the Determination is conſequently both true and neceſſary for limiting the lines given.

The

The Compoſition of the foregoing Probl. 8.



B ——————
P ———
C ——————————
H ———
I ——
K ——

Suppoſ.

85. B = the Baſe of a Triangle is given.
86. P = the Perpendicular is given.
87. C = the ſumm of the leggs is given.
88. C ⊏ $\sqrt{:\square B \dashv\vdash 4\square P:}$ *(Determination.)*

Req. to make the Triangle.

Conſtruction.

89. By *Probl.* 4. *Chap.* 5. find a right line H; ſuch, that its Square may be equal to □C — □B; which Effection is poſſible, as is evident by the Determination preſcribed in 88°, therefore ſuppoſe

□H = □C — □B.

90. Find likewiſe a right line I, ſuch, that its Square may be equal to □C — □B — 4□P, which Effection the Determination ſhews to be poſſible, therefore ſuppoſe

□I = □C — □B — 4□P.

91. Then by *Probl.* 8. *Chap.* 5. let it be made, as the line H to the line I, ſo the line B (the given Baſe) to a fourth Proportional, ſuppoſe it to be the line K, therefore,

H . I :: B . K.

That is, in 29°, $\sqrt{:cc-bb}$. $\sqrt{:cc-bb-4pp:}$ :: *b* . *a.*

In which Analogy, the firſt Term H is greater than the ſecond Term I, (as is evident by *Conſtr.* in 90° and 91°,) therefore the third Term B ſhall be greater than the fourth K, *(per Schol. Prop.* 14. *Elem.* 5.) and conſequently C ⊏ K, for (by the Determination in 88°) C ⊏ B. Thus far that hath been done which is directed by *Theor.* 2. in 34° of this *Probl.* the reſt of the Conſtruction follows.

92. Let a Triangle be made of theſe three right lines, to wit, B, $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K and $\frac{1}{2}$C — $\frac{1}{2}$K, which is poſſible to be done. *(per prop.* 22. *Elem.* 1.) if C ⊏ K, and that the ſumm of every two of thoſe three lines be longer than the third; but that thoſe lines are ſo qualified, I prove thus;

Firſt, by what hath been ſaid in 91°, C ⊏ K, and conſequently $\frac{1}{2}$C — $\frac{1}{2}$K is equal to ſome real right line.

Secondly, the ſumm of B and $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K is manifeſtly greater than the third line $\frac{1}{2}$C — $\frac{1}{2}$K.

Thirdly, the ſumm of the two lines $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K and $\frac{1}{2}$C — $\frac{1}{2}$K makes C, which (by the Determination in 88°) is greater than the third line B.

Fourthly, that the ſumm of B and $\frac{1}{2}$C — $\frac{1}{2}$K is greater than the third line $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K may be proved thus;

It hath been ſhewn in 91°, that . . . . . B ⊏ K.
Therefore by adding $\frac{1}{2}$C to each part, ... B $\dashv\vdash$ $\frac{1}{2}$C ⊏ K $\dashv\vdash$ $\frac{1}{2}$C.
Therefore by ſubtracting $\frac{1}{2}$K from each part, . . B $\dashv\vdash$ $\frac{1}{2}$C — $\frac{1}{2}$K ⊏ $\frac{1}{2}$K $\dashv\vdash$ $\frac{1}{2}$C.

Which was to be Dem.

Now ſince it hath been proved that $\frac{1}{2}$C — $\frac{1}{2}$K is equal to ſome real right line, and that the ſumm of every two of theſe three right lines, to wit, B, $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K and $\frac{1}{2}$C — $\frac{1}{2}$K, is greater than the third, 'tis poſſible to make a Triangle of thoſe three lines, *(per prop.* 22. *Elem.* 1.) Suppoſe then it be done, and that the Triangle ſo made is ARO, (in the preceding Diagram,) having its Baſe AR equal to the given Baſe B, and the greater legg AO equal to $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K, and the leſſer legg RO equal to $\frac{1}{2}$C — $\frac{1}{2}$K. I ſay the Triangle ARO

ARO will ſatisfie the Problem propounded; but to render the Demonſtration thereof the more eaſie to Learners, I ſhall premiſe a few things in eight ſteps next following.

93. If the quantities of the given lines B, P and C be expreſt by numbers, it will be eaſie to diſcover the kind of the Triangle ſought, when the leggs are unequal, (as they were ſuppoſed to be in the Reſolution,) by *Theor.* 3. in 35° of this Problem; for if the fourth Proportional found out by that Theorem be leſs than the Baſe, the Perpendicular falls within the Triangle; if greater, without; if equal to the Baſe, upon the end of the Baſe.

Suppoſing then it be diſcovered, that the Perpendicular falls upon AR within the Triangle ARO, from the Center O, at the diſtance of the leſſer legg OR, (= $\frac{1}{2}$C — $\frac{1}{2}$K) deſcribe the Circle ORGD cutting OA in G; then produce AO to the Circumference in D, draw alſo the Semidiameter OE, and from the Center O let fall OF perpendicular to EB, therefore *(per prop.* 5. *Elem.* 3.) FE = FR. Then,

94. Becauſe *(per deſin.* 15. *Elem.* 1.) . . . . . } OD = OR = OG.
95. Therefore by adding AO to each part, ... } AD = AO $\dashv\vdash$ OR.
96. But by *Conſtr.* in 92°, ... } C = AO $\dashv\vdash$ OR.
97. Therefore from 95° and 96°, *(per Ax.* 1. *Chap.* 2.) . . . } AD = C.
98. Again, by *Conſtr.* in 92°, ... } AO = $\frac{1}{2}$C $\dashv\vdash$ $\frac{1}{2}$K.
99. Alſo by *Conſtr.* in 92°, ... } OR = $\frac{1}{2}$C — $\frac{1}{2}$K.
100. Therefore, by ſubtracting the Equation in 99° from that in 98°, ... } AG = K.

101. Now I ſhall ſhew that the Triangle ARO, made as before, will ſatisfie the Problem. Firſt then by *Conſtruction* in 92°, the Baſe AR is equal to the given Baſe B; and it hath been proved in 96°, that AO $\dashv\vdash$ OR = C the given ſumm of the leggs. So it remains only to ſhew, that the Perpendicular OF is equal to the given Perpendicular P; but that is made manifeſt by the following Demonſtration, which is form'd out of the foregoing Reſolution, by a repetition of its ſteps in a backward (not direct) order.

102. . . *Req. demonſtr.* . . . . . } OF = P.

Demonſtration.

103. By *Conſtr.* in 91°, . . H . I :: B . K.
That is, in 29°, ... $\sqrt{:cc-bb}$. $\sqrt{:cc-bb-4pp:}$ :: *b* . *a.*
104. Therefore, *(per prop.* 22. *El.* 6.) □H . □I :: □B . □K.

Now that the Terms of the laſt Analogy may be converted into their equivalent quantities expreſſible by the lines in the Diagram, theſe ſeven Equations next following are to be well obſerved.

105. By *Conſtr.* in 89°, ... } □C — □B = □H.
106. And from 97°, ... } □AD = □C.
107. And from 92°, ... } □AR = □B.
108. Therefore from 105°, 106°, 107°, . . . } □AD — □AR = □H.
109. Again, by *Conſtr.* in 90°, ... } □C — □B — 4□P = □I.
110. Therefore from 105°, 108° and 109°, . . } □AD — □AR — 4□P = □I.
111. And from 100°, ... } □AG = □K.

112. Therefore the Terms of the Analogy in 104° being exchanged for their equivalent quantities in 108°, 110°, 107° and 111°, that Analogy will be converted into this, *viz.*

□AD — □AR . □AD — □AR — 4□P :: □AR . □AG.

That is, in 28°,

cc — *bb* . *cc* — *bb* — 4*pp* :: *bb* . *aa.*

113. Therefore by *alterna* and *inverſe Reaſon*,

□AR . □AD — □AR :: □AG . □AD — □AR — 4□P.

That is, in 27°,

bb . *cc* — *bb* :: *aa* . *cc* — *bb* — 4*pp.*

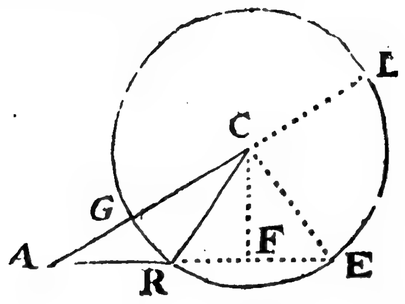
Now to return backwards from the 27th to the 25th ſtep of the Reſolution, by the lines of the Diagram, ſome Analogies not expreſt in the Reſolution muſt be introduced, (which are inferr'd from the Algebraical Fraction $\frac{ccaa-bbaa}{bb}$, as before hath been hinted in 55°,) to wit, the four Analogies next following.

114. By

114. By *Theor.* 2. in 29° of *Probl.* 9. *Chap.* 7. ∴ AD . AR :: AE . AG.
115. Therefore *(per prop.* 22. *Elem.* 6.) . . ∴ □AD . □AR :: □AE . □AG.
116. Therefore, by *Diviſion of Reaſon*,
□AD − □AR . □AR :: □AE − □AG . □AG.
117. And inverſly,
□AR . □AD − □AR :: □AG . □AE − □AG.
118. Therefore from 113° and 114°, *(per prop.* 11, *&* 14. *Elem.* 5.)
□AE − □AG = □AD − □AR − 4□P.
That is, ? in 25°, $\S$ . . $\frac{ccaa}{bb} - aa = cc - bb - 4pp.$
119. And by adding 4□P, alſo □AG to each part, and ſubtracting □AE from each part of the Equation in 118°, this will ariſe,
4□P = □AG + □AD − □AR − □AE.
120. But by *Theor.* 1. in 33° of this Problem,
□AE − □AG = □AD − □AR − 4□OF.
121. Therefore by adding 4□OF, alſo □AG to each part, and ſubtracting □AE from each part of the laſt preceding Equation, this will ariſe,
4□OF = □AG + □AD − □AR − □AE.
122. Therefore from 119° and 121°, *(per Ax.* 1. *Chap.* 2.) ∴ 4□OF = 4□P.
123. Therefore, ... ∴ □OF = □P.
124. Therefore, ... ∴ . . OF = P.
Which was to be Demonſtr. Therefore the Problem is ſatisfied. But for further Illuſtration, all the preceding Canons and Theorems raiſed out of the Reſolution, may be exemplified by the Numbers placed near the Diagram at the beginning of this Problem.

Probl. IX.

The Baſe, Perpendicular and difference of the leggs of a plain Triangle being ſeverally given, to find the Triangle. But the Baſe muſt exceed the difference of the leggs.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Suppoſe the Triangle ARC obtuſangled at R, (the end of the Baſe AR,) to be that which is ſought; then from the Center C, at the diſtance of the leſſer legg CR deſcribe the Circle CRGLE, cutting the greater legg CA in G; ſo ſhall AG be the difference of the leggs CA and CR, for CG = CR.
2. Produce AC and AR to the Circumference in L and E, then is AL equal to the ſumm of the leggs AC and CR, for CL = CR.
3. Draw the Semidiameter CE, and make CF ⊥ RE, ſo ſhall CF cut RE into two equal parts in F, *(per prop.* 3. *Elem.* 3.) Which things being premiſ'd, the Reſolution of the Problem propounded may be reſolved in manner following.

Suppoſ.

4. b = AR the Baſe of △ARC is given.
5. p = CF the Perpendicular is given.
6. d = AG the difference of the leggs AC and RC is given.

Req. to find the Triangle.

Reſolution.

7. Put a for the unknown ſumm of the leggs, *viz.* aſſume
a = AC + RC = AL.

8. Then

8. Then from *Theor.* 2. in 34° of the foregoing *Probl.* 8. of this *Chap.* this Analogy ariſeth,
$\sqrt{:aa - bb:} \;.\; \sqrt{:aa - bb - 4pp:} :: b \;.\; d.$
9. The Squares of which proportional lines ſhall be Proportionals alſo, therefore
$aa - bb \;.\; aa - bb - 4pp :: bb \;.\; dd.$
10. Therefore by Converſion of Reaſon,
$aa - bb \;.\; 4pp :: bb \;.\; bb - dd.$
11. And alternately,
$aa - bb \;.\; bb :: 4pp \;.\; bb - dd.$
12. And by Compoſition,
$aa \;.\; bb :: 4pp + bb - dd \;.\; bb - dd.$
13. And by inverſe and alternate Reaſon,
$bb - dd \;.\; 4pp + bb - dd :: bb \;.\; aa.$
14. But the ſides of proportional Squares are alſo Proportionals, therefore from the laſt preceding Analogy,
$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} :: b \;.\; a.$
15. And becauſe by *Theor.* 2. *Probl.* 10. *Chap.* 7.
$d \;.\; AE :: b \;.\; a.$
16. Therefore from the two laſt preceding Analogies, this ariſeth,
$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} :: d \;.\; AE.$
The Analogy in 14° gives

THEOR. 1.

17. As the right line whoſe Square is equal to the exceſs whereby the Square of the Baſe of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whoſe Square is equal to the ſaid exceſs together with the Square of the double Perpendicular; ſo is the Baſe, to the ſumm of the leggs.

Or thus, which is more convenient for Arithmetical practice.

As the exceſs of the Square of the Baſe above the Square of the difference of the leggs, is to the ſumm of the ſaid exceſs and the Square of the double Perpendicular; ſo is the Square of the Baſe, to the Square of the ſumm of the leggs.

Therefore, the Baſe, Perpendicular, and difference of the leggs of a plain Triangle being ſeverally given, the ſumm of the leggs ſhall be given alſo by the ſaid *Theor.* 1. And conſequently the leggs ſhall be given ſeverally, by *Theor.* 9. *Chap.* 4.

The Analogy in 16° gives

THEOR. 2.

18. As the right line whoſe Square is equal to the exceſs by which the Square of the Baſe of a plain Triangle exceeds the Square of the difference of the leggs, is to the right line whoſe Square is equal to the ſaid exceſs together with the Square of the double Perpendicular; ſo is the difference of the leggs to a fourth Proportional, which exceeds the Baſe, when the Perpendicular falls without the Triangle, for then 'tis the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle at the nearer end of the Baſe; but when the Perpendicular falls within, the ſaid fourth Proportional is leſs than the Baſe, and is the difference of the ſegments of the Baſe made by the Perpendicular: Laſtly, when the ſaid fourth Proportional is equal to the Baſe, the Perpendicular falls upon the end thereof.

Therefore, if the quantities of the lines given in this *Probl.* 9. be expreſt by numbers, we may diſcover by *Theor.* 2. above expreſt, whether the Triangle ſought be obtuſangled, acute-angled, or right-angled at the Baſe.

It is alſo evident by the 14th and 16th ſteps of the Reſolution, that to the end the Problem propounded may be poſſible, the given lines muſt be ſubject to this

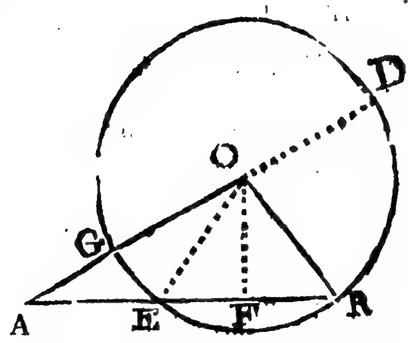
Determination.

The given Baſe muſt exceed the given difference of the leggs.

The truth of the preceding Theorems and Determination, as alſo the Compoſition of this *Probl.* 9. will be obvious to him that underſtands what hath been delivered in the foregoing *Probl.* 8. and therefore I ſhall wave the Compoſition.

Probl. X.

In a plain Triangle having unequal acute angles at the Baſe, the Perpendicular, difference of the leggs, and difference of the ſegments of the Baſe made by the Perpendicular, being ſeverally given, to find the Triangle. But the lines given muſt be ſubject to the Determinnations hereafter expreſt.



AR = 21	FR = 6
AO = 17	OF = 8
RO = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Preparat.

1. Let the Diagram belonging to the preceding *Probl.* 8. of this *Chapt.* be here repeated, and ſuppoſe the △ ARO having unequal acute angles A and R at the ends of the Baſe AR to be the Triangle ſought; then reſpect being had to the Preparatory Conſtruction in the three firſt ſteps of the ſaid *Probl.* 8. the Reſolution of this *Probl.* 10. may be formed thus;

Suppoſ.

2. p = OF the Perpendicular of △ ARO is given.
3. d = AG the difference of the leggs is given.
4. b = AE the difference of the ſegments of the Baſe is given.

Req. to find the Triangle.

Reſolution.

5. It is manifeſt that if AE the given difference of the ſegments FA, FR be eſteemed the Baſe of the △ AEO obtuſangled at E, then AG ſhall be the difference of the leggs AO and EO, as well as of AO and RO, (for EO = RO = OG,) and OF a common Perpendicular to the two Triangles ARO and AEO; therefore in △ AEO, the Baſe AE, the Perpendicular OF, and AG the difference of the leggs AO and EO being ſeverally given, the ſaid △ AEO ſhall be given by the foregoing *Probl.* 9. of this *Chapt.* For, the ſumm of the leggs, to wit, AD = AO + EO = AO + OR ſhall be given by this following Analogy, (according to *Theor.* 1. in 17° of the ſaid *Probl.* 9.) *viz.*

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; b \;.\; \text{AD}.$$

6. Then AD and AG the ſumm and difference of the leggs AO and EO, or of AO and RO, being given, the ſaid leggs ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.
7. Moreover, foraſmuch as AR in reference to △ AEO obtuſangled at E, is compos'd of the Baſe AE and ER, (= 2FE = 2FR,) the ſaid AR, which is alſo the Baſe of △ ARO, ſhall be given by *Theor.* 2. in 18° of the preceding *Probl.* 9. For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; \text{AR}.$$

From the premiſſes tis manifeſt that the Baſe and leggs of the Triangle ſought in this *Probl.* 10. may be found out by the foregoing *Probl.* 9. But that there may be a poſſibility of finding out the Triangle required, the lines given muſt be liable to theſe two following Determinations, *viz.*

Determination 1.

8. The line given for the difference of the ſegments of the Baſe made by the Perpendicular falling within the Triangle, muſt exceed the given difference of the leggs, that is, (in the Figure belonging to this *Probl.* 10.) AE ⊏ AG, the truth whereof is manifeſt by *prop.* 8. *Elem.* 3.

9. Again,

9. Again, becauſe by *Suppoſition* the Triangle ſought hath unequal acute angles at the Baſe, the Perpendicular falls within, and the Baſe muſt neceſſarily exceed the difference of the ſegments of the Baſe made by the Perpendicular; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy before expreſt in 7° muſt exceed the given line AE. Hence,

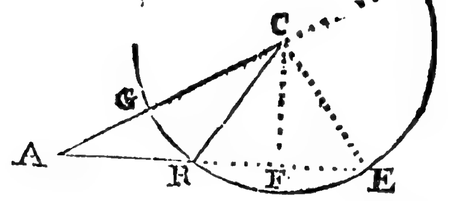
Determination 2.

10. . . . . $\dfrac{\sqrt{:4pp + bb - dd:} \times d}{\sqrt{:bb - dd:}} \sqsubset b.$

Suppoſing then the given Quantities to be qualified according to the tenour of the Determinations before preſcribed, the induſtrious Learner may eaſily apply what hath been ſaid in the foregoing 5th, 6th and 7th ſteps, as well to the Geometrical Effection, as to the Arithmetical Solution of this *Probl.* 10.

Probl. XI.

In a plain Triangle obtuſangled at the Baſe, the Perpendicular, difference of the leggs, and the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle, being ſeverally given, to find the Triangle. But the given lines muſt be ſubject to the Determinations hereafter expreſt.



AR = 9	FE = 6
AC = 17	CF = 8
RC = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Prepar.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and ſuppoſe the △ ARC obtuſangled at R, (the end of the Baſe AR,) to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in 1°, 2° and 3° of *Probl.* 9. the Reſolution of this *Probl.* 11. may be formed thus;

Suppoſ.

2. p = CF the Perpendicular of △ ARC is given.
3. d = AG the difference of the leggs AC and RC is given.
4. b = AE the line compos'd of the Baſe AR and 2FE, (or 2FR,) is given.

Req. to find the Triangle.

Reſolution.

5. It is manifeſt, that if the given line AE be eſteemed the Baſe of the △ AEC having unequal acute angles at A and E, then AG is the difference of the leggs AC and EC, as well as of AC and RC, (for EC = RC,) and CF is a common Perpendicular to the two Triangles AEC and ARC; therefore in △ AEC, the Baſe AE, the Perpendicular CF, and AG the difference of the leggs AC and EC, (or RC,) being given ſeverally, the ſaid △ AEC ſhall be given by *Probl.* 9. of this *Chapt.* For AL = AC + EC, (the ſumm of the leggs of △ AEC,) ſhall be given by this following Analogy, (according to *Theor.* 2. in 17° of the ſaid *Probl.* 9.) *viz.*

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; b \;.\; \text{AL}.$$

6. Then AL and AG the ſumm and difference of the leggs AC and EC being ſeverally given, the leggs themſelves ſhall be alſo given ſeverally, by *Theor.* 9. *Chap.* 4.
7. Moreover, becauſe AR in reference to the △ AEC is the difference of the ſegments FA and FE, made by the Perpendicular CF, and is alſo the Baſe of the △ ARC required,

required, the ſaid A R ſhall be given by *Theor.* 2. in *Probl.* 18. of *Probl.* 9. of this *Chap.* For,

$$\sqrt{:bb - dd:} \;.\; \sqrt{:4pp + bb - dd:} \;::\; d \;.\; \text{AR}.$$

From the premiſſes 'tis evident, that the three ſides of the Triangle required by this *Probl.* 11. are diſcovered by *Probl.* 9. of this *Chapt.* But the lines given muſt be ſubject to the following Determinations, that there may be a poſſibility of finding out a Triangle to ſatisfie the Problem propounded.

Determination 1.

8. The line given for the ſumm of the Baſe and double diſtance from the foot of the Perpendicular to the obtuſe angle, muſt be longer than the line given for the difference of the leggs, that is, $\text{AE} \sqsubset \text{AG}$, as may be eaſily proved; for by *Suppoſition* $\triangle$ ARC is obtuſangled at R, therefore $\text{AE} \sqsubset \text{AR}$, and conſequently AE much greater than AG, for (*per prop* 8. *Elem.* 3.) $\text{AR} \sqsubset \text{AG}$.

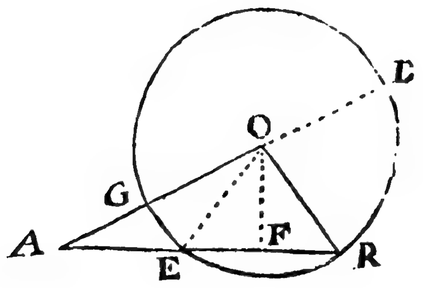
Again, becauſe by *Suppoſition* the Triangle ſought is obtuſangled at the Baſe, the Perpendicular falls without, and the Baſe ſhall neceſſarily be leſs than the line compos'd of the Baſe and the double diſtance from the foot of the Perpendicular to the obtuſe angle; therefore to the end the lines given may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy in 7°, muſt be leſs than the given line AE. Hence,

Determination 2.

$$\frac{\sqrt{:4pp + bb - dd:} \times d}{\sqrt{:bb - dd:}} \sqsupset b.$$

Probl. XII.

In a plain Triangle having unequal acute angles at the Baſe, the Perpendicular, ſumm of the leggs, and difference of the ſegments of the Baſe made by the Perpendicular, being ſeverally given, to find the Triangle. But the lines given muſt be liable to the Determinations hereafter declared.



AR = 21	FR = 6
AO = 17	OF = 8
OR = 10	AF = 15
AE = 9	AG = 7
EF = 6	AD = 27

Prepar.

1. Let the Diagram belonging to the preceding *Probl.* 8. of this *Chapt.* be here repeated, and ſuppoſe the Triangle ARO having unequal acute angles A and R at the ends of the Baſe AR to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in the three firſt ſteps of the ſaid *Probl.* 8. the Reſolution of this *Probl.* 12. may be formed thus;

Suppoſ.

2. p = OF the Perpendicular of $\triangle$ ARO is given.
3. c = AD = AO + RO the ſumm of the leggs is given.
4. b = AE = FA − FR the difference of the ſegments of the Baſe is given.

Req to find the Triangle.

Reſolution.

5. It is evident, that if the given line AE be eſteem'd the Baſe of the $\triangle$ AEO obtuſangled at E, then AD is the ſumm of the leggs AO and EO, as well as of AO and RO, for EO = RO, and OF is a common Perpendicular to the two Triangles AEO and ARO; therefore in $\triangle$ AEO, the Baſe AE, the Perpendicular OF, and AD the ſumm of the leggs AO and EO being ſeverally given, the Triangle AEO

AEO ſhall be given by the foregoing *Probl.* 8. of this *Chapt.* For AG, the difference of the leggs AO and EO, ſhall be given by this following Analogy, (according to *Theor.* 2. in 34° of *Probl* 8.)

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; b \;.\; \text{AG}.$$

6. Then AD and AG the ſumm and difference of the leggs AO and EO, (or RO,) being given ſeverally, the ſaid leggs ſhall be alſo given ſeverally, by *Theor.* 9 *Chap.* 4.
7. Moreover, foraſmuch as AR in reference to the $\triangle$ AEO obtuſangled at E, is compos'd of the Baſe AE and ER, ($= 2\text{FE} = 2\text{FR}$,) the ſaid AR, which is alſo the Baſe of $\triangle$ ARO required, ſhall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\sqrt{:cc - bb:} \;.\; \sqrt{:cc - bb - 4pp:} \;::\; c \;.\; \text{AR}.$$

From the premiſſes 'tis manifeſt, that the Baſe and leggs of the Triangle ſought by this *Probl.* 12. are diſcovered by the foregoing *Probl.* 8. But that there may be a poſſibility of finding out the deſired Triangle, the given lines muſt be ſubject to theſe two following Determinations, *viz.*

Determination 1.

$$c \sqsubset \sqrt{:bb + 4pp:} \quad \text{That is,}$$

8. The line given for the ſumm of the leggs muſt exceed that right line whoſe Square is equal to the ſumm of the Square of the given Baſe and the Square of the double of the given Perpendicular.

This Determination doth openly ſhew it ſelf in the preceding Analogy in 5°, and hath already been demonſtrated in *Probl.* 8. of this Chapter.

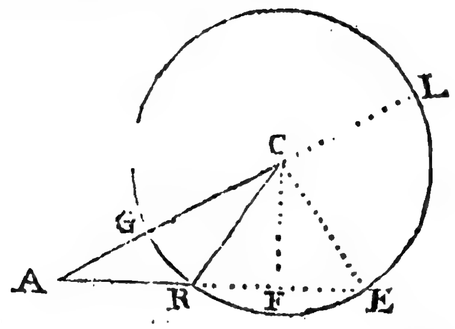
9. Again, becauſe by *Suppoſition* the Triangle ſought hath unequal acute angles at the Baſe, the Perpendicular falls within, and the Baſe muſt neceſſarily exceed the difference of the ſegments of the Baſe made by the Perpendicular, therefore to the end the given lines may be capable of effecting the Problem propounded, the fourth Proportional (or Baſe) found out by the Analogy in 7°, muſt exceed the given line AE. Hence,

Determination 2.

$$\frac{\sqrt{:cc - bb - 4pp:} \times c}{\sqrt{:cc - bb:}} \sqsubset b.$$

Probl. XIII.

In a plain Triangle obtuſangled at the Baſe, the Perpendicular, ſumm of the leggs, and the line compos'd of the Baſe and double diſtance from the foot of the Perpendicular to the obtuſe angle, being given ſeverally, to find the Triangle. But the given lines muſt be ſubject to the Determinations hereafter declared.



AR = 9	FE = 6
AC = 17	CF = 8
CR = 10	AF = 15
AE = 21	AG = 7
FR = 6	AL = 27

Preparat.

1. Let the Diagram belonging to the foregoing *Probl.* 9. of this *Chapt.* be here repeated, and ſuppoſe the $\triangle$ ARC obtuſangled at R, (the end of the Baſe AR,) to be the Triangle ſought; then reſpect being had to the preparatory Conſtruction in 1°, 2° and 3° of the ſaid *Probl.* 9. the Reſolution of this *Probl.* 13. may be formed thus;

Suppoſ.

2. p = CF the Perpendicular of $\triangle$ ARC is given.
3. c = AL = AC + RC the ſumm of the leggs is given.

4. b = AE

4. b = AE the line compos'd of the Baſe AR and 2 FR (or 2 FE) is given.
Req. to find the Triangle.

Reſolution.

5. It is manifeſt, that if the given line AE be eſteem'd the Baſe of the △ AEC having unequal acute angles at A and E, then AL is the ſumm of the leggs AC and EC, as well as of AC and RC, (for EC = RC,) and CF is a common Perpendicular to the two Triangles ARC and AEC; therefore in △ AEC, the Baſe AE, the Perpendicular CF, and AL the ſumm of the leggs AC and EC, (or RC,) being given ſeverally, the △ AEC ſhall be given by the foregoing *Probl.* 8. of this Chapter. For AG the difference of the leggs ſhall be given by this following Analogy, (according to *Theor.* 2. in 34° of *Probl.* 8.)

$$\sqrt{:}\ cc - bb\ :\ .\ \sqrt{:}\ cc - bb - 4pp\ :\ ::\ b\ .\ \text{AG}.$$

6. Then AL and AG the ſumm and difference of the leggs AC and EC being given ſeverally, the leggs ſhall alſo be given ſeverally, by *Theor.* 9. *Chap.* 4.

7. Moreover becauſe AR, in reference to the △ AEC, is the difference of the ſegments FA and FE made by the Perpendicular CF, and is alſo the Baſe of the △ ARC required, the ſaid AR ſhall be given by *Theor.* 3. in 35° of *Probl.* 8. For,

$$\sqrt{:}\ cc - bb\ :\ .\ \sqrt{:}\ cc - bb - 4pp\ :\ ::\ c\ .\ \text{AR}.$$

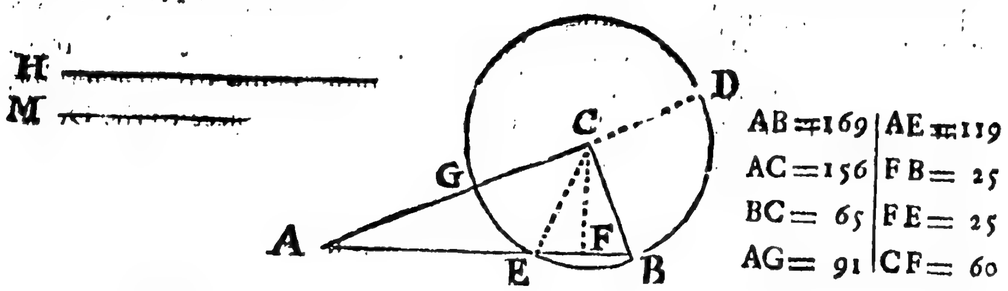
8. From the premiſſes 'tis evident, that the three ſides of the Triangle required by this *Probl.* 13. are diſcovered by *Probl.* 8. of this *Chapt.* But the lines given muſt be ſubject to the following Determinations, *viz.*

Determinat. 1. . . . . $c \sqsubset \sqrt{:}\ bb + 4pp$:

Determinat. 2. . . $\dfrac{\sqrt{:}\ cc - bb - 4pp\ : \times c}{\sqrt{:}\ cc - bb\ :} \sqsupset b.$

Probl. XIV.

The Hypothenuſal of a right-angled Triangle being given, as alſo a mean Proportional between the Baſe and Perpendicular, to find the Triangle. But the right line ariſing by the Application of the Square of the given mean to the given Hypothenuſal, muſt not be greater than half the Hypothenuſal.



AB = 169 | AE = 119
AC = 156 | FB = 25
BC = 65 | FE = 25
AG = 91 | CF = 60

Suppoſ.

1. ABC is a △ right-angled at C.
2. AC ⊏ CB.
3. CBGD is a ⊙, whence AG = AC — CB (CG.)
4. CF ⊥ AB.
5. h = AB the Hypothenuſal is given.
6. m = M a right line given, and ſuch, that AC . m :: m . CB.
Req. to find out △ ABC.

Reſolut. 1.

7. Becauſe by *Suppoſition* in 6° the right line m is a mean Proportional between AC and CB, therefore mm = □ AC, CB, that is, □ AB, CF, for *(per prop.* 41. *Elem.* 1.) each of thoſe Rectangles is equal to 2 △ ABC; therefore $\frac{mm}{h}$ = CF the Perpendicular is given; therefore alſo △ ABC ſhall be given both Geometrically and Arithmetically,

tically, by *Probl.* 16. *Chap.* 5. But here I ſhall frame the Reſolution and Compoſition of the ſame Problem after another manner.

Reſolut. 2.

8. Put a for the difference of the leggs ſought, *viz.* — a = AG = AC — CB.
9. Then becauſe AD is the ſumm, and AG the difference of the leggs AC and CB, therefore, *(per Theor.* 7. *Chap* 4.) . . . . . □M(□AC,CB) + ¼□AG = □½AD.
10. Therefore in the letters belonging to the Reſolution, the Square of half the ſumm of the leggs ſhall be equal to . . . . . $mm + \frac{1}{4}aa$ (= □ ½AD.)
11. Therefore the ſquare Root of that Square of the half ſumm ſhall be the half ſumm of the leggs, to wit, ... $\sqrt{:}\ mm + \frac{1}{4}aa$: (= ½AD.)
12. Therefore from 8° and 11°, (by *Theor.* 9. *Chap.* 4.) the greater legg is . . . . . $\sqrt{:}\ mm + \frac{1}{4}aa$: + ½a (= AC.)
13. And the leſſer legg is . . . . . $\sqrt{:}\ mm + \frac{1}{4}aa$: — ½a (= CB.)
14. Therefore from 12°, (by *Theor.* 2. *Chap.* 4.) the Square of the greater legg ſhall be . . $mm + \frac{1}{2}aa + a \times \sqrt{:}\ mm + \frac{1}{4}aa$:
15. And from 13°, (by *Theor.* 5. *Chap.* 4.) the Square of the leſſer legg ſhall be . . . . $mm + \frac{1}{2}aa - a \times \sqrt{:}\ mm + \frac{1}{4}aa$:
16. Therefore the ſumm of all in 14° and 15° gives the ſumm of the Squares of the legs, to wit, $2mm + aa$ (= □ AC + □ CB.)
17. And becauſe by *Suppoſ.* in 1° < ACB = ⊥, therefore from 5° and 16°, (*per prop.* 47. *Elem.* 1.) this Equation ariſeth, ... $2mm + aa = bb$ (= □ AB.)
18. Therefore by ſubtracting $2mm$ from each part of the laſt Equation, ... $aa = bb - 2mm$.
19. Therefore by extracting the ſquare Root out of each part of the laſt Equation, the difference of the leggs is made known, *viz.* . . . . $a = \sqrt{:}\ bb - 2mm$: (= AG.)
20. Therefore from 12°, 13°, 18° and 19°, the leggs ſhall be given ſeverally, *viz.*

$$\begin{cases} \text{AC} = \sqrt{:}\ \frac{1}{4}bb + \frac{1}{2}mm\ : + \sqrt{:}\ \frac{1}{4}bb - \frac{1}{2}mm\ : \\ \text{CB} = \sqrt{:}\ \frac{1}{4}bb + \frac{1}{2}mm\ : - \sqrt{:}\ \frac{1}{4}bb - \frac{1}{2}mm\ : \end{cases}$$

The Equation in 19° gives

CANON 1.

21. From the Square of the given Hypothen. ſubtract the double Square of the given mean Proportional, ſo the ſquare root of the remainder ſhall be the difference of the leggs ſought.

The Equations in 20° give

CANON 2.

22. To and from the Square of half the given Hypothenuſal add and ſubtract half the Square of the given mean Proportional, and reſerve the ſumm and remainder; then extract the ſquare Root out of the ſaid ſumm and remainder ſeverally; laſtly, the ſumm and difference of the ſaid ſquare Roots ſhall be the ſides about the right angle of the Triangle ſought.

Note. If the values of AC and CB (the ſides about the right angle) before expreſt in 20° be ſeverally ſquared, and the Univerſal ſquare Root extracted out of each Product, there will come forth the Canon delivered in *Sect.* 55. *Probl.* 16. *Chap.* 5. for the Arithmetical Reſolution of ſuch ambiguous Biquadratick Equations as fall under the Form there expounded.

But that the truth of the preceding Canons may more clearly appear, I ſhall propound and demonſtrate them in the form of Thorems, by a repetition of the ſteps of the foregoing Reſolution.

THEOR. 1.

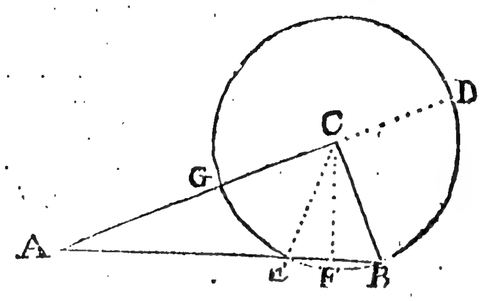
23. In a right-angled Triangle having unequal leggs about the right angle, the difference of thoſe leggs is equal to a right line whoſe Square is equal to the exceſs whereby the Square of the Hypothenuſal exceeds the double Square of a mean Proportional between the ſaid leggs.

THEOR. 2.

24. In a right-angled Triangle having unequal leggs about the right angle, the greater legg is equal to the ſumm of theſe two right lines, to wit, the right line whoſe Square is equal to the Square of half the Hypothenuſal together with half the Square of a mean Proportional between the leggs, and the right line whoſe Square is equal to the exceſs whereby the Square of half the Hypothenuſal exceeds half the Square of the ſaid mean. But the leſſer legg is equal to the difference of the ſaid two right lines.

Suppoſ.

25. A B C is a △ right-angled at C.
26. A C ⊏ C B.
27. C B G D is a ⊙; and A C D is a right line, therefore
28. A D = A C ⊣⊢ C B, and A G = A C − C B.



29. M is a right line, ſuch, that . . . . . A C . M :: M . C B.

Req. demonſtr.

30. *Theor.* 1. $AG = \sqrt{:\square AB - 2\square M:}$

31. *Theor.* 2.
$$\begin{cases} AC = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} \dashv\vdash \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \\ CB = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} - \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \end{cases}$$

Demonſtration.

32. By *Suppoſ.* in 28°, ... ⊦ $AG = AC - CB.$
33. Therefore the Square of that Equation, (*per Theor.* 5. *Chap.* 4.) gives . . . . ⊦ $\square AG = \square AC \dashv\vdash \square CB - 2\square AC,CB.$
34. By *Suppoſ.* in 25° ∠ A C B is ⊥, therefore (*per prop.* 47. *Elem.* 1.) . . . . ⊦ $\square AB = \square AC \dashv\vdash \square CB.$
35. Therefore from 33° and 34°, (*per Ax.* 6. *Chap.* 2.) . . . . . ⊦ $\square AG = \square AB - 2\square AC,CB.$
36. From 29°, (*per prop.* 17. *Elem.* 6.) . ⊦ $\square M = \square AC,CB.$
37. And conſequently, ... ⊦ $2\square M = 2\square AC,CB.$
38. Therefore from 35° and 37°, (*per Ax.* 6. *Chap.* 2.) . . . . . ⊦ $\square AG = \square AB - 2\square M.$
39. But the ſides of equal Squares are alſo equal, therefore from 38°, ... ⊦ $AG = \sqrt{:\square AB - 2\square M:}$

Which was *Theor.* 1. to be Dem.

40. Again, becauſe by *Suppoſ.* in 28°, A D is the ſumm, and A G the difference of the leggs A C and C B, therefore (*per Theor.* 7. *Chap.* 4.) ⊦ $\square\frac{1}{2}AD = \square AC,CB \dashv\vdash \frac{1}{4}\square AG.$
41. And becauſe from 29°, (*per prop.* 17. *Elem.* 6.) . . . . . ⊦ $\square M = \square AC,CB.$
42. And by taking $\frac{1}{4}$ of all in 38°, . . . ⊦ $\frac{1}{4}\square AB - \frac{1}{2}\square M = \frac{1}{4}\square AG.$
43. Therefore from 40°, 41° and 42°, (*per Ax.* 6. *Chap.* 2.) . . . . . ⊦ $\square\frac{1}{2}AD = \frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M.$
44. But the ſides of equal Squares are alſo equal, therefore from 43°, ... ⊦ $\frac{1}{2}AD = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:}$
45. And for the like reaſon, 'tis manifeſt from the Equation in 42°, that . . . . ⊦ $\frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:}$
46. Therefore, by taking the ſumm and difference of the Equations in 44° and 45°, theſe will ariſe,

$$\begin{cases} \frac{1}{2}AD \dashv\vdash \frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} \dashv\vdash \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \\ \frac{1}{2}AD - \frac{1}{2}AG = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} - \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \end{cases}$$

47. And becauſe A D is the ſumm, and A G the difference of A C and C B; therefore (*per Theor.* 9. *Chap.* 4.)

$$AC = \tfrac{1}{2}AD \dashv\vdash \tfrac{1}{2}AG; \text{ and } CB = \tfrac{1}{2}AD - \tfrac{1}{2}AG.$$

48. There-

48. Therefore from 46° and 47°, (*per Ax.* 1. *Chap.* 2.)

$$\begin{cases} AC = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} \dashv\vdash \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \\ CB = \sqrt{:\frac{1}{4}\square AB \dashv\vdash \frac{1}{2}\square M:} - \sqrt{:\frac{1}{4}\square AB - \frac{1}{2}\square M:} \end{cases}$$

Which was *Theor.* 2. to be Demonſtr.

In the next place, to the end the Geometrical Effection of the foregoing *Probl.* 14. may meet with no obſtruction, I ſhall prove the truth of the Determination annex'd to the Problem, by demonſtrating this following

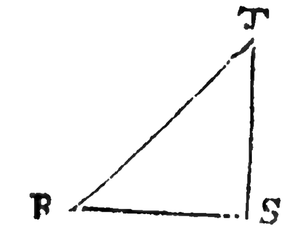
LEMMA.

49. In a right-angled Triangle, if the Square of a mean Proportional between the ſides about the right angle, (that is, if the Rectangle of thoſe ſides) be applied to the Hypothenuſal, the line thence ariſing ſhall ſometimes be equal to half the Hypothenuſal, and ſometimes leſs, but never greater than the ſaid half.

The ſides about the right angle are either equal to one another, or elſe unequal; I ſhall begin with with the firſt Caſe.

Suppoſ. in *Caſe* 1.

50. R S T is a △ right-angled at S.
51. R S = S T.
52. M is a right line, ſuch, that
53. R S . M :: M . S T.



54. . . *Req. demonſtr.* . . $\frac{\square M}{RT} = \frac{1}{2}RT.$

Demonſtration.

55. By *Suppoſition* in 51°, ... ⊦ . . R S = S T.
56. Therefore by drawing S T as a common altitude into each part, ... ⊦ $\square RS,ST = \square ST = \square RS.$
57. Therefore, (*per Ax.* 8. *Chap.* 2.) . . ⊦ $2\square RS,ST = \square ST \dashv\vdash \square RS.$
58. And becauſe by *Suppoſ.* in 50° ∠ S is ⊥, therefore, (*per prop.* 47. *Elem.* 1.) . . ⊦ . . $\square RT = \square ST \dashv\vdash \square RS.$
59. Therefore from 57° and 58°, (*per Ax.* 1. *Chap* 2.) . . . . . ⊦ $2\square RS,ST = \square RT.$
60. And conſequently, ... ⊦ $\square RS,ST = \frac{1}{2}\square RT.$
61. But from 53°, (*per prop.* 17. *Elem.* 6.) . ⊦ $\square RS,ST = \square M.$
62. Therefore from 60° and 61°, (*per Ax.* 1. *Chap.* 2.) . . . . . ⊦ . . $\square M = \frac{1}{2}\square RT = \square RT,\frac{1}{2}RT.$
63. Therefore from 62°, by Application of each part to R T, ... ⊦ . . $\frac{\square M}{RT} = \frac{1}{2}RT.$

Which was *Caſe* 1. to be Dem.

Suppoſ. in *Caſe* 2.

64. A B C is a △ right-angled at C.
65. A C ⊏ C B.
66. M a right line, ſuch, that . . . A C . M :: M . C B.

[*See the* Diagr. *in the precedent Page.*]

67. . . *Req. demonſtr.* . . . . . $\frac{\square M}{AB} \sqsupset \frac{1}{2}AB.$

Demonſtration.

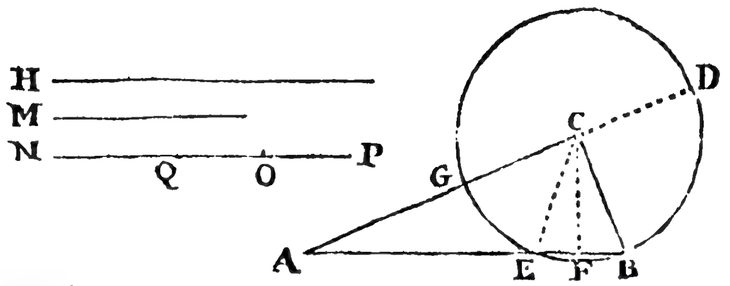
68. By the preceding *Theor.* 1. before demonſtrated 'tis evident that . . . . . ⊦ $2\square M \sqsupset \square AB.$
69. Therefore, by taking the half of each part, it follows that . . . . . ⊦ $\square M \sqsupset \frac{1}{2}\square AB.$
70. Therefore from 69°, by Application of each part to A B, ... ⊦ $\frac{\square M}{AB} \sqsupset \frac{1}{2}AB.$

Which was *Caſe* 2. to be Dem.

71. Now becauſe in every right-angled Triangle, the ſides about the right angle are either equal or unequal between themſelves, and it hath been demonſtrated, that when the ſaid ſides are equal to one another, the right line ariſing by the Square of a mean

Proportional between the ſaid ſides to the Hypothenuſal is equal to half the Hypothenuſal; but when the ſaid ſides are unequal, the ſaid right line is leſs than half the Hypothenuſal, it is evident that the right line ariſing by the ſaid Application can never be greater than half the Hypothenuſal: Therefore the truth of the *Lemma* is manifeſt, and conſequently, the lines given in *Probl.* 14. muſt be ſubject to the Determination annex'd to it, that there may be a poſſibility of effecting the Problem.

The Compoſition of the foregoing Probl. 14.



Suppoſ.

72. H = the Hypothenuſal of a right-angled Triangle is given.
73. M = a mean Proportional between the ſides about the right angle is given.
74. $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2}H$. *(Determination.)*

Req to find the Triangle.

Conſtruction.

75. By *Probl.* 2. *Chap.* 5. find a right line NO, ſuch, that its Square may be equal to $\frac{1}{4}\square H + \frac{1}{2}\square M$, therefore

$$NO = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:}$$

76. By the Determination in 74°, $\frac{\square M}{H}$ not $\sqsubset \frac{1}{2}H$; ſuppoſe then it be granted, or diſcovered by H and M given in numbers, that $\frac{\square M}{H}$ is leſs than $\frac{1}{2}H$, and conſequently, (by multiplying each part into H,) that $\square M \sqsupset \frac{1}{2}\square H$; then it evidently follows, that 'tis poſſible *(per Probl.* 4. *Chap.* 5. *)* to find out a right line OP, ſuch, that its Square may be equal to $\frac{1}{4}\square H - \frac{1}{2}\square M$, ſuppoſe therefore

$$OP = \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

77. Make NP = NO + OP, then from the Conſtruction in 75° and 76°, 'tis manifeſt that

$$NP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} + \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

78. From NO cut off OQ = OP, which may be done, for 'tis evident by *Conſtr.* in 75° and 76°, that NO $\sqsubset$ OP, ſuppoſe therefore OQ = OP, then from 75° and 76° it follows that

$$NQ = NO - OP = \sqrt{:\tfrac{1}{4}\square H + \tfrac{1}{2}\square M:} - \sqrt{:\tfrac{1}{4}\square H - \tfrac{1}{2}\square M:}$$

79. Make AC = NP; alſo CB = NQ, and CB $\perp$ AC; laſtly, draw AB.
80. I ſay ABC is the right-angled Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. Firſt then by *Conſtruction* in 79°, CB $\perp$ AC, and conſequently the angle ACB is a right angle. But that the Hypothenuſal AB is equal to the given Hypothenuſal H, and that the given right line M is a mean Proportional between AC and CB, (the ſides about the right angle,) the following Demonſtration will make manifeſt.

81. . . *Req. demonſtr.* . . . . . { AB = H. Alſo, that AC . M :: M . CB.

Demonſtration.

82. By *Conſtr.* in 77°, ... NP = NO + OP.
83. Therefore *(per Theor.* 2. *Chap.* 4.*)* . . □NP = □NO + □OP + 2□NO,OP.
84. By *Conſtr.* in 78°, ... NQ = NO − OP.
85. Therefore *(per Theor.* 5. *Chap.* 4.*)* . . □NQ = □NO + □OP − 2□NO,OP.
86. Therefore, by adding together the Equations in 83° and 85°, ... { □NP + □NQ = 2□NO + 2□OP.

87. And

87. And from the *Conſtr.* in 79°, *(per prop.* 47. *Elem.* 1.*)* . . . . . { □NP + □NQ = □AC + □CB.
88. Therefore from 86° and 87°, *(per Ax.* 1. *Chap.* 2. *)* . . . . . { 2□NO + 2□OP = □AC + □CB.
89. But by *Conſtr.* in 79°, ∠ ACB is ⊥, therefore *(per prop.* 47. *Elem.* 1. *)* . . . . { . . . . □AB = □AC + □CB.
90. Therefore from 88° and 89°, *(per Ax* 1. *Chap.* 2. *)* . . . . . { 2□NO + 2□OP = □AB.
91. But from the *Conſtr.* in 75° and 76°, (by adding together the double Squares of the Equations there expreſt,) 'tis evident that . . { 2□NO + 2□OP = □H.
92. Therefore from 90° and 91°, *(per Ax.* 1. *Chap.* 2. *)* . . . . . { □AB = □H.
93. But the ſides of equal Squares are alſo equal, therefore from 92°, ... { AB = H. Which was to be Dem.
94. Again, becauſe by *Conſtr.* in 77° and 78°, NP is the ſumm, and NQ the difference of NO and OP, therefore *(per Theor.* 8. *Chap.* 4. *)* . . { □NP,NQ = □NO − □OP.
95. But from the *Conſtr.* in 79°, ... { □NP,NQ = □AC,CB.
96. Therefore from 94° and 95°, *(per Ax.* 1. *Chap.* 2. *)* . . . . . { □AC, CB = □NO − □OP.
97. By *Conſtr.* in 75°, ... { $\frac{1}{4}\square H + \frac{1}{2}\square M = \square NO$.
98. And by *Conſtr.* in 76°, ... { $\frac{1}{4}\square H - \frac{1}{2}\square M = \square OP$.
99. And from 97°, by ſubtracting □OP from each part, ... { $\frac{1}{4}\square H + \frac{1}{2}\square M - \square OP = \square NO - \square OP$.
100. And from 98°, by adding $\frac{1}{2}\square M$ to each part, ... { $\frac{1}{2}\square M + \square OP = \frac{1}{4}\square H$.
101. And by adding the Equation in 100° to that in 99°, ... { $\frac{1}{4}\square H + \square M = \square NO - \square OP + \frac{1}{4}\square H$.
102. And from 101°, by ſubtracting $\frac{1}{4}\square H$ from each part, there will remain . . . { □M = □NO − □OP.
103. But it hath been proved in 96°, that . . { □AC, CB = □NO − □OP.
104. Therefore from 102° and 103°, *(per Ax.* 1. *Chap.* 2. *)* . . . . . { □AC, CB = □M.
105. Wherefore from 104°, *(per prop.* 14. *Elem.* 6.*)* . . . . . { AC . M :: M . CB.

Which was to be Demonſtr. Therefore that is done which the Problem required.

Note. The foregoing Problem is the ſame in effect with this, *viz.* The ſumm of the Squares and the Rectangle of two right lines being given ſeverally, to find out thoſe lines.

Probl. XV.

The Hypothenuſal and Area of a right-angled Triangle being given ſeverally, to find out the Triangle. But the right line ariſing by the Application of the double Area to the Hypothenuſal, muſt not be greater than half the Hypothenuſal.

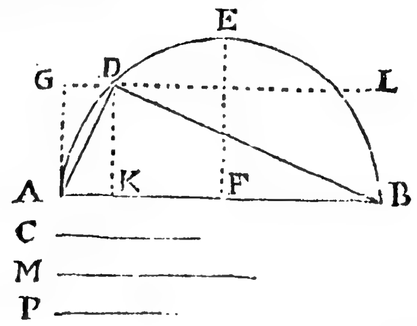
This Problem differs but little from the preceding 14th, for there, the Rectangle of the ſides about the right angle, (that is, the double Area,) and Hypothenuſal are given; but here, half the ſaid Rectangle, (that is, the Area of the right-angled Triangle ſought,) and the Hypothenuſal are given.

Suppoſ.

1. ABD is a △ right-angled at D.
2. *k* = AB the Hypothenuſal is given.
3. *c* = C a given right line, whoſe Square is equal to △ ABD, that is, □$\frac{1}{2}$BD,DA.

Req. to find out the Triangle.

Reſolu-



AB = 169	C = √5070
BD = 156	M = √10140
AD = 65	P = 60
DK = 60	
BK = 144	
KA = 25	

Reſolution.

4. Becauſe (by *prop.* 41. *Elem.* 1.) the Rectangle of BD into DA is equal to the double Area of △ ABD, therefore from 3° the ſaid Rectangle or double Area is . . . . . } $2cc$.
5. And out of 2° and 3°, (by *Theor.* 1. in 23° of the foregoing *Probl.* 14. of this *Chapt.*) the Square of the difference of the leggs about the right angle ſhall be . . . . . } $hh - 4cc$.
6. And by adding $8cc$, (to wit, four Rectangles of the leggs,) to $hh - 4cc$, (that is, the Square of the difference of the leggs,) the ſumm of that Addition gives (*per Theor.* 7. *Chap.* 4.) the Square of the ſumm of the leggs, to wit, ... } $hh + 4cc$.
7. And becauſe (by *Theor.* 3. *Chap.* 4.) a quarter of the Square of any whole right line is equal to the Square of the half; therefore from 6°, the Square of half the ſumm of the leggs ſhall be . . . . . } $\frac{1}{4}hh + cc$.
8. And conſequently from 7°, half the ſumm of the leggs is . . . } $\sqrt{:\frac{1}{4}hh + cc:}$
9. And from 5°, (by *Theor.* 3. *Chap.* 4.) the Square of half the difference of the leggs is . . . . . } $\frac{1}{4}hh - cc$.
10. And conſequently from 9°, half the difference of the leggs is . . } $\sqrt{:\frac{1}{4}hh - cc:}$
11. Therefore from 8° and 10°, (by *Theor.* 9. *Chap.* 4.) the leggs ſhall be given ſeverally, *viz.*

$$BD = \sqrt{:\tfrac{1}{4}hh + cc:} + \sqrt{:\tfrac{1}{4}hh - cc:}$$
$$DA = \sqrt{:\tfrac{1}{4}hh + cc:} - \sqrt{:\tfrac{1}{4}hh - cc:}$$

From 6° and 5° ariſeth

THEOR. 1.

12. In every right-angled Triangle having unequal ſides about the right angle, the Square of the ſumm of thoſe ſides is equal to the Square of the Hypothenuſal together with the quadruple of the Area: But the Square of the difference of the ſame ſides is equal to the exceſs whereby the Square of the Hypothenuſal exceeds the quadruple of the Area.

The Equations in 11° give

THEOR. 2.

13. In every right-angled Triangle having unequal ſides about the right-angle, if to and from the Square of half the Hypothennſal, the Area be added and ſubtracted ſeverally, and out of the ſumm and remainder ſeverally the ſquare Root be extracted, the ſumm and difference of thoſe ſquare Roots ſhall be equal to the ſides about the right angle.

The truth of the Determination annex'd to this *Probl.* 15. hath already been demonſtrated in the preceding *Probl.* 14. and the reaſon thereof will appear in the following Conſtruction.

The Compoſition of the foregoing Probl. 15.

Suppoſ.

14. AB = the Hypothenuſal of a right-angled Triangle is given.
15. C is a right line given, whoſe Square is equal to the Area of that Triangle.
16. $\frac{2\square C}{AB}$ not $\sqsubset \frac{1}{2}AB$.

Req. to find the Triangle.

Conſtru-

Conſtruction.

17. This Problem might be effected according to the direction of the foregoing Theorem in 13°, but more compendiouſly thus; Firſt, by *Probl.* 2. *Chap.* 5. find a right line M, ſuch, that its Square may be equal to $2\square C$, therefore

$$M = \sqrt{2\square C}.$$

18. Then by *Probl.* 7. *Chap.* 5. let it be made as AB to M, ſo M to a third proportional line, ſuppoſe it to be the line P, therefore

$$AB \cdot M :: M \cdot P.$$

19. Upon AB deſcribe the Semicircle FADB.
20. Make AG ⊥ AB; alſo AG = P; and GL ∥ AB, which Parallel GL ſhall neceſſarily either touch the Semicircle FADB, or cut the ſame; for by *Suppoſ.* in 16°, $\frac{2\square C}{AB}$ not $\sqsubset \frac{1}{2}AB$, and by *Conſtr.* in 17° and 18°, P is equal to $\frac{2\square C}{AB}$; therefore P is not greater than $\frac{1}{2}AB$, (= the Semidiameter FE.) But GL was before drawn parallel to AB at the diſtance of the right line P, (= AG,) and therefore the ſaid Parallel ſhall either touch the Semicircle in E, or elſe cut the ſame. Suppoſing then the Parallel GL to cut the Semicircle in D, draw the right lines AD and DB, ſo ſhall ADB be the right-angled Triangle required. But now we muſt ſhew that it will ſatisfie the Problem.
21. Firſt then by *Conſtruction* in 19°, AB the Baſe of the Triangle ADB is that which in 14° was preſcribed for the Hypothenuſal of the right-angled Triangle ſought; ſecondly, by *Conſtr.* in 19° and 20° the angle ADB is in the Semicircle FADB, and therefore 'tis a right angle, (*per prop.* 31. *Elem.* 3.) thirdly and laſtly, that the Area of the right-angled Triangle ADB is equal to the Square of the given right line C, the following Demonſtration will make manifeſt.

Prepar.

22. From the point D in the Circumference, let fall DK perpendicular to the Diameter AB.
23. . . *Req. demonſtr.* . . . . . △ADB = □C.

Demonſtration.

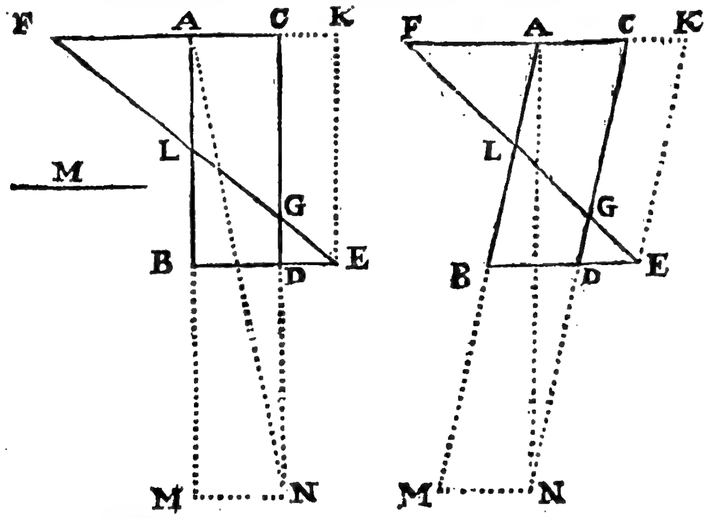
24. By *Conſtr.* in 18°, ... } AB . M :: M . P.
25. And from the *Conſtr.* in 20° and 22°, (*per prop.* 34. *Elem.* 1.) . . . . . } P = AG = DK.
26. Therefore from 24° and 25°, by taking DK inſtead of P, ... } AB . M :: M . DK.
27. Therefore from 26°, (*per prop.* 17. *Elem.* 6.) } □AB,DK = □M.
28. But by *Conſtr.* in 17°, ... } 2□C = □M.
29. Therefore from 27° and 28°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □AB,DK = 2□C.
30. And becauſe by *Conſtr.* ∠ADB is ⌟, therefore (*per prop.* 41. *Elem.* 1.) . . . . . } □AB,DK = 2△ADB.
31. Therefore from 29° and 30°, (*per Ax.* 1. *Chap.* 2.) . . . . . } 2△ADB = 2□C.
32. Therefore from 31°, (*per Ax.* 9. *Chap.* 2.) } △ADB = □C.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. XVI. (*Prop.* 164. *Lib.* 7. Pappi.)

A Parallelogram BACD being given by Poſition, from a given point E in BD produced, to draw a right line EF to concurr with CA produced in F, ſo as to make the Triangle FCG equal to the given Parallelogram BACD.

Suppoſ.



AB	=	75
BD	=	32
DE	=	20
CK	=	20
FA	=	48
EK	=	75
FK	=	100
CG	=	60
GD	=	15
AL	=	36
LB	=	39
EF	=	125 } *in Fig.* 1.

Suppoſ.

1. BACD is a Parallelogram given by Poſition.
2. E is a point given in BD produced.
3. b = AC = BD is given.
4. d = CD = AB is given.
5. c = DE is given.

Req. to find

6. AF a right line to be added to CA in a direct line, that EF being drawn, it may make △FCG = BACD.

Prepar.

7. By the given point E draw EK ‖ DC, and to concurr with AC produced in K, whence CK = DE, and EK = DC.
8. Let CD be continued to N, ſo, that DN = DC, whence CN = 2DC, and therefore △ACN = □AC,CD = BACD, *(per prop.* 41. *Elem.* 1.)

Reſolution.

9. Suppoſe that done which is required, and put — a = FA.
10. Then becauſe by *Conſtr.* in 7° EK ‖ DC, the Triangles FCG and FKE are equiangular, *(per prop.* 29. *Elem.* 1.) therefore *(per prop.* 4. *Elem.* 6.) theſe are Proportionals, *viz.* — FK . FC :: KE . CG, i.e. $a+b+c \,.\, a+b :: d \,.\, \frac{da+db}{a+b+c}$.
11. By *Conſtr.* in 8°, — △ACN = BACD.
12. And the Problem requires — △FCG = BACD.
13. Therefore from 11° and 12°, *(per Ax.* 1. *Chap.* 2.) — △ACN = △FCG.
14. And becauſe thoſe equal Triangles ACN and FCG have a common angle FCN, the ſides about that angle ſhall be reciprocally proportional, *(per prop.* 15. *Elem.* 6.) therefore — FC . AC :: CN . CG, i.e. $a+b \,.\, b :: 2d \,.\, \frac{da+db}{a+b+c}$.
15. Therefore, by halving the Antecedents in the laſt Analogy, — $\frac{1}{2}a+\frac{1}{2}b \,.\, b :: d \,.\, \frac{da+db}{a+b+c}$.
16. But it hath been ſhewn above in 10°, that — $a+b+c \,.\, a+b :: d \,.\, \frac{da+db}{a+b+c}$.
17. Therefore from 15° and 16°, *(per prop.* 11. *Elem.* 5.) — $a+b+c \,.\, a+b :: \frac{1}{2}a+\frac{1}{2}b \,.\, b$.
18. And by doubling the two latter Terms, their Reaſon is not alter'd, therefore — $a+b+c \,.\, a+b :: a+b \,.\, 2b$.
19. Therefore from the laſt Analogy by *Diviſion of Reaſon*, — $c \,.\, a+b :: a-b \,.\, 2b$.
20. Therefore by comparing the Rectangle of the means to the Rectangle of the extremes, — $aa-bb = 2bc$.

21. Therefore by adding bb to each part, — $aa = bb+2bc$ (= □FA.)
22. Therefore, by extracting the ſquare Root out of each part, — $a = \sqrt{:bb+2bc:}$ = FA.
23. Again, from 3° and 5°, reſpect being had to the Diagram, — BE (= BD + DE) = $b+c$.
24. Therefore by ſquaring each part, — □BE = $bb+cc+2bc$.
25. And by ſubtracting □DE = cc from each part, — □BE − □DE = $bb+2bc$.
26. But from 21°, — . . . □FA = $bb+2bc$.
27. Therefore from 25° and 26°, *(per Ax.* 1. *Chap.* 2.) — . . . □FA = □BE − □DE.
28. Therefore by extracting the ſquare Root out of each part, — . . . . FA = $\sqrt{:}$□BE − □DE:

The Equations in 21° and 27° do afford this

THEOREM.

29. If FC ‖ BE, and AB ‖ CD, and △FCG = BACD, then the Square of FA is equal to the Square of BD together with twice the Rectangle of BD into DE. Moreover, the Square of FA is equal to the exceſs by which the Square of BE exceeds the Square of DE.

Therefore BD and DE being given ſeverally, FA ſhall be given alſo, and conſequently EF may be drawn to ſolve the Problem propounded.

But to manifeſt the truth of the ſaid Theorem, I ſhall form a Demonſtration thereof by a repetition of the ſteps of the preceding Reſolution in a direct order, to which end, let reſpect be had to the *Diagram*, *Suppoſ.* and *Prepar.* at the beginning of the Problem.

30. . . *Req. demonſtr.* . . . □FA = □BD + 2□BD,DE = □BE − □DE.

Demonſtration.

31. Foraſmuch as △FKE and △FCG are equiangular, (for by *Conſtr.* in 7°, EK ‖ CD,) therefore *(per prop.* 4. *Elem.* 6.) — FK . FC :: KE . CG.
32. By *Conſtr.* in 8°, — △ACN = BACD.
33. And by *Suppoſ.* in 29°, — △FCG = BACD.
34. Therefore from 32° and 33°, *(per Ax.* 1. *Chap.* 2.) — △ACN = △FCG.
35. And becauſe ∠FCN is common to thoſe equal Triangles ACN and FCG, therefore, *(per prop.* 15. *Elem.* 6.) — FC . AC :: CN (or 2CD) . CG.
36. Therefore from 35°, by halving the Antecedents, — $\frac{1}{2}$FC . AC :: CD (or KE) . CG.
37. But it hath been ſhewn in 31°, that — FK . FC :: KE . CG.
38. Therefore from 36° and 37°, *(per prop.* 11. *Elem.* 5.) — FK . FC :: $\frac{1}{2}$FC . AC.
39. And from 38°, by doubling the two latter Terms, — FK . FC :: FC (or FA+AC) . 2AC.
40. And from 39°, by *Diviſion of Reaſon*, — CK . FC :: FA − AC . 2AC.
41. That is, (as is evident by the Diagram,) — DE . FA + AC :: FA − AC . 2BD.
42. Therefore from 41°, *(per prop.* 16. *Elem.* 6.) — □{FA + AC, FA − AC} = 2□BD,DE.
43. But by *Theor.* 8. *Chap.* 4. — □{FA + AC, FA − AC} = □FA − □AC (□BD.)
44. Therefore from 42° and 43°, *(per Ax.* 1. *Chap.* 2.) — □FA − □BD = 2□BD,DE.
45. Therefore from 44°, by adding □BD to each part, — □FA = □BD + 2□BD,DE.

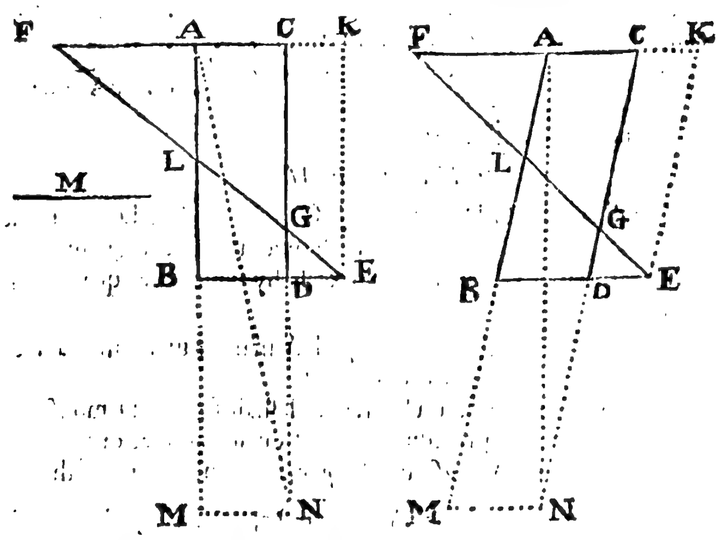
Which was to be Dem.

46. Again, becauſe by *Suppoſition* in 2°, and by the Diagram, — . BE = BD + DE.
47. Therefore by ſquaring each part of that Equation, *(per Theor.* 2. *Chap.* 4.) — □BE = □BD + □DE + 2□BD,DE.

48. And from 47°, by ſubtracting □ DE from each part, ... } □ BE — □ DE = □ BD + 2□ BD,DE.
49. Therefore from 45° and 48°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □ FA = □ BE — □ DE.

Which was alſo to be Dem. Therefore the truth of the preceding Theorem is manifeſt.

The Compoſition of the foregoing Probl. 16.



Suppoſ.

50. BACD is a Prallelogram given by Poſition.
51. E is a point given in BD continued,
52. . . *Req.* to draw EF a right line, ſuch, that △ FCG = BACD.

Conſtruction.

53. By *Probl.* 9. *Chap.* 5. find a mean proportional line M between BD and BD + 2DE, therefore BD . M :: M . BD + 2DE.

54. Produce CA to ſuch a point F, that AF may be equal to the line M, (to wit, the mean Proportional found out in 53°,) then draw a right line from E to F, ſo ſhall the Triangle FCG be equal to the Rectangle BACD, as was required; the truth whereof will evidently appear by the following Demonſtration, form'd out of the foregoing Reſolution by a repetition of its ſteps in a backward (not direct) order. But by way of Preparation, draw EK ‖ and = DC; alſo make CN = 2CD; draw AN, and produce FC to K.

55. . . *Req. demonſtr.* . . . . . △ FCG = BACD.

Demonſtration.

56. By *Conſtr.* in 53° and 54°, . . } BD . M (or FA) :: M . BD + 2DE.
57. Therefore (*per prop.* 17. *Ele.* 6.) } □ FA = □ BD + 2□ BD,DE.
58. Therefore from 57°, by ſubtracting □ BD from each part, } □ FA — □ BD = 2□ BD,DE.
59. But by *Theor.* 8. *Chap.* 4. . } □ FA — □ AC (□ BD) = □ {FA + AC, FA — AC.
60. Therefore from 58° and 59°, (*per Ax.* 1. *Chap.* 2.) . . . } □ {FA + AC, FA — AC} = 2□ BD, DE.
61. Therefore from 60°, (*per prop.* 14. *Elem.* 6.) . . . . } DE(CK) . FA+AC :: FA—AC . 2BD(2AC.)
62. That is, as is evident by the Diagram, ... } CK . FC :: FA — AC . 2AC.
63. Therefore from 62°, (by *Compoſ. of Reaſon,*) . . . . . } FK . FC :: FC(FA+AC) . 2AC.
64. And from 63°, by halving the two latter Terms, ... } FK . FC :: ½FC . AC.

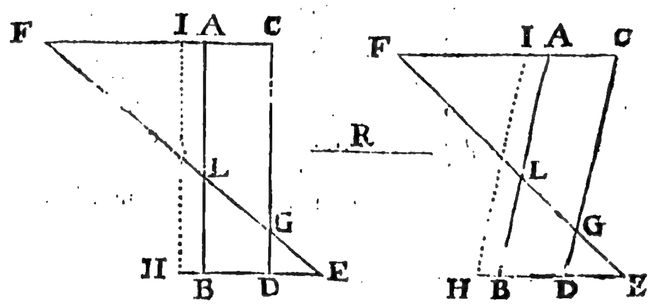
65. But

65. But becauſe △ FKE and △ FCG are like, (for EK ‖ DC,) therefore, (*per prop.* 4. *Elem.* 6.) } FK . FC :: KE (CD) . CG.
66. Therefore from 64° and 65°, (*per prop.* 11. *El.* 5.) } ½FC . AC :: CD . CG.
67. And from 66°, by doubling the Antecedents, . } FC . AC :: CN(2CD) . CG.
68. And becauſe < FCN is common to △ FCG and △ ACN, and it appears in 67°, that the ſides about that common angle are reciprocally proportional, therefore (*per prop.* 15. *Elem.* 6.) . } △ FCG = △ ACN.
69. But by *Conſtr.* in 8°, ... } BACD = △ ACN.
70. Therefore from 68° and 69°, (by *Ax.* 1. *Ch* 2.) } △ FCG = BACD.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. XVII.

A Parallelogram BACD being given by Poſition, from a given point E in BD produced, to draw a right line EF to meet with CA produced in F, that the Triangle FCG may have a given Reaſon to the Parallelogram BCAD, ſuppoſe as HD to BD.



Conſtruction.

1. By the point H draw HI ‖ BA or DC (*per prop.* 31. *Elem.* 1.) then by the laſt preceding Problem draw a right line EF, ſo as to make the Triangle FCG equal to the Parallelogram HICD, ſo ſhall △ FCG be to BACD as HD to BD, which was required; the truth whereof will be manifeſt by the following Demonſtration.

2. . . . *Req. demonſtr.* . . . . . △FCG . BACD :: HD . BD.

Demonſtration.

3. Becauſe (*per prop.* 1. *Elem* 6.) . . . . . } HICD . BACD :: HD . BD.
4. And by *Conſtr.* in 1°, ... } △ FCG = HICD.
5. Therefore from 3° and 4°, ... } △FCG . BACD :: HD . BD.

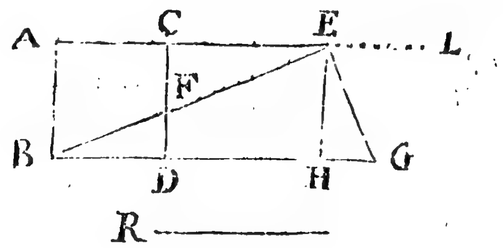
Which was to be Dem.

6. After the ſame manner, from the given point E a right line may be drawn ſo as to make the Triangle FCG equal to a given Space, ſuppoſe the Square of the right line R, by making the Parallelogram HICD equal to the Square of R, and △ FCG = HICD: For,

If by *Conſtruction* . . . . . HICD = □R,
And by *Conſtr.* . . . . . HICD = △FCG.
Then it follows (*per Ax.* 1.) that . . △FCG = □R.

Probl. XVIII. (*Prop.* 71. *Lib.* 7. Pappi.)

A Square BACD (whoſe ſide is BD or DC) being given, to draw a right line from the angle B, as BE, that may ſo cut the ſide DC, and concurr with the ſide AC produced towards L, that FE may be equal to a given right line R.



BD = HE = 60
R = FE = 91
BF = EG = 65
DG = 109
DF = HG = 25
FC = 35
DH = CE = 84

Prepar.

1. Suppoſe that done which is required, *viz.* that BE is a right line ſo drawn from the angle B that it cuts DC in F, and concurrs with AC produced in E, and makes FE equal to the given right line R.
2. Make EG perpendicular to BE, and let BD be continued until it concurr with EG in G, and from E let fall EH perpendicular to BG, whence it follows (*per prop.* 8, *&* 2. *Elem.* 6.) that

△ BEG
△ BHE
△ GHE
△ BDF
} are like (that is, equiangular) right-angled Triangles, and therefore the ſides about the equal angles are Proportionals, (*per prop* 4. *Elem.* 6.)

3. And becauſe the right-angled Triangles BDF and GHE are like, and the ſide BD (= DC) in the one, is equal to the ſide HE in the other, and the angle BFD oppoſite to BD is equal to the angle G oppoſite to HE, therefore the remaining ſides of △BDF ſhall be alſo equal to the remaining ſides of △ GHE, *viz.* each ſide to its correſpondent ſide, (*per prop.* 26. *Elem.* 1.) therefore

BF = GE, and DF = HG.

Theſe things being premiſed, the Reſolution of the Problem may be formed thus:

Suppoſ.

4. b = BD = DC = HE is given.
5. d = FE = R is given.

Reſolution.

6. For DG put a, *viz.* ſuppoſe . . . . . } . . . . a = DG.
7. And for BF (= GE) put e, *viz.* ſuppoſe . . . } . . . . e = BF = GE.
8. Then from 4° and 6°, ... } . . . $b+a$ = BG.
9. And from 7° and 5°, ... } . . . $e+d$ = BE.
10. The Square of the Equation in 7° gives . . . } . . . . ee = □ BF.
11. And the Square of the Equation in 8° gives . . . } $bb+2ba+aa$ = □ BG.
12. And the Square of the Equation in 9° gives . . . } $ee+2de+dd$ = □ BE.
13. Now becauſe by *Conſtr.* in 2°, the Triangle BEG is right-angled at E, and from 3°, □ BF = □GE, therefore from 10°, 11°, 12°, (*per prop.* 47. *Elem.* 1.) this Equation ariſeth, *viz.*

$$\underbrace{\square\,BG}_{bb+2ba+aa} = \underbrace{\square\,BE}_{ee+2ed+dd} + \square\,GE\ (\square\,BF.) + ee.$$

14. And becauſe from 2°, ... } △BEG and △BHE are like.
15. Therefore (*per prop.* 4. *Elem.* 6.) . . . . . } BG . GE :: BE . EH.
16. That is, in the letters of the Reſolution, . . . } $b+a$. e :: $e+d$. b.
17. Which Analogy being reduced to an Equation, gives } $bb+ba = ee+ed$.
18. And by ſubtracting the Equation in 17° from that in 13°, this remains, ... } $ba+aa = ee+ed+dd$.
19. And if inſtead of $ee+ed$ in the laſt preceding Equation, there be taken $bb+ba$, which in 17° appears to be equal to $ee+ed$, then the Equation in 18° will be reduced to this . . . . . } $ba+aa = bb+ba+dd$.
20. Whence, by ſubtracting ba from each part, there remains . . . . . } $aa = bb+dd$.
21. Therefore by extracting the ſquare Root out of each part of the laſt Equation, it gives . . . . . } $a = \sqrt{:bb+dd:}$ = DG.

Hence

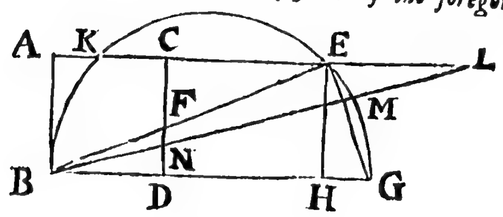
Hence this

THEOREM.

22. The right line DG is equal to that right line whoſe Square is equal to the ſumm of the Squares of BD and FE. Therefore if BD (or DC) and FE be given ſeverally, then DG is given alſo, by the help whereof the Problem may be effected.

This Theorem is demonſtrated in *Prop.* 71. of the 7th Book of *Pappus*'s Mathematical Collections, and the truth thereof is alſo manifeſt by the foregoing Reſolution, wherein the Argumentation is clearly Geometrical as well as Algebraical, and therefore there is no need of any further Demonſtration of the ſaid Theorem.

The Compoſition of the foregoing Probl. 18.



R ——
P ——

BD = HE = 60
R = FE = 91
BF = EG = 65
DG = 109
DF = HG = 25
FC = 35
DH = CE = 84

Suppoſ.

23. BACD is a Square given, whoſe ſide is BD or DC.
24. R is a right line given.
25. . . *Req.* { To draw a right line from the angle B, as BE, that may ſo cut the ſide DC, and concurr with the ſide AC produced towards L, that FE may be equal to a given right line R.

Conſtruction.

26. By *Probl.* 2. *Chap.* 5. find a right line P, ſuch, that its Square may be equal to □ BD + □ R, therefore } $P = \sqrt{:\square\,BD + \square\,R:}$
27. To BD add the line P, ſo, that BD and P may make a ſtraight line, as BDG, therefore . . . } DG = P, and BG = BD + P.
28. Upon BG deſcribe the Semicircle BKEG.
29. Let AC be continued towards L, ſo ſhall the line produced cut the Semicircle BKEG, I ſay cut it, not touch it, nor lye without it; for ½ BG is greater than BD or BA, as may be proved thus:

Becauſe by *Conſtr.* in 27°, ... DG = P.
And by *Conſtr.* in 26°, ... P ⊏ BD.
Therefore (*per Ax.* 4. *Chap.* 2.) . . . . . DG ⊏ BD.
And by adding BD to each part, ... BD + DG ⊏ 2 BD.
But by *Conſtr.* in 27°, ... BD + DG = BG.
Therefore (*per Ax.* 3. *Chap.* 2.) . . . . . BG ⊏ 2 BD.
And conſequently, ... ½ BG ⊏ BD or BA.

Which was to be proved. And therefore ACE, which is parallel to BG at the diſtance of BA ſhall neceſſarily cut the Semicircle BKEG in two points, as in K and E.

30. Laſtly, draw the right line BE, ſo ſhall FE be equal to the given right line R, as was required. But that FE = R, I demonſtrate thus;
31. . . *Req. demonſtr.* . . . . . FE = R.

Demonſtration.

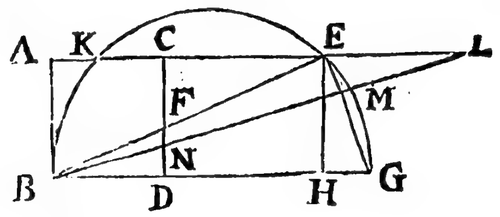
32. By *Conſtr.* in 26°, ... } □BD + □R = □P.
33. And from the *Conſtr.* in 27°, ... } . . □DG = □P.
34. Therefore (*per Ax.* 1. *Chap.* 2.) . . . . . } . . □DG = □BD + □R.
35. But by the *Theor.* in 22° of this Problem, . . } . . □DG = □BD + □FE.
36. Therefore from 34° and 35°, (*per Ax.* 1. *Chap.* 2.) } □BD + □FE = □BD + □R.
37. And from 36°, by ſubtracting □ BD from each part, } . . . □FE = □R.
38. Therefore, ... } . . . . FE = R.

Which was to be Dem.

39. But

39. But that FE is the only right line that can be found equal to the given right line R; and in such a Position as the Problem requires; I shall demonstrate in the next place; to which end, let some other line besides BE, as BL, be drawn from B to concurr with AC produced beyond E in L: Now if BL doth effect the Problem, then NL must be equal to FE; but NL is greater than FE, as will be made manifest by the following Demonstration.

40. . . . *Req. demonstr.* . . . . . NL ⊏ FE.



R ——

P ——

Demonstration.

41. Because by *Suppos.* . . . . . ∠BAL is ⌟.
42. Therefore (*per prop.* 47. *Elem.* 1.) . . . . . □BL = □BA + □AL.
43. And in like manner, ... □BE = □BA + □AE.
44. By *Suppos.* in 39°, AL ⊏ AE, and consequently, . □AL ⊏ □AE.
45. Therefore from 42°, 43° and 44°, (*per Ax.* 4. *Chap.* 2.) □BL ⊏ □BE.
46. Therefore from 45°, ... BL ⊏ BE.
47. Again, because by *Suppos.* . . . . . ∠BDC is ⌟.
48. Therefore (*per prop.* 47. *Elem.* 1.) . . . . . □BE = □BD + □DF.
49. And in like manner, ... □BN = □BD + □DN.
50. And because DF ⊏ DN, and consequently, . . □DF ⊏ □DN.
51. Therefore from 48°, 49° and 50°, ... □BF ⊏ □BN.
52. Therefore from 51°, ... BF ⊏ BN.
53. And consequently, ... BN ⊐ BF.
54. And because it hath been shewn in 46°, that . . . BL ⊏ BE.
55. Therefore from 52°, 53°, 54°, (*per Ax.* 16. *Chap.* 2.) BL − BN ⊏ BE − BF.
56. That is, (as is evident by the Diagram,) . . . NL ⊏ FE.

Which was to be Dem. And therefore BL will not effect the Problem propounded. The like Demonstration will hold good in comparing BE to any other right line that shall be drawn from B to cut DC, and to concurr with AC produced.

57. Here the Learner may observe, that in resolving a Problem by the Algebraick Art, there may often-times be found out various Equations so constituted, that every one of them may be capable of solving the Problem, but the simplest of those Equations is to be preferr'd before the rest, and chiefly to be aim'd at, though for the most part 'tis much harder to be discovered than those more compounded. As, in the foregoing *Probl.* 18. among various Equations that may be found out to solve the same, that in 21° at the end of the preceding Resolution is the simplest. But who would think, that the way to solve that Problem is to search out the quantity of the line DG, and not rather of one of these lines, to wit, BE, BF, AE, CE, DF? for by any one of these lines, from the consideration of the like right-angled Triangles BAE, FCE, BDF we may come to an Equation more easily than by the line DG, but the Geometrical Construction of such Equation will be much harder than that of the Equation whereby DG is before discovered: And because the Equation resulting upon the search of any of the said five lines, to wit, BF, BE, AE, CE, DF falls under a higher Form than any of the Equations expounded in this Book, I shall referr the more curious Reader for satisfaction concerning the same, to *Pag.* 82, 83, 84 of *Renatus des Cartes*'s Geometry, set forth by *Fran. van Schooten* in 1659. yet I shall here shew how the quantities of those five lines before mentioned are also deducible from the preceding Resolution.

First then, the same things being supposed as before, draw EG, and make EH ⊥ BG; then let the Equations in the preceding 17th and 21th steps be here repeated, *viz.*

58. It

58. It hath been shewn in 17°, that . . . . . } $bb + ba = ee + ed.$
59. And in 21°, that . . . . . } $a = \sqrt{:bb + dd}:$
60. Therefore if $\sqrt{:bb+dd}:$ instead of a be drawn into b, the Equation in 58° will be reduced to this, *viz.* . . . . . } $bb + b\sqrt{:bb+dd}: = ee + ed.$
61. Which last Equation may be reduced into these three Proportionals, *viz.* . . . . . } $e \mid d, \sqrt{:bb+b\sqrt{bb+dd}}, e \div.$

62. Of which three Proportionals, the mean, to wit, $\sqrt{:bb+b\sqrt{bb+dd}}:$ is given, as also d the difference of the extremes $e+d$ and e, therefore the extremes shall be given severally, by the *Theor* in 24° of *Probl.* 12. *Chap.* 5. *viz.*

$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb+dd}}: - \tfrac{1}{2}d = e = BF.$$
$$\sqrt{:\tfrac{1}{4}dd + bb + b\sqrt{bb+dd}}: + \tfrac{1}{2}d = e + d = BE.$$

63. And because EH (= DC = DB) is a mean Proportional between BH and HG, whose summ is BG; which mean and summ of the extremes are represented in the preceding Resolution by b and $b+a$, whereof b is given in 4°, and a in 21°, for 'tis there found equal to $\sqrt{:bb+dd}:$ and consequently $b+a = b+\sqrt{:bb+dd}:$ therefore by the help of the said given mean b, and the said given summ of the extremes, to wit, $b+\sqrt{:bb+dd}:$ the extremes BH (= AE) and HG (= DF) shall be given severally by the *Theor.* in 21° of *Probl.* 13. *Chap.* 5. *viz.*

$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd}: + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: = BH = AE.$$
$$\tfrac{1}{2}b + \sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd}: - \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: = HG = DF.$$

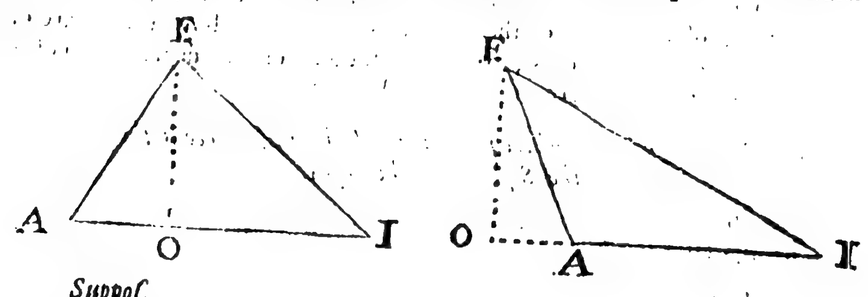
64. And because CE = DH = BH − BD, and BH and BD are given as before, therefore CE (= DH) is given also, *viz.*

$$\sqrt{:\tfrac{1}{4}bb + \tfrac{1}{4}dd}: + \sqrt{:\tfrac{1}{4}dd - \tfrac{1}{2}bb + \tfrac{1}{2}b\sqrt{bb+dd}}: - \tfrac{1}{2}b = CE.$$

Lastly, for the better illustration of the premisses, I have calculated whole Numbers (placed near the Diagram) to express the Quantities of all the Right lines given and sought in this *Probl.* 18.

A LEMMA, *leading to the following* Probl. 19.

If in any oblique-angled plain Triangle, any one of the three sides be called the Base, and the other two the leggs; Then, as the Radius, (or total Sine,) is to the Sine complement of the angle contain'd under the leggs; so is the double Rectangle of the leggs, to the difference between the summ of the Squares of the leggs, and the Square of the Base.



Suppos.

1. Let EI be the Base, } of the oblique-angled Triangle AEI.
2. AE and AI the leggs }
3. ∠A (that is, ∠EAI) is contain'd under the leggs AE, AI.
4. EO ⊥ AI.
5. R = the Radius, or total Sine.
6. Sc. ∠A = the Sine complement of the Angle A, (or ∠EAI,) that is, the Sine of the angle AEO.

Req. demonstr.

7. { If ∠A be acute, then, R . Sc. ∠A :: 2 □AE,AI . □AE + □AI − □EI.
   { If ∠A be obtuse, then, R . Sc. ∠A :: 2 □AE,AI . □EI − □AE − □AI.

Demonstr.

Demonſtration.

8. By a vulgar Axiom in the Doctrine of plain Triangles, . . } R . Sc.<A :: AE . AO.
9. Therefore by taking the common altitude AI, ... } R . Sc.<A :: □AE, AI . □AO, AI.
10. And by doubling the two latter Terms, ... } R . Sc.<A :: 2□AE,AI . 2□AO,AI.
11. And becauſe by *Suppoſition* in *Caſe* 1. <A is acute, therefore (*per prop.* 13. *Elem.* 2.) . . } ⧈AE + ⧈AI — ⧈EI = 2 □AO, AI.
12. Therefore, from 10° and 11°, by exchanging equal quantities, } R . Sc.<A :: 2□AE,AI . ⧈AE+□AI—⧈EI.

Which was to be Dem.

13. But when <A is obtuſe, then (*per prop.* 12. *Elem.* 2.) . . . } ⧈EI — □AE — ⧈AI = 2 □AO, AI.
14. Therefore from 10° and 13°, by exchanging equal quantities, } R . Sc.<A :: 2□AE,AI . ⧈EI—□AE—⧈AI

Which was alſo to be Dem. Therefore the truth of the *Lemma* is manifeſt. Hence this

COROLLARY.

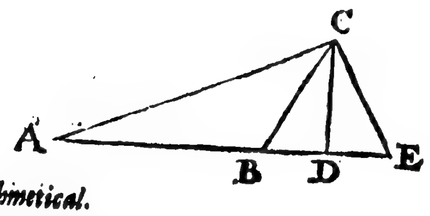
15. If in an oblique-angled plain Triangle the three ſides be given ſeverally, the angles ſhall alſo be given ſeverally, without the help of the Perpendicular; for if the ſide oppoſite to an angle ſought be called the Baſe, and the other two ſides the leggs, then as the double Rectangle of the leggs is to the difference between the ſumm of the Squares of the leggs and the Square of the Baſe; ſo is the Radius to the Sine complement of the angle oppoſite to the Baſe. Which angle is acute when the Square of the Baſe is leſs than the ſumm of the Squares of the leggs; but obtuſe when greater.

An Example in Numbers, where the ſumm of the Squares of the leggs exceeds the Square of the Baſe.

Suppoſ. in △ AEC.

16. AC = 7, the Baſe is given.
17. AE = 8, } the leggs are given.
18. EC = 3, }

Req. to find <E.



Solution Arithmetical.

19. By the preceding Corollary; As 48 the double Rectangle of the leggs, AE, EC, is to 24 the exceſs of the ſumm of the Squares of the leggs above the Square of the Baſe AC; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whoſe complement 60. degrees is the meaſure of the angle E ſought.

An Example in Numbers, where the Square of the Baſe exceeds the ſumm of the Squares of the leggs.

Suppoſ. in △ ABC.

20. AC = 7, the Baſe is given.
21. AB = 5, } the leggs are given.
22. BC = 3, }

Req. to find < ABC.

Solution Arithmetical.

23. By the preceding Corollary; As 30, the double Rectangle of the leggs, AB, BC, is to 15, the exceſs of the Square of the Baſe above the ſumm of the Squares of the leggs; So is 100000 the Radius, to 50000 the Sine of 30. degrees, whoſe complement 60. degrees ſubtracted from 180. degrees, leaves 120. degrees for the angle ABC ſought.

Note. Becauſe in this ſecond Example the Square of the Baſe exceeds the ſumm of the Squares of the leggs, the angle ſought is obtuſe; and therefore the complement of the angle relating to the Sine which is the fourth Proportional of the before-mentioned Analogy, being ſubtracted from 180. degrees, leaves the angle ſought. But when the ſumm of the

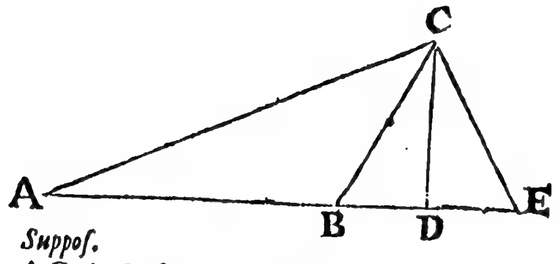
the Squares of the leggs exceeds the Square of the Baſe, then the complement it ſelf of the angle relating to the ſaid Sine (or fourth Proportional) is the angle ſought; as in the firſt Example.

Probl. XIX.

The Baſe (that is, any ſide) of a plain Triangle being given, as alſo the angle oppoſite to the Baſe, and the ſumm of the ſides (or leggs) containing that angle, to find the Triangle. But the given ſumm of the leggs muſt exceed the given Baſe, (*per prop.* 22. *Elem.* 1.)

Note. Becauſe in this Problem the given angle is not of the ſame kind with the right lines given, a right line is to be found out, by the help of that angle, which may ſtand inſtead of the angle: To which end, let a Circle be deſcribed at any diſtance, and make an angle at the Center equal to the given angle, then from one end of that arch of the Circumference which is the meaſure of the ſaid angle at the Center, let fall a Perpendicular upon a Diameter drawn to the other end of the ſaid arch, ſo is the ſaid Perpendicular the Sine of the given angle, and the ſegment of the Diameter between the foot of the ſame Perpendicular and the Center of the Circle is the Sine-complement of the given angle. Now inſtead of the given angle the ſaid Sine-complement may be taken, by the help whereof, and of the Radius (or Semidiameter) of the ſaid Circle, the Reſolution of the Problem propounded may be formed in manner following.

The Problem hath three Caſes, for the given angle oppoſite to the Baſe given is either right, or acute, or obtuſe; the firſt of thoſe Caſes hath already been ſolved in *Probl* 4. of this *Chapt.* I ſhall therefore begin with the ſecond Caſe, which ſuppoſeth the given angle to be acute, and the leggs containing that angle to be unequal.



AC = 7
AE = 8
EC = 3 = BC
AB = 5
<E = 60 degrees.
<ABC = 120 degrees.

Suppoſ.

1. *b* = AC the Baſe of △ ACE is given.
2. *c* = AE + EC the ſumm of the leggs is given.
3. <E oppoſite to the Baſe AC is acute, and given.
4. *r* = the Radius (or total Sine) is given.
5. *d* = the Sine-complement of <E is given.

Req. to find out the Triangle.

Reſolution.

6. Suppoſe ACE to be the △ ſought, and put *a* for the difference of the leggs AE, EC; *viz.* aſſume . . . . . } *a* = AE — EC.
7. Therefore from 2° and 6°, (*per Theor.* 9. *Chap.* 4.) the greater legg ſhall be . . . . . } ½*c* + ½*a* = AE.
8. And (by the ſame *Theor.*) the leſſer legg ſhall be . . . . . } ½*c* — ½*a* = EC.
9. Therefore the double Product of the leggs is . . . . . } ½*cc* — ½*aa*.
10. And the ſumm of the Squares of the leggs is . . . . . } ½*cc* + ½*aa*.
11. And becauſe the given angle E is acute, the ſumm of the Squares of the leggs exceeds the Square of the Baſe, (*per prop.* 13. *Elem.* 2.) therefore ½*cc* + ½*aa* exceeds *bb*, and the exceſs it ſelf is . . } ½*cc* + ½*aa* — *bb*.
12. And from 4°, 5°, 9° and 11°, this Analogy is manifeſt, (by the *Lemma* prefixt before this *Probl.*) viz.

r . *d* :: ½*cc* — ½*aa* . ½*cc* + ½*aa* — *bb*.

13. Therefore from that Analogy, by Compoſition of Reaſon converſe,

r + *d* . *r* :: *cc* — *bb* . ½*cc* — ½*aa*.

14. And

14. And by doubling the two latter Terms of the laſt Analogy,

$$r + d \;.\; r \;::\; 2cc - 2bb \;.\; cc - aa.$$

Hence

CANON 1.

15. When the given angle is acute, let it be made, as the ſumm of the Radius and the Sine-complement of the given angle is to the Radius, ſo the exceſs by which the double Square of the given ſumm of the leggs exceeds the double Square of the Baſe, to a fourth Proportional. Then ſubtract that fourth Proportional from the Square of the ſumm of the leggs, and the ſquare Root of the remainder ſhall be the difference of the leggs ſought. Laſtly, the ſumm, as alſo the difference of the leggs being given, the leggs ſhall be given ſeverally by *Theor. 9. Chap. 4.*

16. But when the given angle is obtuſe, then the Square of the Baſe exceeds the ſumm of the Squares of the leggs, (*per prop. 12. Elem. 2.*) in which Caſe, $\frac{1}{2}cc + \frac{1}{2}aa$ in the 10th ſtep muſt be ſubtracted from bb the Square of the Baſe, and the remainder will be $bb - \frac{1}{2}cc - \frac{1}{2}aa$, ſo inſtead of the Analogy in the 12th ſtep, this ariſeth,

$$r \;.\; d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; bb - \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

17. And becauſe $r \sqsubset d$, (for the Radius or total Sine is greater than any other Sine,) therefore from the laſt preceding Analogy, by Converſion of Reaſon,

$$r \;.\; r - d \;::\; \tfrac{1}{2}cc - \tfrac{1}{2}aa \;.\; cc - bb.$$

18. Therefore inverſly,

$$r - d \;.\; r \;::\; cc - bb \;.\; \tfrac{1}{2}cc - \tfrac{1}{2}aa.$$

19. And by doubling the two latter Terms of the laſt Analogy,

$$r - d \;.\; r \;::\; 2cc - 2bb \;.\; cc - aa.$$

Hence

CANON 2.

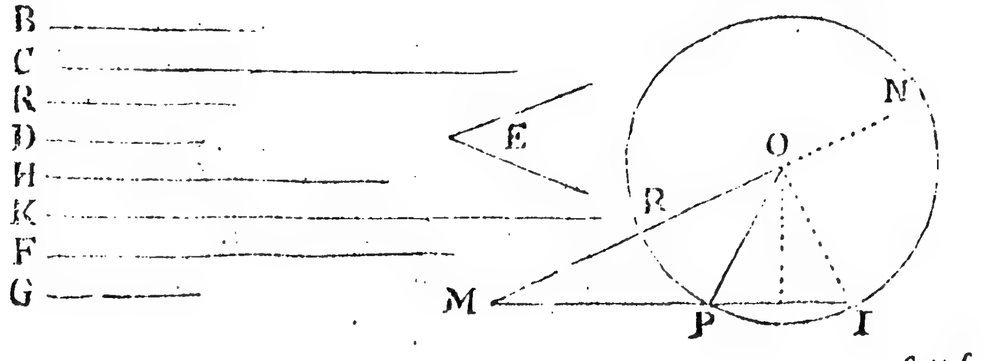
20. When the given angle is obtuſe, let it be made, As the exceſs by which the Radius exceeds the Sine-complement of the given angle is to the Radius; So the exceſs by which the double Square of the given ſumm of the leggs exceeds the double Square of the Baſe, to a fourth Proportional. Then ſubtract that fourth Proportional from the Square of the ſumm of the leggs; and the ſquare Root of the remainder ſhall be the difference of the leggs ſought. Laſtly, from the ſumm and difference of the leggs, the leggs ſhall be given ſeverally, by *Theor. 9. Chap. 4.*

From the preceding Canons in 15° and 20° this following Theorem is deducible, and eaſie to be demonſtrated by the ſteps of the foregoing Reſolution in a direct order, *viz.* by proceeding from the beginning to the end of the Reſolution.

THEOREM.

21. If any one of the three ſides of an oblique-angled plain Triangle be called the Baſe, and the other two ſides (or leggs) be unequal, then the exceſs of the double Square of the ſumm of the leggs above the double Square of the Baſe, ſhall be to the exceſs of the Square of the ſumm of the leggs above the Square of their difference, as the ſumm of the Radius and Sine-complement of the angle oppoſite to the Baſe is to the Radius, when the ſaid angle is acute; but as the exceſs of the Radius above the ſaid Sine-complement is to the Radius, when the ſaid angle is obtuſe.

The Compoſition of the foregoing Probl. 19. *when the given angle is acute.*



Suppoſ.

Suppoſ.

22. B = the Baſe of a Triangle is given.
23. C = the ſumm of the leggs is given.
24. C ⊏ B. (*Determination.*)
25. ∠ E = to the angle oppoſite to the Baſe is acute, and given.
26. R = the Radius or Semidiameter of a Circle is given.
27. D = the Sine-complement of the angle E, where the Radius is equal to the line R, is given.

Req. to make the Triangle.

Conſtruction.

28. Find a right line H that may be equal to R + D.
29. By *Probl. 4. Chap. 5.* find a right line K, ſuch, that its Square may be equal to $2\square C - 2\square B$, which effection is poſſible, for by *Suppoſition* C ⊏ B, therefore

$$\square K = 2\square C - 2\square B.$$

30. By *Probl. 11. Chap. 5.* let it be made as H to R, ſo □K to another Square, whoſe ſide ſuppoſe to be found F, therefore

$$H \;.\; R \;::\; \square K \;.\; \square F.$$

That is, in 14°, $r + d \;.\; r \;::\; 2cc - 2bb \;.\; cc - aa.$

31. Find a right line G, ſuch, that its Square may be equal to $\square C - \square F$, which effection is poſſible if C ⊏ F; but that C is greater than F, I prove thus,

By the Theorem in the preceding 21th ſtep, □F the laſt Term of the Analogy in 30° is equal to the exceſs of □C above the Square of the difference of the leggs, therefore □C = □F + the Square of the ſaid difference, whence 'tis manifeſt that □C ⊏ □F, and conſequently C ⊏ F, therefore 'tis poſſible to find a right line G, ſuch, that

$$\square G = \square C - \square F \;(= aa.)$$

Thus far the Conſtruction hath been made according to the direction of *Canon 1.*

32. Now let a Triangle be made of theſe three right lines, to wit, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, which effection is poſſible (*per prop. 22. Elem. 1.*) if C ⊏ G, and the ſumm of every two of thoſe three lines be greater than the third; but theſe things may be made manifeſt thus,

Firſt, it hath been proved in 31° that C ⊏ G, and conſequently $\frac{1}{2}C - \frac{1}{2}G$ (one of the above-mentioned three lines,) is greater than nothing, and therefore equal to ſome real right line.

Secondly, it is manifeſt that the ſumm of B and $\frac{1}{2}C + \frac{1}{2}G$ is greater than $\frac{1}{2}C - \frac{1}{2}G$.

Thirdly, the ſumm of $\frac{1}{2}C + \frac{1}{2}G$ and $\frac{1}{2}C - \frac{1}{2}G$ makes C, which by *Suppoſition* is greater than B.

Fourthly, that the ſumm of B and $\frac{1}{2}C - \frac{1}{2}G$ is greater than $\frac{1}{2}C + \frac{1}{2}G$, that is, B ⊏ G, I prove thus;

By *Conſtr.* in 30°, ...	H . R :: □K . □F.
That is, as appears by *Conſtr.* in 28°,	R + D . R :: $2\square C - 2\square B$. $\square C - \square G$.
29° and 31°, ...	
Therefore by Diviſion of Reaſon, .	D . R :: $\square C + \square G - 2\square B$. $\square C - \square G$.
And inverſly, ...	R . D :: $\square C - \square G$. $\square C + \square G - 2\square B$.
But R ⊏ D, therefore from the laſt preceding Analogy, (*per Schol prop. 14. Elem 5.*)	$\square C - \square G \sqsubset \square C + \square G - 2\square B$.
And by adding □G to each part, .	$\square C \sqsubset \square C + 2\square G - 2\square B$.
And by adding $2\square B$ to each part,	$2\square B + \square C \sqsubset \square C + 2\square G$.
And by ſubtracting □C from each part,	$2\square B \sqsubset 2\square G$.
And by halving each part, ...	$\square B \sqsubset \square G$.
Therefore, ...	B ⊏ G. Which was to be Dem.

33. Now ſince it hath been ſhewn that the ſumm of every two of theſe three right lines, B, $\frac{1}{2}C + \frac{1}{2}G$, and $\frac{1}{2}C - \frac{1}{2}G$, is greater than the third, 'tis poſſible to make a Triangle of thoſe three lines; ſuppoſe it therefore done, and that the Triangle ſo made is MOP, and that MP is equal to B, $MO = \frac{1}{2}C + \frac{1}{2}G$, and $OP = \frac{1}{2}C - \frac{1}{2}G$, then ſhall MOP be the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem. First then by *Conſtruction*, MP = B the given Baſe; likewiſe by *Conſtr.*

. MO -|- OP = C the line preſcribed for the ſumm of the leggs; it remains only to prove that the angle MOP is equal to the given angle E; but that will be made manifeſt by the following Demonſtration, which is formed out of the ſteps of the foregoing Reſolution, by returning backwards from the 14th ſtep to the 11th.

Prepar.

See the laſt preceding Diagram.

34. From the Center O, at the diſtance of OP, deſcribe the Circle OPRNS, and and produce MO to N in the Circumference; therefore MN = MO -|- OP (ON) = C, and RM = OM — OP (OR) = G.

35 . . *Req. demonſtr.* . . . . . ∠MOP = ∠E.

Demonſtration.

36. By *Conſtr.* in 30°, ... H . R :: □K . □F.

37. Therefore from 36°,28°,29°, 31°, by exchange of equal quantities, ... R-|-D . R :: 2□C—2□B . □C—□G.

That is, in 14°, ... *r*-|-*d* . *r* :: 2*cc* — 2*bb* . *cc* — *aa*.

38. And becauſe by *Conſtr.* in 33° and 34°, ... MN = C . MP = B . MR = G.

39. Therefore from 37° and 38°, by exchanging equal quantities, R-|-D . R :: 2□MN—2□MP . □MN—□MR.

40. And by halving the two latter Terms, ... R-|-D . R :: □MN—□MP . ½□MN—½□MR.

That is, in 13°, ... *r*-|-*d* . *r* :: *cc* — *bb* . ½*cc* — ½*aa*.

41. Therefore from 40°, by Diviſion of Reaſon,
D . R :: ½□MN -|- ½□MR — □MP . ½□MN — ½□MR.

42. Therefore inverſly,
R . D :: ½□MN — ½□MR . ½□MN -|- ½□MR — □MP.

That is, in 12°, ... *r* . *d* :: ½*cc* — ½*aa* . ½*cc* -|- ½*aa* — *bb*.

43. And becauſe by *Theor.* 7. *Chap.* 4. . . . . 2□OM,OP = ½□MN — ½□MR.

44. And by *Theor.* 6. *Chap.* 4. □OM -|- □OP = ½□MN -|- ½□MR.

45. Therefore from 42°,43°,44°, by exchanging equal quantities, R . D :: 2□OM,OP . □OM-|-□OP—□MP.

46. By the laſt Term of the laſt preceding Analogy 'tis evident that . . . . . □OM -|- □OP ⊏ □MP.

47. Therefore in △MOP, (*per prop* 13. *Elem.* 2.) . . . ∠MOP is acute.

48. Therefore in △MOP, (by the *Lemma* prefixt before this *Probl.*) Rad. Sin.comp.∠MOP :: 2□OM,OP. □OM+□OP—□MP

49. Therefore from 45° and 48°, (*per prop.* 11. *Elem.* 5.) . . R . D :: Rad. Sin. comp. ∠MOP.

50. But by *Suppoſ.* in 26° and 27°, R . D :: Rad. Sin. comp. ∠E.

51. Therefore from 49° and 50°, (*per prop.* 11. *Elem.* 5.) . . Rad. Sin. comp. ∠MOP :: Rad. Sin. comp. ∠E.

52. Therefore, (*per prop.* 14. *El.* 5.) Sin. comp. ∠MOP = Sin. comp. ∠E.

53. Therefore from 51° and 52°, (*per prop* 28. *Elem.* 3.) . . ∠MOP = ∠E. Which was to be Dem.

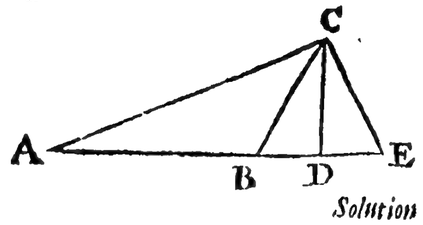
After the ſame manner, the third *Caſe* of *Probl.* 19. (*viz.* when the given angle is obtuſe,) may be Geometrically effected and demonſtrated.

Examples in Numbers, to illuſtrate the foregoing Reſolution of Probl. 19.

Example 1. *where the given angle is acute.*

Suppoſ. in △ AEC.

54. . . AC = 7, the Baſe is given.
55. AE-|-EC = 11, the ſumm of the leggs is given.
56. . . ∠E = 60 degrees is given.
Req. to find AE and EC ſeverally.



Solution

Solution Arithmetical.

57. Suppoſe the Radius of a Circle to be . . . . . 100000.
58. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees, is . . . . . 50000.
59. Therefore the ſumm of the Radius and that Sine-complement is . 150000.
60. Then (by *Canon* 1. in the preceding 15th ſtep,) As the ſaid ſumm 150000 is to the Radius 100000; So is 144, (the exceſs whereby 242 the double Square of 11 the given ſumm of the leggs exceeds 98 the double Square of the given Baſe 7,) to a fourth Proportional 96, which ſubtracted from 121 the Square of the given ſumm of the leggs, leaves 25, whoſe ſquare Root 5 is the difference of the leggs: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AE and EC ſhall be 8 and 3.

Example 2. *where the given Angle is obtuſe.*

Suppoſ. in △ ABC.

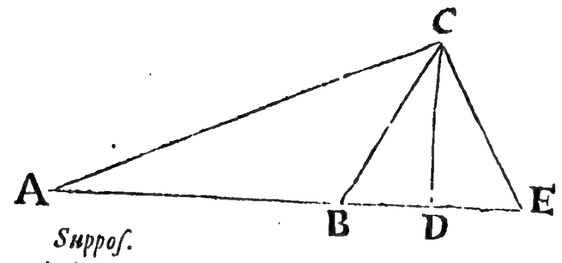
61. . . . AC = 7, the Baſe is given.
62. AB -|- BC = 8, the ſumm of the leggs is given.
63. . ∠ABC = 120 degrees is given.
Req. to find AB and BC ſeverally.

Solution Arithmetical.

64. Suppoſe the Radius of a Circle to be . . . . . 100000.
65. Then the Sine-complement of the given angle ABC 120. degrees, that is, the Sine of 30. degrees, is . . . . . 50000.
66. Therefore the exceſs of the Radius above the Sine-complent is . . 50000.
67. Then (by *Canon* 2. in the preceding 20th ſtep,) As the ſaid exceſs 50000 is to the Radius 100000, So is 30 (the exceſs whereby 128 the double Square of 8 the given ſumm of the leggs exceeds 98 the double Square of the given Baſe 7,) to a fourth Proportional 60; which ſubtracted from 64 the Square of the given ſumm of the leggs, leaves 4, whoſe ſquare Root 2 is the difference of the leggs ſought: Therefore (*per Theor.* 9. *Chap.* 4.) the leggs themſelves, to wit, AB and BC ſhall be 5 and 3.

Probl. XX.

The Baſe of a plain Triangle being given, as alſo the angle oppoſite to the Baſe, and the difference of the ſides (or leggs) containing that angle, to find the Triangle. But the given Baſe muſt be greater than the given difference.



Suppoſ.

AC = 7
AE = 8
EC = 3 = BC.
AB = 5
∠E = 60 degrees.
∠ABC = 120 degrees.

1. *b* = AC the Baſe of △ ACE is given.
2. *c* = AE — EC the difference of the leggs is given.
3. ∠E oppoſite to the Baſe AC is acute, and given.
4. *r* = the Radius (or total Sine) is given.
5. *d* = the Sine-complement of ∠E is given.
Req. to find AE and EC ſeverally.

Reſolution.

6. For the ſumm of the leggs AE and EC, put *a*, *viz.* ſuppoſe . . . . . *a* = AE + EC.

7. Then out of 1°,2°,4°,5° and 6°, (by the *Theor.* in 21° of the foregoing *Probl.* 19.) this following Analogy will ariſe, ... *r*+*d* . *r* :: 2*aa* — 2*bb* . *aa* — *cc*.

8. Therefore

8. Therefore by doubling the Consequents, (*per Schol.* Clavii *in prop.* 22. *Elem.* 5.) . . } $r+d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$
9. And by halving each of the two latter Terms, } $r+d \,.\, 2r :: aa-bb \,.\, aa-cc.$
10. And inversly, ... } $2r \,.\, r+d :: aa-cc \,.\, aa-bb.$
11. Therefore by Conversion and Inversion of Reason, ... } $r-d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 1.

12. When the given angle is acute, let it be made, As the excess of the Radius above the Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs, to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ, as also the difference of the leggs being given, the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

13. But when the given angle is obtuse, then by the *Theorem* in 21° of the preceding *Probl.* 20. this Analogy will arise, ... } $r-d \,.\, r :: 2aa-2bb \,.\, aa-cc.$
14. Therefore by doubling the Consequents, . } $r-d \,.\, 2r :: 2aa-2bb \,.\, 2aa-2cc.$
15. And by halving each of the two latter Terms, } $r-d \,.\, 2r :: aa-bb \,.\, aa-cc.$
16. Therefore inversly, ... } $2r \,.\, r-d :: aa-cc \,.\, aa-bb.$
17. Therefore by Conversion and Inversion of Reason, ... } $r+d \,.\, 2r :: bb-cc \,.\, aa-cc.$

Hence

CANON 2.

18. When the given angle is obtuse, let it be made, As the summ of the Radius and Sine-complement of that angle, is to the double Radius; So the excess of the Square of the given Base above the Square of the given difference of the leggs to a fourth Proportional: Then to that fourth Proportional add the Square of the difference of the leggs, and the square Root of the summ shall be the summ of the leggs sought. Lastly, the summ and difference of the leggs being given, the leggs shall be given severally, by *Theor.* 9. *Chap.* 4.

From the preceding Canons in 12° and 18° this following Theorem is deducible, and easie to be demonstrated by the steps of the Resolution in a direct order.

THEOREM.

19. If any one of the three sides of an oblique-angled plain Triangle be called the Base, and the other two sides (or leggs) be unequal; Then the excess of the Square of the Base above the Square of the difference of the leggs, shall be to the excess of the Square of the summ of the leggs above the Square of their difference; As the excess of the Radius above the Sine-complement of the angle opposite to the Base, is to the double Radius, when the said angle is acute; but as the summ of the Radius and the Sine-complement is to the double Radius, when the said angle is obtuse.

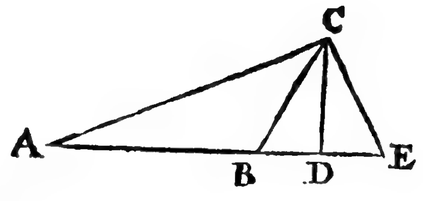
The Composition of this *Probl.* 20. may be made like that of the preceding *Probl.* 19. But waving the Composition, I shall illustrate the Resolution by Examples in Numbers.

Example 1. *where the given angle is acute.*

Suppos. in △ AEC.

20. . . AC = 7 the Base is given.
21. AE − EC = 5 the difference of the leggs is given.
22. . . ∠E = 60. degrees is given.

Req. to find AE and EC severally.



Solution Arithmetical.

23. Suppose the Radius of a Circle to be . . . . . } 100000.
24. Then the Sine-complement of the given angle E, 60. degrees, that is, the Sine of 30. degrees is . . . . . } 50000.
25. Therefore the excess of the Radius above the Sine-complement is . . } 50000.

26. The

26. Then by *Canon* 1. (in the preceding 12th step,) As the said excess 50000 is to the double Radius 200000; So is 24, (the excess whereby 49 the Square of the Base exceeds 25 the Square of the difference of the leggs,) to a fourth Proportional 96, which increased with 25 the Square of the difference of the leggs, makes 121, whose square Root 11 is the summ of the leggs sought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, AE and EC shall be 8 and 3.

Example 2. *where the given angle is obtuse.*

Suppos. in △ ABC.

27. . . . AC = 7 the Base is given.
28. AB − BC = 2 the difference of the leggs is given.
29. . . ∠ABC = 120. degrees is given.

Req. to find AB and BC severally.

Solution Arithmetical.

30. Suppose the Radius of a Circle to be . . . . . } 100000.
31. Then the Sine-complement of the given angle ABC, 120. degrees, that is, the Sine of 30. degrees is . . . . . } 50000.
32. Therefore the summ of the Radius and that Sine-complement is . . } 150000.
33. Then (by *Canon* 2. in the preceding 18th step,) As the said summ 150000 is to the double Radius 200000; So is 45, (the excess of 49 the Square of the Base above 4 the Square of the difference of the leggs,) to a fourth Proportional 60, which increased with 4 the Square of the difference of the leggs, makes 64, whose square Root 8 is the summ of the leggs sought. Therefore (by *Theor.* 9. *Chap.* 4.) the leggs themselves, to wit, AB and BC shall be 5 and 3.

CHAP. IX.

The third Classis *of Examples of the Resolution and Composition of Plane Problems.*

IN which Examples, the Resolution ends in an Analogy, wherein the Mean of three proportional right lines is given, as also the Difference or else the Summ of the Extremes, to find the Extremes severally.

Probl. I.

To find two right lines, such, that their difference may be equal to a right line given; and that the Rectangle made of the lines found out may be equal to a given Space.

D ——— 5
M ——— 6
L ——— 4
K ——— 9

Suppos.

1. d = D the difference of two right lines is given.
2. m = M a right line given, whose Square is equal to a given Space.

Req. to find

3. L and K two right lines, such, that K − L = D. Also, that
4. □ KL = □ M.

Resolution.

5. Put a for the lesser of the two right lines sought, *viz.* suppose . . . . . } $a = L.$
6. Therefore from 1° and 5° the greater right line sought shall be } $a + d \;(= K.)$
7. And from 5° and 6°, the Rectangle (or Product of their multiplication) shall be . . . . . } $aa + da.$

8. But

8. But the ſaid Rectangle muſt be equal to the given Space, ſuppoſed to be equal to the Square of m, therefore . . . $aa + da = mm$.

9. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a + d \,.\, m :: m \,.\, a$.

10. Of which three Proportionals $a + d$, m and a, the mean m is given, as alſo d the difference of the extremes $a + d$ and a, therefore the extremes ſeverally, which are the right lines ſought by this Problem, ſhall be given alſo, *per Probl.* 12. *Chap.* 5. whence alſo, (reſpect being had to the Theorem in 24° of the ſame *Probl.*) there will ariſe this following

CANON.

11. . . . . $\left\{\begin{array}{l} \sqrt{:\frac{1}{4}dd + mm} : + \frac{1}{2}d = \mathrm{K}, \\ \sqrt{:\frac{1}{4}dd + mm} : - \frac{1}{2}d = \mathrm{L}, \end{array}\right\}$ the lines ſought.

That is, in words,

To the Square of half the given difference add the Square equal to the given Space, and extract the ſquare Root of the ſumm: Then adding half the ſaid difference to the ſaid ſquare Root, this ſumm ſhall be equal to the greater of the two right lines ſought; but ſubtracting the ſaid half difference from the ſaid ſquare Root, the remainder ſhall be equal to the leſſer line ſought.

This Canon may be alſo Synthetically inferr'd from the given lines, by the help of *Theor.* 7, and 9. *Chap.* 4 For,

12. By conſidering the things given and ſought in 1°, 2°, 3° and 4° of this Problem it follows by *Theor.* 7. *Chap.* 4. that . . . . . $\frac{1}{4}dd + mm = \square : \overline{\frac{1}{2}\mathrm{K} + \frac{1}{2}\mathrm{L}}:$

13. Therefore by extracting the ſquare Root out of each part in 12°, ... $\sqrt{:\frac{1}{4}dd + mm} : = \frac{1}{2}\mathrm{K} + \frac{1}{2}\mathrm{L}.$

14. And from 1° and 3°, ... $\frac{1}{2}d = \frac{1}{2}\mathrm{K} - \frac{1}{2}\mathrm{L}.$

15. Therefore (by *Theor.* 9. *Chap.* 4.) the ſumm and difference of the two laſt preceding Equations gives the two lines ſought by this Problem, *viz.* . . . $\left\{\begin{array}{l} \sqrt{:\frac{1}{4}dd + mm} : + \frac{1}{2}d = \mathrm{K}. \\ \sqrt{:\frac{1}{4}dd + mm} : - \frac{1}{2}d = \mathrm{L}. \end{array}\right.$

Thus you ſee the ſame Canon is diſcovered as before.

The Compoſition of Probl. 1.

D ——————— 5
M ——————— 6
L ——————— 4
K ——————— 9

Suppoſ.

16. D a right line equal to the difference of two right lines ſought, is given.

17. M a right line given, whoſe Square is equal to a given Space.

Req. to find,

18. Two ſuch right lines, that their difference may be equal to the given difference D, and that the Rectangle made of them may be equal to the Square of the given right line M.

Conſtruction.

19. Let the given right-line M be eſteemed the mean of three Proportionals, and the given right line D the difference of the extremes; then by *Probl.* 12. *Chap.* 5. find the extremes, the leſſer whereof ſuppoſe to be L, therefore the greater ſhall be equal to L + D, and conſequently theſe ſhall be Proportionals, *viz.*

L + D . M :: M . L.

20. Make K = L + D, whence K − L = D.

21. I ſay K and L are the two right lines ſought, but that they will ſatisfie the Problem propounded, I prove thus: Firſt by *Conſtruction* in the 20th ſtep, the difference of the ſaid right lines K and L is equal to the given difference D; ſo it remains only to prove that the Rectangle made of the ſaid right lines K and L is equal to the Square of the given right line M, but that is here-under demonſtrated by returning backwards from the 9th ſtep, (to wit, the laſt of the Reſolution) to the 8th.

22. . . *Req. demonſtr.* . . . . . □ K, L = □ M.

Demonſtr.

Demonſtration.

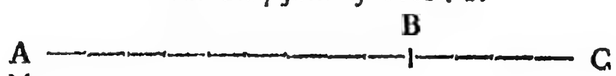
23. By *Conſtr.* in 19° and 20°, ... K (L + D) . M :: M . L.
That is, in 9°, ...

Determination.

16. The side of a Square equal to the given Space must not be greater than half the right line given to be cut into two parts. But that this Determination is necessary, I demonstrate thus;

Forasmuch as by the 9th step of the Resolution the right line m is found to be the mean of three Proportionals whereof the summ of the extremes is b, and in 20° of *Probl.* 13. *Chap.* 5. it hath been proved that the mean of three Proportionals cannot be greater than half the summ of the extremes, it follows, that m must not be greater than $\frac{1}{2}b$. If therefore m happens to be greater than $\frac{1}{2}b$, the Problem propounded cannot possibly be solved; for then mm cannot be subtracted from $\frac{1}{4}bb$, as the Canon directs. But if m be equal to, or less than $\frac{1}{2}b$, then that which the Canon bids to be done is possible; by which also 'tis easie to perceive, that when $m = \frac{1}{2}b$, then the extreme Proportionals (which are equal to the parts required,) will be equal between themselves, that is, each of them will be equal to $\frac{1}{2}b$, to wit, half the line given to be cut into two parts.

The Composition of Probl. 2.



Suppos.

17. AC is a right line given to be cut into two parts.
18. M is a right line given, whose Square is equal to a Space given.
19. M not $\sqsubset \frac{1}{2}$AC. (*Determination.*)

Req. to find

20. AB and BC such parts of AC, that AB + BC = AC. Also,
21. $\square$ AB, BC = $\square$ M.

Construction.

22. By *Probl.* 14. *Chap.* 5. cut the given right line AC into two such parts in B, that the line M may be a mean Proportional between the parts; which effection is possible, for by the Determination in 19°, M not $\sqsubset \frac{1}{2}$AC; suppose therefore

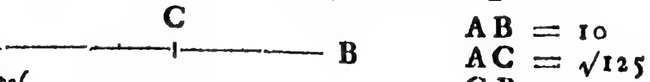
AB . M :: M . BC.

23. I say AB and BC are the parts required, for their summ by *Construction* is equal to the given right line AC; and because by *Constr.* also it hath been made, As AB to M, so M to BC, therefore (*per 17. prop. 6. Elem.*) $\square$ AB, BC = $\square$ M.

Which was required to be done.

Probl. III.

To cut a given right line according to the extreme and mean Reason; that is, into two such parts, that the Rectangle made of the whole line and lesser part may be equal to the Square of the greater.



AB = 10
AC = $\sqrt{125} - 5$
CB = $15 - \sqrt{125}$

Suppos.

1. b = AB is a right line given.

Req to find

2. AC and CB such parts of AB, that $\square$ AB, CB = $\square$ AC.

Resolution.

3. Put a for the greater part sought, *viz.* . . . . } a = AC.
4. Therefore from 1° and 3° the lesser part shall be . . } $b - a$ = CB.
5. Therefore from 1° and 4° the Rectangle (or Product) of the whole line and lesser part is . . . . . } $bb - ba$.
6. Which Rectangle, (according to the tenour of the Problem,) must be equal to the Square of the greater part, therefore . . . . . } $bb - ba = aa$.
7. Therefore, by adding ba to each part of that Equation, } $bb = aa + ba$.
8. Which last Equation may be resolved into this Analogy, ... } $a . b :: b . b + a$.

9. But

9. But that Analogy doth manifestly consist of three Proportionals, whereof the mean b is given, which is also the difference of the extremes a and $b + a$; therefore the extremes, whereof the lesser is equal to the greater part sought, shall be given also, by *Probl.* 12. *Chap.* 5. Whence also there will arise this following

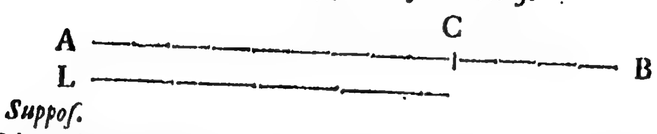
CANON.

10. . . . . $\sqrt{: bb + \frac{1}{4}bb :} - \frac{1}{2}b$ = AC, the greater segment.

That is, in words,

To the Square of the given line, add the Square of half the same line; then from the square Root of that summ subtract half the given line, and the remainder shall be the greater part sought, which subtracted from the given line leaves the lesser part.

The Composition of Probl. 3.



Suppos.

11. Let AB be a right line given to be cut into two such parts, that the Rectangle made of the whole line and the lesser part may be equal to the Square of the greater.

Construction.

12. Let the given line AB be esteemed the mean, as also the difference of the extremes of three Proportionals; then by *Probl.* 12. *Chap.* 5. find out the extremes severally, the lesser whereof we may suppose to be the right line L, and consequently the greater extreme is AB + L, therefore these are Proportionals, *viz.*

L . AB :: AB . AB + L.

13. From AB cut off AC = L, which is possible to be done, for by *Construction* in 12°, AB $\sqsubset$ L, because AB is the mean of three Proportionals whose lesser extreme is L. So the given line AB is cut into two parts in C, according to extreme and mean Reason, *viz.* the Rectangle made of the whole line AB and the lesser part CB is equal to the Square of AC the greater part, as will be made manifest by the following Demonstration, formed out of the preceding Resolution by a repetition of its steps in a backward order.

14. . . *Req. demonstr.* . . . . . $\square$ AB, CB = $\square$ AC.

Demonstration.

15. By *Constr.* in 12° and 13°, ... } AC . AB :: AB . AB + AC.
That is, in 8°, ... } $a . b :: b . b + a$.
16. Therefore (*per prop.* 17. *Elem.* 6.) . . . } $\square$AB = $\square$AC + $\square$AB,AC.
That is, in 7°, ... } $bb = aa + ba$.
17. Therefore by subtracting $\square$ AB, AC from each part, ... } $\square$AB — $\square$AB,AC = $\square$AC.
That is, in 6°, ... } $bb - ba = aa$.
18. And because (as is evident by the Diagram,) } AB — AC = CB.
19. Therefore, by drawing AB as a common altitude into each part of the last Equation, . } $\square$AB — $\square$AB,AC = $\square$AB,CB.
20. Therefore from 17° and 19°, (*per Ax.* 1. *Chap.* 2.) . . . . . } $\square$ AB, CB = $\square$ AC.

Which was to be Demonstr. Therefore the Problem is satisfied.

COROLL. 1.

21. From the preceding Resolution, the invention of a right-angled Triangle whose three sides shall be Proportionals is discovered; for if a given right line b be cut according to extreme and mean Reason, and the greater segment be a, 'tis manifest by the 7th step of the foregoing Resolution, that

$bb = ba + aa$.

22. And because (by *prop.* 48. *Elem.* 1.) if a Square be equal to two Squares, the sides of those three Squares will constitute a right-angled Triangle, therefore the square Roots

of the three Planes in the preceding Equation in 21°, viz. $b, \sqrt{ba}, a$ ſhall be the three ſides of a right-angled Triangle, and be Proportionals alſo, for the Rectangle of the extremes is manifeſtly equal to the Square of the mean. Hence therefore a Canon is diſcovered to find out a right-angled Triangle, whoſe three ſides ſhall be Proportionals.

CANON.

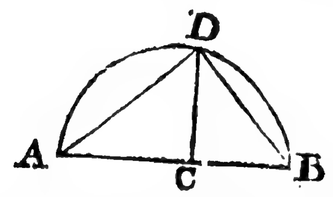
23. . . . . { Hyp. = b a right line or number taken at pleaſure,
Baſe = $\sqrt{:bb + \frac{1}{4}bb : - \frac{1}{2}b}$.
Perp. = $\sqrt{:b \times \sqrt{bb + \frac{1}{4}bb} - \frac{1}{2}b :}$

That is, in words,

Take any right line (or number) for the Hypothenuſal of a right-angled Triangle, and cut it into two parts according to extreme and mean Reaſon; then the greater part ſhall be one of the ſides about the right angle, and a mean Proportional between the greater part and the Hypothenuſal ſhall be the other ſide about the right angle; and thoſe three ſides ſhall be Proportionals.

Conſtruction.

∠ADB is ⌟.	AB = 10
AB, AD, DB ÷÷.	AD = 7.8615
AC = DB.	DB = 6.1803



24. Take any right line, as AB, and (by the foregoing *Probl.* 3.) cut the ſame by extreme and mean Reaſon in C, therefore

AB . AC :: AC . CB.

25. Then upon the line AB deſcribe the Semicircle ADB, and from C raiſe CD ⊥ AB. Laſtly, draw AD and DB, ſo ſhall ADB be a right-angled Triangle whoſe three ſides are Proportionals: For firſt, the ∠ADB being in the Semicircle, is a right angle, (*per prop.* 31. *Elem.* 3.) But that the three ſides AB, AD, DB are Proportionals, I prove thus;

26. . . *Req. demonſtr.* . . . . . AB . AD :: AD . DB.

Demonſtration.

27. Becauſe by *Conſtr.* in 25° ADB is a Semicircle, and CD ⊥ AB, therefore (*per Coroll. prop.* 8. *Elem.* 6.) . . . . . } AB . DB :: DB . CB.
28. And becauſe by *Conſtr.* in 24°, ... } AB . AC :: AC . CB.
29. Therefore out of 27° and 28°, (*per prop.* 17. *El.* 6.) } □AB, CB = □DB = □AC.
30. And conſequently, ... } . . . . . DB = AC.
31. And becauſe (by *Conſtr.* in 25°,) △ADB is right-angled at D, and DC ⊥ AB, therefore (*per prop.* 8. *Elem* 6.) . . . . . } AB . AD :: AD . AC.
32. Therefore from 30° and 31°, ... } AB . AD :: AD . DB.

Which was to be Dem.

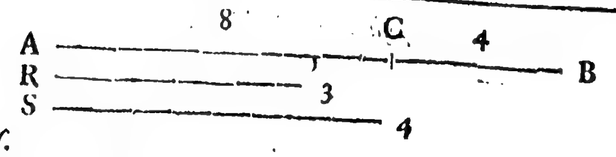
COROLL. 2.

33. From the premiſſes the way of ſolving this following Problem is alſo deducible; *viz.* To cut a right line given into three ſuch Proportionals, that the Square of the greateſt ſhall be equal to the Squares of the other two: For if it be made as the ſumm of the three Proportionals AB, AD, DB (being the ſides of a right-angled Triangle, and alſo Proportionals,) is to every one of them; ſo the right line given to three others; the Problem will be ſatisfied.

Probl. IV.

To cut a given right line into two ſuch parts, that the Rectangle made of the whole line and one of the parts, to the Square of the other part may have a given Reaſon.

A ——



Suppoſ.

1. b = AB is a right line given to be cut into two parts.
2. $\{r = R, s = S\}$ are the Terms of a given Reaſon.

Req. to find

3. AC and CB ſuch parts of AB, that AC + CB = AB. Alſo, that
4. □AB, CB . □AC :: R . S.

Reſolution.

5. Put a for one of the parts ſought, *viz.* . . . } a = AC.
6. Therefore from 1° and 5° the other part ſhall be . . . . . } $b - a$ = CB.
7. Therefore the Square of the firſt part is . . . } aa (= □AC.)
8. And the Rectangle of the given line b and latter part is . . . . . } $bb - ba$ (= □AB, CB.
9. Therefore to anſwer what the Problem requires, theſe muſt be Proportionals, *viz.* . } $r . s :: bb - ba . aa$.
10. Now to avoid an Equation between Solids, let it be made, as r to s, ſo b to a fourth Proportional, which may be called d, therefore . } $r . s :: b . d$.
11. Therefore from 9° and 10°, (*per prop.* 11. *Elem.* 5.) . . . . . } $b . d :: bb - ba . aa$.
12. And by drawing $b - a$ as a common altitude into b and d ſeverally, this Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.) . . . . . } $b . d :: bb - ba . db - da$.
13. Therefore from 11° and 12°, (*per prop.* 11. *Elem.* 1.) . . . . . } $bb - ba . aa :: bb - ba . db - da$.
14. And becauſe the firſt Term of the laſt Analogy is equal to the third, the ſecond ſhall be equal to the fourth, (*per prop.* 14. *Elem.* 5.) therefore, ... } $aa = db - da$.
15. Therefore by adding da to each part of the laſt Equation, it gives . . . . . } $aa + da = db$.
16. Which laſt Equation may be reſolved into this Analogy, ... } $a + d . \sqrt{db} :: \sqrt{db} . a$.

In which Analogy (conſiſting of three Proportionals,) the mean, to wit, $\sqrt{db}$ is given, as alſo d the difference of the extremes $a + d$ and a, therefore the extremes ſeverally, (the leſſer whereof is one of the Parts ſought by this Problem,) ſhall be given alſo by *Probl.* 12. *Chap.* 5. Whence alſo this

CANON.

17. . . . . $\sqrt{:\frac{1}{4}dd + db : - \frac{1}{2}d}$ = AC.

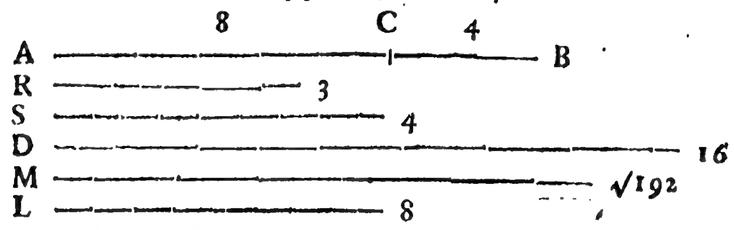
That is, in words,

Let it be made, As R the firſt Term of the given Reaſon, to the latter S; ſo AB the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then to the Square of half that fourth Proportional, add the Rectangle made of the ſame Proportional and the given line AB, and from the ſquare Root of the ſumm ſubtract half the ſaid fourth Proportional D: The remainder ſhall be one of the parts ſought, which may be called the firſt. Laſtly, the ſaid firſt part being ſubtracted from the given line AB, gives alſo the other part.

18. *Note.* Whether the firſt Term of the given Reaſon be greater or leſs than the latter, it ſhall always be; As the firſt Term is to the latter; ſo the Rectangle made of the given line and the latter part found out by the Canon, to the Square of the firſt part.

The

The Compoſition of Probl. 4.



Suppoſ.

19. A B is a right line given to be cut into two parts.
20. R and S are the Terms of a given Reaſon.

Req. to find

21. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo, that
22. □ A B, C B . □ A C :: R . S.

Conſtruction.

23. By *Probl.* 8. *Chap.* 5. let it be made, as R to S, ſo A B to a fourth, which may be called D, therefore } R . S :: A B . D.
24. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . . . } A B . M :: M . D.
25. Therefore *(per prop.* 17. *Elem.* 6.) . . . . ∴ □ M = □ D, A B.
26. Then eſteeming the line M to be the mean of three Proportionals, and D the difference of the extremes, find out the extremes, *(per Probl.* 12. *Chap.* 5.) the leſſer whereof ſuppoſe to be L, then conſequently the greater is equal to L + D, therefore theſe are Proportionals, *viz.*

L + D . M :: M . L.

27. Therefore, *per Theor.* in 24° of *Probl.* 12. *Chap.* 5. ∴ $L = \sqrt{:\frac{1}{4}\square D + \square M:} - \frac{1}{2}D$.
28. From A B cut off A C = L, which effection is poſſible, if A B ⊏ L, but that A B is greater than L, I prove thus,

Firſt, it is manifeſt that . . . . ∴	$\square AB + \frac{1}{4}\square D + \square D,AB \sqsubset \frac{1}{4}\square D + \square D,AB.$
And by extracting the ſquare Root out of each part, ... }	$AB + \frac{1}{2}D \sqsubset \sqrt{:\frac{1}{4}\square D + \square D,AB:}$
And by ſubtracting $\frac{1}{2}$D from each part, ∴	$AB \sqsubset \sqrt{:\frac{1}{4}\square D + \square D,AB:} - \frac{1}{2}D.$
But by *Conſtr.* in 25° and 27°, . . ∴	$L = \sqrt{:\frac{1}{4}\square D + \square D,AB:} - \frac{1}{2}D.$
Therefore from the two laſt preceding ſteps, *(per Ax.* 3. *Chap.* 2.) . . }	A B ⊏ L. Which was to be Dem.

29. I ſay the given line A B is cut in C into two parts, according to the tenour of the Problem propounded. Now we muſt ſhew, that the Rectangle made of the whole line A B and one of the parts, is to the Square of the other part as R to S. But that Analogy is made manifeſt by the following Demonſtration, formed out of the ſteps of the preceding Reſolution, by returning backwards from the 16th ſtep to the Analogy in the 9th ſtep.

30. . . *Req. demonſtr.* . . . . . R . S :: □ AB,CB . □ AC.

Demonſtration.

31. Becauſe by *Conſtruction* in 26°, ... ∴	. L + D . M :: M . L.
32. And by *Conſtr.* in 28°, ... ∴	A C = L.
33. Therefore from 31° and 32°, by exchange of equal quantities, ... }	A C + D . M :: M . A C.
That is, in 16°, ... ∴	. *a* + *d* . √*db* :: √*db* . *a*.
34. Therefore, *(per prop.* 17. *Elem.* 6.) . . . ∴	□ AC + □ D,AC = □ M.
35. And becauſe by *Conſtr.* in 24° and 25°, . ∴	. . . . □ D,AB = □ M.
36. Therefore from 34° and 35°, *(per Ax.* 1. *Chap.* 2.) . . . . . }	□ AC + □ D,AC = □ D,AB.
That is, in 15°, ... ∴	*aa* + *da* = *db*.
37. Therefore by ſubtracting □ D,AC from each part of the Equation in 36°, ... }	□ AC = □ D,AB − □ D,AC.
That is, in 14°, ... ∴	*aa* = *db* − *da*.

38. And

38. And from 37°, *(per prop.* 7. *Elem.* 5.) this Analogy is manifeſt, ... }	□AB − □AB,AC . □AC :: □AB − □AB,AC . □D,AB − □D,AC. } that is, in 13°, } *bb* − *ba* . *aa* :: *bb* − *ba* . *db* − *da*.
39. And by reaſon of the common altitude AB − AC, this following Analogy is manifeſt, *(per propoſ.* 1. *Elem.* 6.) . . . . }	AB . D :: □AB − □AB,AC . □D,AB − □D,AC. } that is, in 12°, } *b* . *d* :: *bb* − *ba* . *db* − *da*.
40. Therefore from 38° and 39°, *(per prop.* 11. *Elem.* 5.) . . . . . }	AB . D :: □AB − □AB,AC . □AC.
That is, in 11°, ... ∴	*b* . *d* :: *bb* − *ba* . *aa*.
41. And becauſe by *Conſtr.* in 23°, . . . ∴	AB . D :: R . S.
That is, in 10°, ... ∴	*b* . *d* :: *r* . *s*.
42. Therefore from 40° and 41°, *(per prop.* 11. *Elem.* 5.) . . . . . }	R . S :: □AB − □AB,AC . □AC.
That is, in 9°, ... ∴	*r* . *s* :: *bb* − *ba* . *aa*.
43. And becauſe *(per Diagram,)* . . . ∴	. . . CB = AB − AC.
44. And conſequently, *(per prop.* 1. *Elem.* 6.) by drawing A B into each part, . . . ∴	□AB, CB = □ AB − □AB, AC.
45. Therefore from 42° and 44°, by exchanging equal quantities, ... }	R . S :: □ AB, CB . □ AC.

Which was to be Demonſtr. Therefore the Problem is ſatisfied.

Probl. V.

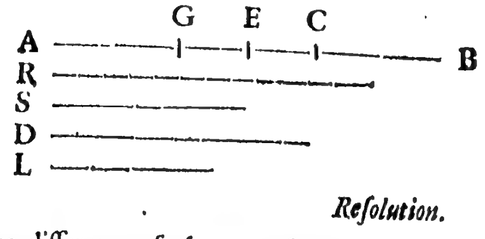
To cut a given right line into two ſuch parts, that the difference of the Squares of the parts, to the Rectangle made of the ſame parts, may have a given Reaſon.

Suppoſ.

1. *b* = AB is a right line given to be cut into two parts.
2. *r* = R }
3. *s* = S } are the Terms of a given Reaſon.

Req. to find

4. A C and C B, ſuch parts of A B, that A C + C B = A B. Alſo, that
5. □ A C − □ C B . □ A C, C B :: R . S.



	AB = 12
	R = 3
	S = 2
	D = 8
	L = 4
	AC = 8
	CB = 4

Reſolution.

6. For the difference of the parts ſought put . . ∴	*a*.
7. Then out of 1° and 6°, (by *Theor.* 9. *Chap.* 4.) the greater part ſhall be . . . . . }	$\frac{1}{2}b + \frac{1}{2}a$.
8. And (by the ſame Theorem) the leſſer part ſhall be ∴	$\frac{1}{2}b - \frac{1}{2}a$.
9. Therefore from 7° and 8°, the Product or Rectangle of the parts ſhall be . . . . . }	$\frac{1}{4}bb - \frac{1}{4}aa$.
10. And (from 1° and 6°, *per Theor.* 8. *Chap.* 4.) the difference of the Squares of the parts is . . }	*ba*.
11. Then according to the import of the Problem propounded, this Analogy ariſeth, (out of 2°, 3°, 10° and 9°, *viz.* . . . . . }	*r* . *s* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.
12. Now to avoid an Equation between Solids, let it be made, as *r* to *s*, ſo *b* to a fourth Proportional, which may be called *d*, therefore . . . . }	*r* . *s* :: *b* . *d*.
13. Therefore out of 11° and 12°, *(per pro.* 11. *El.* 5.) ∴	*b* . *d* :: *ba* . $\frac{1}{4}bb - \frac{1}{4}aa$.

14. And

14. And by drawing a as a common altitude into b and d ſeverally, this following Analogy will be manifeſt, (*per prop.* 1. *Elem.* 6.) *viz.* . . . } $b \;.\; d :: ba \;.\; da.$

15. Therefore from the 13th and 14th ſteps, (*per prop.* 11. *Elem.* 5.) . . . . . } $ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa :: ba \;.\; da.$

16. Therefore from the 15th ſtep this Equation ariſeth, (*per prop.* 14. *Elem.* 5.) *viz.* . . . . } $\frac{1}{4}bb - \frac{1}{4}aa = da.$

17. And by multiplying each Term of the Equation in the 16th ſtep by 4, this Equation is produced, *viz.* } $bb - aa = 4da.$

18. Therefore by adding aa to each part, . . . } $bb = aa + 4da.$

19. Which Equat. may be reſolved into this Analogy, } $a + 4d \;.\; b :: b \;.\; a.$

20. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean b is given, as alſo $4d$ the difference of the extremes $a + 4d$ and a; therefore the extremes ſeverally, the leſſer whereof is the difference of the parts ſought by this Problem, ſhall be given alſo, by *Probl.* 12. *Chap.* 5. whence alſo, (reſpect being had to the Theorem in 24° of the ſame Problem,) there will ariſe this

CANON.

21. . . . . $\sqrt{:bb + 4dd} : -2d = a$, the difference of the parts ſought.

That is, in words,

Let it be made, as the firſt Term (whether it be the greater or leſſer) of the given Reaſon, is to the ſecond; ſo the line given to be divided, to a fourth Proportional. Then add the Square of the double of that Proportional to the Square of the line given to be divided, and from the ſquare Root of the ſumm ſubtract the double of the ſaid Proportional, ſo ſhall the remainder be the difference of the parts ſought. Then by the ſumm and difference of the parts, the parts ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.

The Compoſition of Probl. 5.

Suppoſ.

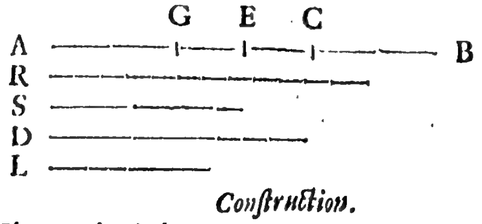
22. A B is a right line given.

23. R and S are the Terms of a given Reaſon.

Req. to find

24. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,

25. □ A C − □ C B . □ A C, C B :: R . S.



Conſtruction.

26. By *Probl.* 8. *Chap.* 5. let it be made, as R to S, ſo the given line A B to a fourth Proportional, which may be called D, therefore

R . S :: A B . D.

27. Then eſteeming the given line A B to be the mean of three Proportionals, and 4 D the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the leſſer whereof ſuppoſe to be L, then conſequently the greater is equal to L + 4 D, and theſe are Proportionals, *viz.*

L + 4D . A B :: A B . L.

Whence, (*per Theor.* in 24° of *Probl.* 12. *Chap.* 5.) $L = \sqrt{:\square AB + 4\square D} : -2D.$

Thus far the Conſtruction hath been made according to the direction of the Canon in 21°.

28. Divide the given line A B into two equal parts in E, whence $EA = EB = \frac{1}{2}AB$.

29. From E A and E B cut off E G and E C, ſuch parts, that each may be equal to $\frac{1}{2}L$, which is poſſible to be done, if $\frac{1}{2}AB \sqsubset \frac{1}{2}L$; but that $\frac{1}{2}AB \sqsubset \frac{1}{2}L$, I prove thus,

Firſt, it is manifeſt that . . . . } $\square AB + 4\square AB,D + 4\square D \sqsubset \square AB + 4\square D.$

Therefore by extracting the ſquare Root out of each part, ... } $AB + 2D \sqsubset \sqrt{:\square AB + 4\square D}:$

And

And by ſubtracting 2D from each part, . . . } $AB \sqsubset \sqrt{:\square AB + 4\square D} : -2D.$

But by *Conſtr.* in 27°, ... } $L = \sqrt{:\square AB + 4\square D} : -2D.$

Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . } $AB \sqsubset L$; and $\frac{1}{2}AB \sqsubset \frac{1}{2}L.$

Therefore it is poſſible to cut off from $\frac{1}{2}$ A B, that is, from E B = E A, a right line equal to $\frac{1}{2}$ L; ſuppoſe therefore $EC = EG = \frac{1}{2}L.$

30. I ſay the given right line A B is cut into two parts in C, as the Problem requires; *viz.* The difference of the Squares of the parts A C and C B, is to the Rectangle made of the ſame parts, as R to S; which Analogy may be demonſtrated by a retrograde repetition of the ſteps of the preceding Reſolution, in manner following.

31. . . . *Req. demonſtr.* . . . R . S :: □ AC − □ CB . □ AC,CB.

Demonſtration.

32. Foraſmuch as by *Conſtr.* in 27°, } L + 4 D . A B :: A B . L.

That is, in 19°, ... } $a + 4d \;.\; b :: b \;.\; a.$

33. Therefore from 32°, (*per prop.* 17. *Elem.* 6.) . . . . . } □ AB = □ L + 4 □ DL.

That is, in 18°, ... } $bb = aa + 4da.$

34. Therefore from 33°, by ſubtracting the □ L from each part, . . } □ AB − □ L = 4 □ DL.

That is, in 17°, ... } $bb - aa = 4da.$

35. And by taking a quarter of the Equation in 34°, it gives . . . } $\frac{1}{4}\square AB - \frac{1}{4}\square L = \square DL.$

That is, in 16°, ... } $\frac{1}{4}bb - \frac{1}{4}aa = da.$

36. Therefore from 35° this following Analogy will be manifeſt, (*per* 7. *prop.* 5. *Elem.*) . . . . . } □AB,L . $\frac{1}{4}\square AB - \frac{1}{4}\square L$ :: □AB,L . □DL.

That is, in 15°, ... } $ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa :: ba \;.\; da.$

37. And by reaſon of the common altitude L, this following Analogy will be manifeſt, (*per prop.* 1. *Elem.* 6.) } AB . D :: □AB, L . □DL.

That is, in 14°, ... } $b \;.\; d :: ba \;.\; da.$

38. Therefore from 36° and 37°, (*per prop.* 11. *Elem.* 5.) . . . . . } AB . D :: □AB, L . $\frac{1}{4}\square AB - \frac{1}{4}\square L.$

That is, in 13°, ... } $b \;.\; d :: ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa.$

39. And becauſe by *Conſtr.* in 26°, . . } AB . D :: R . S.

That is, in 12°, ... } $b \;.\; d :: r \;.\; s.$

40. Therefore out of 38° and 39°, (*per* 11. *prop.* 5. *Elem.*) . . . . . } R . S :: □AB, L . $\frac{1}{4}\square AB - \frac{1}{4}\square L.$

That is, in 11°, ... } $r \;.\; s :: ba \;.\; \frac{1}{4}bb - \frac{1}{4}aa.$

41. And becauſe by *Conſtr.* in 29°, . . } . . . EC = $\frac{1}{2}$ L = EG.

42. Therefore by taking 2EC, that is, GC, inſtead of L in the Analogy in 40°, this will thence ariſe, *viz.* . } R . S :: □AB,GC . $\frac{1}{4}\square AB - \frac{1}{4}\square GC.$

43. And becauſe (*per Diagram,*) . . } . . . AB = AC + CB.

44. And by *Conſtr.* in 28° and 29°, . . } . . . GC = AC − CB (AG.)

45. Therefore from 43° and 44°, (*per Theor.* 8. *Chap.* 4.) . . . . . } □AC−□CB = □AB,GC.

46. And becauſe by *Conſtr.* in 29°, . . } . . . . AC = $\frac{1}{2}$AB + $\frac{1}{2}$GC.

47. And by *Conſtr.* alſo in 29°, . . . } . . . . CB = $\frac{1}{2}$AB − $\frac{1}{2}$GC.

48. Therefore from 46° and 47°, (*per Theor.* 8. *Chap* 4. & *Prop.* 1. *Elem.* 6.) } □AC, CB = $\frac{1}{4}\square AB - \frac{1}{4}\square GC.$

49. Therefore if inſtead of the two latter Terms of the Analogy in 42°, you take their equivalent quantities, to wit, the firſt parts of the Equations in 45° and 48°, this Analogy will ariſe, *viz.* . . . . . } R . S :: □ AC − □CB . □ AC,CB.

Which was to be demonſtr. Therefore the Problem is ſatisfied.

Probl. VI.

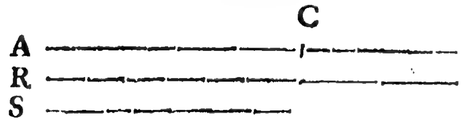
To cut a given right line into two ſuch parts, that the Rectangle made of the whole line and the difference of the parts, to the Square of the leſſer part may have a given Reaſon.

Suppoſ.

1. b = A B is a right line given to be cut into two parts.
2. $\left\{\begin{matrix} r = \mathrm{R} \\ s = \mathrm{S} \end{matrix}\right\}$ are the Terms of a given Reaſon.

Req. to find

3. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,
4. □ A B × $\overline{\mathrm{AC} - \mathrm{CB}}$. □ C B :: R . S.



A B = 16 | A C = 10
R = 16 | C B = 6
S = 9

Reſolution.

5. Put a for the greater part ſought, *viz.* aſſume . . . . . a = A C.
6. Therefore out of 1° and 5°, the leſſer part ſhall be . . . . . $b - a$ = C B.
7. And from 5° and 6°, the difference of the parts is . . . . . $2a - b$.
8. And from 1° and 7°, the Rectangle of the given line into the difference of the parts is $2ba - bb$.
9. And from 6°, the Square of the leſſer part is . . . . . $bb + aa - 2ba$.
10. Therefore from 2°, 8° and 9°, according to the import of the Problem, theſe muſt be Proportionals, *viz.* . . . . . $r . s :: 2ba - bb . bb + aa - 2ba$.
11. Therefore inverſly, ... $s . r :: bb + aa - 2ba . 2ba - bb$.
12. And by Compoſition, ... $s + r . r :: aa . 2ba - bb$.
13. And inverſly, ... $r . s + r :: 2ba - bb . aa$.
14. Now to avoid an Equation between Solids, let it be made, as r to $s + r$; ſo b to a fourth Proportional, which may be called d, therefore . . . . . $\ldots r . s + r :: b . d$.
15. Therefore from 13° and 14°, (*per prop.* 11. *Elem.* 5.) . . . . . $\ldots\ldots b . d :: 2ba - bb . aa$.
16. And by drawing $2a - b$ as a common altitude into b and d ſeverally, this Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.) . . . . $b . d :: 2ba - bb . 2da - db$.
17. Therefore from 15° and 16°, (*per prop.* 11. *Elem.* 5.) . . . . . $2ba - bb . 2da - db :: 2ba - bb . aa$.
18. And from 17°, (*per prop.* 14. *Elem.* 5.) $2da - db = aa$.
19. And by adding db to each part of that Equation, ... $2da = aa + db$.
20. And by ſubtracting aa from each part, $2da - aa = db$.
21. Which laſt Equation may be reſolved into this Analogy, ... $2d - a . \sqrt{db} :: \sqrt{db} . a$.
22. But that Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean $\sqrt{db}$ is given, as alſo $2d$ the ſumm of the extremes $2d - a$ and a; therefore the extremes ſeverally, the leſſer whereof is the greater part ſought, ſhall be given alſo, by *Probl.* 13. *Ch.* 5. And from the premiſſes and the Theorem in 21° of the ſame Problem there will ariſe this following

CANON.

23. . . . . $d - \sqrt{:dd - db:}$ = A C, the greater part ſought.

That

That is, in words,

Let it be made as R the firſt Term of the given Reaſon, to R + S the ſumm of both Terms; ſo A B the line given to be cut into two parts, to a fourth Proportional, which may be called D. Then from the Square of that fourth Proportional, ſubtract the Rectangle made of the ſaid fourth Proportional and the given line A B, and extract the ſquare Root of the remainder. Laſtly, ſubtract the ſaid ſquare Root from the ſaid fourth Proportional, and this remainder ſhall be the greater part ſought.

24. *Note.* Although the Equation found out in 20° may be expounded by either of the two extreme Proportionals mentioned in 22°, yet the leſſer of them is only capable of ſolving the Problem propounded; for the greater extreme is greater than the line given to be cut into two parts, and therefore cannot be equal to either of the parts; which I prove thus,

If $\sqrt{db}$ be the mean of three Proportionals, and $2d$ the ſumm of the extremes, then (by *Probl.* 13. *Chap.* 5.) the greater extreme ſhall be equal to . . . . . $d + \sqrt{:dd - db:}$

Now we muſt prove that . . . . . $d + \sqrt{:dd - db:} \sqsubset b$.

Firſt, by *Conſtruction* in 14°, (and by *Schol. prop.* 14. *Elem.* 5.) 'tis evident that . . . . . $d \sqsubset b$.

And conſequently, by drawing d into each part, ... $dd \sqsubset db$.

Therefore, ... $\sqrt{:dd - db:} \sqsubset d$.

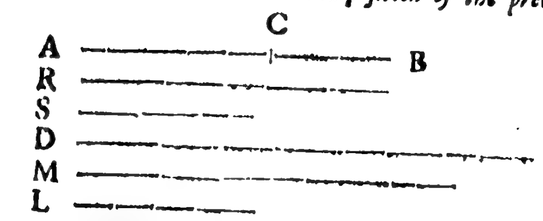
And conſequently, by adding d to each part, ... $d + \sqrt{:dd - db:} \sqsubset 2d$.

But by *Conſtr.* in 14°, ... $d \sqsubset b$.

Therefore (*per Ax.* 5. *Chap.* 2.) . . . . . $d + \sqrt{:dd - db:} \sqsubset b$.

Which was to be Dem.

The Compoſition of the preceding Probl. 6.



A B = 16 | M = 20
R = 16 | L = 12
S = 9 | A C = 10
D = 25 | C B = 6

Suppoſ.

25. A B is a right line given to be cut into two parts.
26. R and S are the Terms of a given Reaſon.

Req. to find

27. A C and C B ſuch parts of A B, that A C + C B = A B. Alſo,
28. □ A B × $\overline{\mathrm{AC} - \mathrm{CB}}$. □ C B :: R . S.

Conſtruction.

29. By *Probl.* 8. *Chap.* 5. let it be made, as R to S + R; ſo A B to a fourth Proportional, which we may ſuppoſe to be the line D, therefore . . . . . R . S + R :: A B . D.
30. By *Probl.* 9. *Chap.* 5. find a mean proportional line, as M, between A B and D, therefore . . . . A B . M :: M . D.
31. Therefore it follows from the laſt Analogy, (*per prop.* 17. *Elem.* 6.) that . . . . . □ A B, D = □ M.
32. Then by *Probl* 14. *Chap.* 5. cut the double of the right line D (before found in 29°) into two ſuch parts, that the line M may be a mean Proportional between them; which Effection is poſſible, if M be not greater than D; but that M is leſs than D, I prove thus,

By the *Conſtruct.* in 29° 'tis manifeſt that D = A B, and by *Conſtr.* in 30°, M is a mean Proportional between A B and D, therefore M = D. Therefore 'tis poſſible to cut 2D into two ſuch parts that M may be a mean Proportional between the parts. Suppoſe then it be done, and that the leſſer part is found L, therefore theſe ſhall be Proportionals, *viz.*

2D − L . M :: M . L.

33. And

33. And conſequently, (*per Theor.* in 21° of *Probl.* 13. *Chap.* 5.) ⟩ $L = D - \sqrt{:\square D - \square M:}$

34. From A B cut off A C = L, which is poſſible to be done if A B ⊏ L, but that A B ⊏ L, I prove thus;

Firſt, from the *Conſtr.* in 29°, ...	⟩ D ⊏ AB.
Therefore, by drawing A B into each part,	⟩ □ D, A B ⊏ □ A B.
And by adding □ D to each part, . .	⟩ □ D + □ D, A B ⊏ □ D + □ A B.
And by ſubtracting 2 □ D, A B from each part, ...	□D − □D,AB ⊏ □D + □AB − 2□D,AB.
And by extracting the ſquare Root out of each part, ...	$\sqrt{:\square D - \square D, AB:} \sqsubset D - AB.$
And by adding A B to each part, . .	$AB + \sqrt{:\square D - \square D, AB:} \sqsubset D.$
And by ſubtracting $\sqrt{:\square D - \square D, AB:}$ from each part, ...	$AB \sqsubset D - \sqrt{:\square D - \square D, AB:}$
But it hath been ſhewn in 31° and 33°, that	$L = D - \sqrt{:\square D - \square D, AB:}$
Therefore from the two laſt preceding ſteps,	⟩ A B ⊏ L.

Which was to be Dem. Therefore 'tis poſſible to cut off from A B a ſegment equal to L, as A C.

35. I ſay the given right line A B, in the point C is cut into two parts, A C and C B, which will ſolve the Problem; *viz.* the Rectangle made of the whole line A B, and the exceſs of A C above C B, is to the Square of C B; as the line R to the line S: As will be made manifeſt by the following Demonſtration, form'd out of the foregoing Reſolution, by a retrograde repetition of the ſteps thereof.

36. . . . *Req. demonſtr.* . . . R . S :: □ AB × $\overline{AC - CB}$. □ CB.

Demonſtration.

37. By *Conſtr.* in 32°, ...	⟩ 2 D − L . M :: M . L.
38. And by *Conſtr.* in 34°, ...	⟩ A C = L.
39. Therefore from 37° and 38°, by exchanging equal quantities, ...	2 D − A C . M :: M . AC.
That is, in 21, ...	⟩ $2d - a \;.\; \sqrt{db} :: \sqrt{db} \;.\; a.$
40. Therefore from 39°, (*per prop.* 17. *El.* 6.)	⟩ 2 □ D, A C − □ A C = □ M.
41. But by *Conſtr.* in 31°, ...	⟩ □ D, A B = □ M.
42. Therefore from 40° and 41°, (*per Ax.* 1. *Chap.* 2.)	2 □ D, A C − □ A C = □ D, AB.
That is, in 20°, ...	⟩ . . $2da - aa = db.$
43. And by adding □ A C to each part of the Equation in 42°, ...	2 □ D, A C = □ A C + □ D, A B.
That is, in 19°, ...	⟩ . . $2da = aa + db.$
44. And by ſubtracting □ D, A B from each part in 43°, ...	2 □ D, A C − □D,AB = □ AC.
That is, in 18°, ...	⟩ . . $2da - db = aa.$

45. And from 44°, (*per prop.* 7. *Elem.* 5.) this Analogy will be manifeſt,

2 □ A B, A C, − □ A B .
2 □ D, A C, − □ D, A B ::
2 □ A B, A C, − □ A B .
□ A C .
} that is, in 17°, {
$2ba - bb$.
$2da - db$ ::
$2ba - bb$.
aa .

46. And by reaſon of the common altitude 2 A C − A B, this following Analogy is manifeſt, (*per prop.* 1. *Elem.* 6.)

2 □ A B, A C − □ A B .
2 □ D, A C − □ D, A B ::
A B .
D .
} that is, in 16°, {
$2ba - bb$.
$2da - db$ ::
b .
d .

47. Therefore from 45° and 46°, (*per prop.* 11. *Elem.* 5.) . . . . . ⟩ AB . D :: 2 □ AB,AC, − □ AB . □AC.

That is, in 15°, ... ⟩ $b \;.\; d ::$ $2ba - bb \;.\; aa.$

48. But by *Conſtruction* in 29°, . . ⟩ AB . . D :: R . S + R.

That

That is, in 14°, ... ⟩ $b \;.\; d ::\; r \;.\; s + r.$

49. Therefore from 47° and 48°, (*per prop.* 11. *Elem.* 5.) . . . . . ⟩ R . S + R :: 2□AB,AC − □AB . □AC.

That is, in 13°, ... ⟩ $r \;.\; s + r ::$ $2ba - bb \;.\; aa.$

50. Therefore inverſly, ... ⟩ S + R . R :: □AC . 2□AB,AC, − □AB.

That is, in 12°, ... ⟩ $s + r \;.\; r ::\; aa \;.\; 2ba - bb.$

51. Therefore by Diviſion of Reaſon,

S .
R ::
□AB + □AC − 2□AB, AC .
2□AB, AC, − □AB .
} that is, in 11°, {
s .
r ::
$bb + aa - 2ba$.
$2ba - bb$.

52. Therefore inverſly,

R .
S ::
2□AB, AC, − □AB .
□AB + □AC − 2□AB, AC .
} that is, in 10°, {
r .
s ::
$2ba - bb$.
$bb + aa - 2ba$.

Now the Scope in the ſix ſteps next following is to prove, that □ AB × $\overline{AC - CB}$ is equal to 2 □ AB, AC − □ AB, to wit, the third Term of the Analogy in 52°.

53. By *Conſtr.* in 34°, ...	⟩ A B = A C + C B.
54. And by adding A C to each part,	⟩ A B + A C = 2 A C + C B.
55. And by ſubtracting C B from each part in 54°, ...	A B + A C − C B = 2 A C.
56. It is evident by the firſt part of the Equation in 44°, that	2 A C ⊏ A B.
57. Therefore A B may be ſubtracted from each part of the Equation in 55°, and this Equation between two real right lines will remain, *viz.* . .	A C − C B = 2 A C − A B.
58. Therefore by drawing AB as a common altitude into each part of the laſt Equation, it will produce (by *Theor* 1. *Chap.* 4.)	□AB × $\overline{AC - CB}$ = 2□AB,AC, − □AB.

Which was to be ſhewn. It remains to prove that □ C B is equal to □ A B + □ A C − 2 □ A B, A C, to wit, the fourth Term of the Analogy in 52°.

59. By *Conſtr.* in 34°, ...	⟩ C B = A B − A C.
60. And becauſe the Squares of equal right lines are alſo equal, therefore from 59°, (*per Theor.* 5. *Chap.* 4.)	□ C B = □ AB + □ AC − 2 □ AB, AC.

Which was to be Demonſtr.

61. Laſtly, inſtead of the third and fourth Terms of the Analogy in 52°, their equivalent quantities, to wit, thoſe in the firſt parts of the Equations in 58° and 60° being taken, this following Analogy ariſeth, *viz.*

R . S :: □ AB × $\overline{AC - CB}$. □ CB.

Which was *Req. demonſtr.* in 36°. Therefore the Problem is ſatisfied.

Probl. VII.

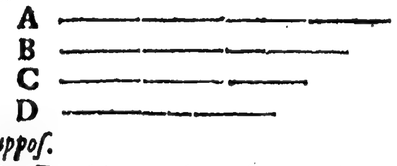
To find two right lines that their ſumm may be equal to a right line given, and that the difference of the Squares of thoſe two lines, to the Square of the leſſer of them, may have a given Reaſon.

This Problem is the ſame in effect with the preceding ſixth; for the difference of the Squares of the two right lines ſought by this Problem, is equal to the Rectangle of the ſumm and difference of the parts ſought by the laſt preceding Problem.

A LEMMA,

A LEMMA, *leading to the following* Probl. 8.

If four right lines be in continual proportion, the ſumm of the means is a mean Proportional between the ſumm of the firſt and ſecond, and the ſumm of the third and fourth Proportionals.



A = 125
B = 100
C = 80
D = 64

Suppoſ.

1. A, B, C, D ÷÷, *viz.* A . B :: B . C :: C . D.
2. . . *Req. demonſtr.* . . . . . A+B . B+C :: B+C . C+D.

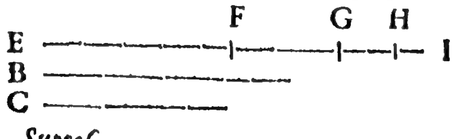
Demonſtration.

3. By *Suppoſ.* in 1°, ... A . B :: B . C.
4. Therefore by Compoſition of Reaſon, . . A+B . B :: B+C . C.
5. And alternately, ... A+B . B+C :: B . C.
6. Again, by *Suppoſ.* . . . . . B . C :: C . D.
7. Therefore by Compoſition, ... B+C . C :: C+D . D.
8. And alternately, ... B+C . C+D :: C . D.
9. Therefore from 6° and 8°, (*per prop.* 11. *Elem.* 5.) . . . . . B+C . C+D :: B . C.
10. Likewiſe from 5° and 9°, (*per prop.* 11. *Elem.* 5.) . . . . . A+B . B+C :: B+C . C+D.

Which was to be Demonſtr.

Probl. VIII.

The ſumm of the extremes, and ſumm of the means of four right lines in Continual proportion being given ſeverally, to find the Proportionals. But the firſt ſumm muſt be greater than the latter; the reaſon whereof is manifeſt, (*per prop.* 25. *Elem.* 5.)



EF = 16
FG = 8
GH = 4
HI = 2

Suppoſ.

1. EF, FG, GH, HI ÷÷, *viz.* EF . FG :: FG . GH :: GH . HI.
2. b = EF +- HI is given. Alſo,
3. c = FG +- GH = FH is given: Therefore,
4. d = b +- c = EI is given.
5. b ⊏ c. (*Determination.*)

Req. to find EF, FG, GH, HI.

Reſolution.

6. By *Suppoſ.* in 1°, ... EF, FG, GH, HI ÷÷.
7. Therefore by the *Lemma* prefixt before this Problem, . . . EF+FG . FG+GH :: FG+GH . GH+HI.
8. That is, as is evident by the Diagram, ... EG . FH :: FH . GI.
9. Of which three continual Proportionals the mean FH, that is, c, is given; as alſo EI, (= EG + GI,) that is, d, the ſumm of the extremes. Threfore, by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes ſhall be given ſeverally, *viz.*

10. . . . . $\begin{cases} \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc}: = EG = EF + FG. \\ \frac{1}{2}d - \sqrt{:\frac{1}{4}ad - cc}: = GI = GH + HI. \end{cases}$

11. Therefore, from 10° and 3° 'tis manifeſt, that of theſe three Proportionals, EF, FG, GH, the ſumm of the firſt and ſecond, to wit, EG, is given; alſo the ſumm of the ſecond and third, to wit, FH, is given. Likewiſe of theſe three Proportionals, FG, GH,

GH, HI, the ſumm of the firſt and ſecond, to wit, FH, is given; alſo GI the ſumm of the ſecond and third is given; therefore, (according to the Canon in 44° of *Probl.* 5. *Chap.* 7.) FG and GH ſhall be given ſeverally by theſe following Analogies, *viz.*

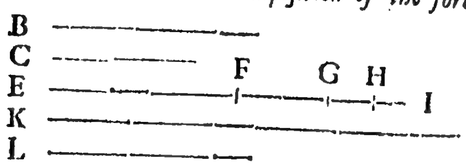
12. . . . $\begin{cases} c + \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc}: . c :: \frac{1}{2}d + \sqrt{:\frac{1}{4}dd - cc}: . FG. \\ c + \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc}: . c :: \frac{1}{2}d - \sqrt{:\frac{1}{4}dd - cc}: . GH. \end{cases}$

Which Analogies, reſpect being had to the Equations in 10°, and to the Diagram, will give this following

CANON.

13. From the Square of half the aggregate of the given ſumm of the extremes and the given ſumm of the means, ſubtract the Square of the ſumm of the means, and extract the ſquare Root of the remainder. Then add and ſubtract that ſquare Root to and from the ſaid half aggregate, and reſerve the ſumm and remainder. Then it ſhall be, as the ſumm reſerved together with the ſumm of the means, is to the ſumm of the means; ſo the ſumm reſerved to the greater mean ſought. Or, as the remainder reſerved together with the ſumm of the means, is to the ſumm of the means; ſo the remainder reſerved to the leſſer mean. Laſtly, the ſumm before reſerved being leſſened by the greater mean, gives the greater extreme: or, the remainder reſerved being leſſened by the leſſer mean, gives the leſſer extreme.

The Compoſition of the foregoing Probl. 8.



B = 18
C = 12
EI = 30
EG = 24
GI = 6

K = 36
L = 18
GF = 8
GH = 4
M = 4

Suppoſ.

14. B = the ſumm of the extremes of four right lines in continual proportion is given.
15. C = the ſumm of the means is given.
16. B ⊏ C. (*Determination.*)

Req. to find the four Proportionals ſeverally.

Conſtruction.

17. Make EI = B +- C.
18. Divide EI into two ſuch parts in G, that the line C may be a mean Proportional between the parts; which effection is poſſible, (*per Probl.* 14. *Chap.* 5.) if C be not greater than ½ EI, but that C is leſs than ½ EI, I prove thus;

By the Determination in 16°, ... C ⊐ B.
Therefore by adding C to each part, ... 2C ⊐ B+C.
But by Conſtruction in 17°, ... EI = B+C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . . 2C ⊐ EI.
And conſequently, ... C ⊐ ½EI.
Which was to be ſhewn.

Therefore 'tis poſſible to cut EI into two ſuch parts, that C ſhall be a mean Proportional between them. Suppoſe then EI to be ſo cut in G, and that EG is greater than GI; therefore, ... EG . C :: C . GI.

19. Make . . . . . K=EG+C; alſo, L=GI+C.
20. Let it alſo be made (by *Probl.* 8. *Chap.* 5.) as K to C, ſo EG to a fourth Proportional GF, therefore, ... K . C :: EG . GF.
21. Again, let it be made as L to C, ſo GI to a fourth Proportional GH, therefore, ... L . C :: GI . GH.
22. Then from EG cut off GF, and from GI cut off GH; (which ſubtractions are poſſible, for by Conſtruction in 19°, K is greater than C; therefore from the Analogy in 20°, EG is greater than GF: Likewiſe by Conſtruction in 19°, L is greater than C, and conſequently from the Analogy in 21°, GI is greater than GH;) ſo the remainders EF and HI are the extreme Proportionals ſought. I ſay EF, FG, GH and HI are

four

four continual Proportionals, which will ſatisfie the Problem propounded. But to make the truth thereof evident, I ſhall prove three things, *viz.* Firſt, that FH the ſumm of the means FG and GH is equal to the given ſumm C: Secondly, that the ſumm of the extremes EF and HI is equal to the given ſumm B: Thirdly and laſtly, that the ſaid EF, FG, GH, HI are in continual proportion in this order, *viz.*

EF . FG :: FG . GH :: GH . HI.

First, that FH the ſumm of the means FG and GH is equal to the given ſumm C, I prove thus;

Preparat.

23. To the lines K and C find a third Proportional, as M, *(per Probl. 7. Chap. 5.)* therefore, ... } K . C :: C . M.

24. . . . *Req. demonſtr.* . . . . . FH = C.

Demonſtration.

25. By *Conſtr.* in 18°, ... } EG . C :: C . GI.
26. Therefore by Compoſition of Reaſon, } EG+C . C :: C+GI . GI.
27. That is, by exchanging equal right lines according to the Conſtruction in 19°, } K . C :: L . GI.
28. Therefore alternly, ... } K . L :: C . GI.
29. And by drawing C as a common altitude into the two latter Terms, ... } K . L :: □C . □C,GI
30. But from the *Conſtr.* in 23° and 21°, *(per prop. 17, & 16. Elem. 6.)* . . . } □KM = □C; and □L,GH = □C,GI.
31. Therefore from 29° and 30°, by exchange of equal Rectangles, ... } K . L :: □K,M . □L,GH
32. But *(per prop. 1. Elem. 6.)* . . . } K . L :: □K,M . □L,M.
33. That from 31° and 32°, *(per prop. 11. Elem. 5.)* . . . . . } □K,M . □L,GH :: □K,M . □L,M.
34. Therefore from 33°, *(per prop. 14. El. 5.)* } . . . □L,GH = □L,M.
35. Therefore from 34°, *(per prop. 14. El. 6.)* } L . M :: L . GH.
36. Therefore from 35°, *(per prop. 14. El. 5.)* } . . . . . M = GH.
37. Again, by *Conſtr.* in 20°, . . . } K . C :: EG . GF.
38. And by *Conſtr.* in 23°, ... } K . C :: C . M.
39. Therefore from 37° and 38°, *(per Coroll. Herigon. in prop. 12. Elem. 5.)* . } 2K . 2C :: EG+C . GF+M.
40. And becauſe by *Conſtr.* in 19°, . . } . . . . . K = EG+C.
41. Therefore from 39° and 40°, . . . } 2K . 2C :: K . GF+M.
42. But *(per prop. 15. Elem. 5.)* . . . } 2K . 2C :: K . C.
43. Therefore from 41° and 42°, *(per prop. 11. Elem. 5.)* . . . . . } K . GF+M :: K . C.
44. Therefore from 43°, *(per prop. 14. El. 5.)* } GF + M = C.
45. But it hath been proved in 36°, that . } GH = M.
46. Therefore from 44° and 45, *(per Ax. 6. Chap. 2.)* . . . . . } GF + GH = C.
47. But 'tis evident by the Diagram, that } GF + GH = FH.
48. Therefore from 46° and 47°, *(per Ax. 1. Chap. 2.)* . . . . . } FH = C. Which was to be Dem.

Secondly, that the ſumm of the extremes EF and HI is equal to the given ſumm B, I demonſtrate thus;

49. . . . *Req. demonſtr.* . . . . . EF + HI = B.

Demonſtration.

50. By *Conſtr.* in 17°, ... } EI = B + C.
51. And it hath been proved in 48°, that . } FH = C.
52. Therefore by ſubtracting C or FH from each part in 50°, ... } EI — FH = B.
53. But 'tis evident by the Diagram, that . } EI — FH = EF + HI.

45. There-

54. Therefore from 52° and 53°, *(per Ax. 1. Chap. 2.)* . . . . . } EF+HI = B. Which was to be Dem.

Thirdly and laſtly, that EF, FG, GH, HI are in continual proportion, I demonſtrate thus;

55. . . . *Req. demonſtr.* . . . . . EF . FG :: FG . GH :: GH . HI.

Demonſtration.

56. By *Conſtr.* in 20°, ... } . . K . C :: EG . GF.
57. And by *Conſtr.* in 19°, ... } EG + C = K
58. Therefore from 56° and 57°, ... } EG+C . C :: EG . GF.
59. And becauſe it hath been proved in 48°, that . . } FH = C.
60. Therefore from 58° and 59°, ... } EG+FH . FH :: EG . GF.
61. It is manifeſt by the Diagram, that of the three right lines EF, FG, GH the ſumm of the firſt and ſecond is EG, and the ſumm of the ſecond and third is FH; therefore the laſt preceding Analogy is qualified in every reſpect according to the Theorem in 45° of *Probl.* 5. *Chap.* 7. whence EF, FG, GH ſhall be Proportionals, *viz.*

EF . FG :: FG . GH.

62. Again, by *Conſtr.* in 21°, ... } . . L . C :: GI . GH.
63. And by *Conſtr.* in 19°, ... } L = GI + C.
64. Therefore from 62° and 63°, ... } GI+C . C :: GI . GH.
65. But it hath been proved in 48°, that . . . } FH = C.
66. Therefore from 64° and 65°, ... } GI+FH . FH :: GI . GH.
67. Therefore from 66°, in like manner as before in 61°, *(per Theor. in 45° of Probl. 5. Chap. 7.)* . } FG . GH :: GH . HI.
68. And from 61°, ... } EF . FG :: FG . GH.
69. Wherefore from 67° and 68°, *(per prop. 11. Elem. 5.)* . . . . . } EF . FG :: FG . GH :: GH . HI.

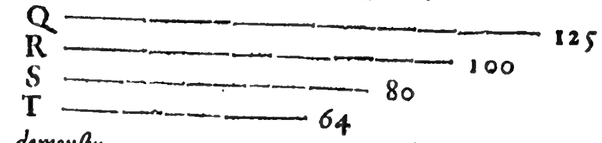
Which was to be demonſtrated in the laſt place. Therefore the Problem is ſatisfied.

A LEMMA, *leading to the following* Probl. 9.

If four right lines be in continual proportion, the difference of the means is a mean Proportional between the difference of the firſt and ſecond, and difference of the third and fourth Proportionals.

Suppoſ.

1. Q, R, S, T ÷÷; *viz.* Q . R :: R . S :: S . T.
2. Q ⊏ R. Whence R ⊏ S, and S ⊏ T.



3. . . . *Req. demonſtr.* . . . . . Q—R . R—S :: R—S . S—T.

Demonſtration.

4. By *Suppoſ.* in 1°, ... } Q . R :: R . S.
5. Alſo by *Suppoſ.* in 2°, ... } Q ⊏ R.
6. Therefore by Diviſion of Reaſon, . . . } Q—R . R :: R—S . S.
7. And alternly, ... } Q—R . R—S :: R . S.
8. Again, by *Suppoſ.* . . . . . } R . S :: S . T.
9. Therefore by Diviſion of Reaſon, . . . } R—S . S :: S—T . T.
10. And alternly, ... } R—S . S—T :: S . T.
11. Therefore from 8° and 10°, *(per prop. 11. Elem. 5.)* . . . . . } R—S . S—T :: R . S.
12. Likewiſe from 7° and 11°, *per prop. 11. El. 5.* } Q—R . R—S :: R—S . S—T.

Which was to be Demonſtr.

Probl. IX.

The difference of the extremes, and difference of the means of four right lines in Continual proportion being given ſeverally, to find the

Proportionals. But the given difference of the extremes muſt be greater than the triple difference of the means.

Q —————————— 125
R —————————— 100
S —————————— 80
T —————————— 64

Suppoſ.

1. Q, R, S, T ∴; *viz.* Q . R :: R . S :: S . T.
2. Q ⊏ R, whence R ⊏ S; and S ⊏ T.
3. b = Q — T is given.
4. c = R — S is given.
5. d = b — c is given.
6. b ⊏ 3c. (*Determination.*)

Req. to find Q, R, S, T.

Reſolution.

7. By *Suppoſ.* in 1° and 2°, ... } Q, R, S, T ∴; alſo Q ⊏ R.
8. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } Q—R . R—S :: R—S . S—T.
9. Of which three continual Proportionals in 8° the mean R — S, that is, c, is given; as alſo d (= Q + S — T — R,) the ſumm of the extremes Q — R and S — T; therefore by the Theorem in 21° of *Probl.* 13. *Chap.* 5. the extremes ſhall be given ſeverally, *viz.*

10. . . . . { ½d + √: ¼dd — cc : = Q — R.
{ ½d — √: ¼dd — cc : = S — T.

11. Therefore from 4°, 5° and 10° 'tis manifeſt, that of theſe three Proportionals Q, R, S, the difference of the firſt and ſecond, to wit, Q — R is given, alſo R — S the difference of the ſecond and third is given. Likewiſe of theſe three Proportionals R, S, T, the difference of the firſt and ſecond, to wit, R — S is given, alſo S — T the difference of the ſecond and third is given; therefore, (according to the Canon in 13. of *Probl.* 6. *Chap.* 7.) R and S ſhall be given ſeverally by theſe following Analogies, *viz.*

12. . . . { ½d + √: ¼dd — cc : — c . c :: ½d + √: ¼dd — cc : . R.
{ c — ½d — √: ¼dd — cc : . c :: ½d — √: ¼dd — cc : . S.

Which Analogies, reſpect being had to the Equations in 10°, do afford this

CANON.

13. From the Square of half the exceſs by which the given difference of the extremes exceeds the given difference of the means, ſubtract the Square of the given difference of the means, and extract the ſquare Root of the remainder; then add and ſubtract that ſquare Root to and from the ſaid half-exceſs, and reſerve the Summ and Remainder. Then it ſhall be, as the exceſs of the Summ reſerved above the given difference of the means, is to the difference of the means; ſo is the Summ reſerved to the greater mean ſought. Or, as the exceſs of the difference of the means above the Remainder reſerved, is to the difference of the means; ſo is the Remainder reſerved to the leſſer mean ſought. Laſtly, if the Summ reſerved be added to the greater mean it gives the greater extreme, and if the Remainder reſerved be ſubtracted from the leſſer mean it gives the leſſer extreme.

But to the end there may be a poſſibility of effecting the Problem propounded, the given lines muſt be liable to this

Determination.

14. The given difference of the extremes muſt be greater than the triple of the given difference of the means.

The truth of this Determination will be made manifeſt by the following Theorem, and that 'tis neceſſary, will be evident in 28° of the following Conſtruction of the Problem.

THEOREM.

15. If four right lines be in continual proportion, the difference of the extremes is greater than the triple difference of the means.

Suppoſ.

Suppoſ.

16. Q, R, S, T ∴; *viz.* Q . R :: R . S :: S . T.
17. Q ⊏ R. Whence R ⊏ S; and S ⊏ T.
18. . . . *Req. demonſtr.* . . . . . Q — T ⊏ 3R — 3S.

Demonſtration.

19. By *Suppoſ* in 16° and 17°, ... } Q, R, S, T ∴; alſo, Q ⊏ R.
20. Therefore by the *Lemma* prefixt before this *Probl.* 9. . . . . } Q—R . R—S :: R—S . S—T.
21. But if four quantities be Proportionals, the ſumm of the extremes is greater than the ſumm of the means, (*per prop.* 25. *Elem.* 5.) therefore from 20°, ... } Q + S — R — T ⊏ 2R — 2S.
22. And by adding R to each part in 21°, . . } Q + S — T ⊏ 3R — 2S.
23. Wherefore by ſubtracting S from each part in 22°, ... } Q — T ⊏ 3R — 3S.

Which was to be Demonſtr.

The Compoſition of the foregoing Probl. 9.

B ——————
C —— G
E ——— I ——
K —
L —
R ——————————
S ——————————
Q ——————————————
T ——————

B = 61 | L = 4
C = 20 | R = 100
EI = 41 | S = 80
EG = 25 | Q = 125
GI = 16 | T = 64
K = 5

Suppoſ.

24. B = the difference of the extremes of four right lines in continual proportion is given.
25. C = the difference of the means is given.
26. B ⊏ 3 C. (*Determination*)

Req. to find the Proportionals ſeverally.

Conſtruction.

27. Make EI = B — C.
28. Divide EI into two ſuch parts in G that C may be a mean Proportional between the parts, which may be done, (*per Probl.* 14. *Chap.* 5.) if C be not greater than ½ EI. But that C is leſs than ½ EI, I prove thus;

By the Determination in 26°, ... 3 C ⊐ B.
Therefore by ſubtracting C from each part, . . . 2 C ⊐ B — C.
But by *Conſtr.* in 27°, ... EI = B — C.
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . . 2 C ⊐ EI.
And conſequently, ... C ⊐ ½ EI.

Which was to be ſhewn. Therefore 'tis poſſible (*per Probl.* 14. *Chap.* 5.) to cut EI into two ſuch parts that C may be a mean Proportional between them. Suppoſe then that EI is ſo cut in G, and that EG is the greater part, and GI the leſſer; therefore,

EG . C :: C . GI.

29. Make K = EG — C, which is poſſible to be done, for by *Conſtr.* in 28°, EG ⊏ C.
30. Make L = C — GI, which is poſſible to be done, for by *Conſtr.* in 28°, C ⊏ GI.
31. By *Probl.* 8. *Chap.* 5. let it be made as K to C, ſo EG to a fourth Proportional, ſuppoſe it be found R, therefore

K . C :: EG . R.

32. Again, let it be made, as L to C, ſo GI to a fourth Proportional S, therefore

L . C :: GI . S.

33. Make Q = EG + R, whence Q — R = EG.
34. Make T = S — GI, whence S — T = GI; but that GI is leſs than S, as is implied by this Effection, I prove thus;

By *Conſtr.* in 30°, ... L = C — GI.
Therefore by adding GI to each part, ... L + GI = C.
Therefore, ... L ⊐ C.
Therefore from the Analogy in 32°, (*per Schol. prop.* 14. *Elem.* 5.) GI ⊐ S.
Which was to be proved.

35. I ſay Q, R, S, T (found out by *Conſtr.* in 33°, 31°, 32° and 34°,) are the four continual Proportionals ſought. But to make it manifeſt that they will ſatisfie the Problem, I ſhall prove three things, *viz.* Firſt, that the difference of the means R and S is equal to the given difference C: Secondly, that the difference of the extremes Q and T is equal to the given difference B: Thirdly and laſtly, that the ſaid Q, R, S, T are in continual proportion in this order, *viz.* Q . R :: R . S :: S . T.

Firſt then, that the difference of the means R and S is equal to the given difference C, I demonſtrate thus:

Prepar.

36. By *Probl.* 7. *Chap.* 5. let it be made as K to C, ſo C to a third proportional line M, therefore . . . . . K . C :: C . M.

Req. demonſtr. . . . . . R — S = C.

Demonſtration.

37. By *Conſtr.* in 28°, ... EG . C :: C . GI.
38. And by *Conſtr.* in 28°, ... EG ⊏ C.
39. Therefore from 37°, by Diviſion of Reaſon, EG—C . C :: C—GI . GI.
40. That is, by exchanging equal right lines, according to the *Conſtr.* in 29° and 30°, . K . C :: L . GI.
41. Therefore alternly, ... K . L :: C . GI.
42. And by drawing C as a common altitude into each of the two latter Terms in 41°, . K . L :: □C . □C,GI.
43. But from the *Conſtr.* in 36° and 32°, . . □K,M = □C; and □L,S = □C,GI.
44. Therefore from 42° and 43°, by exchange of equal Rectangles, ... K . L :: □K,M . □L,S.
45. But by reaſon of the common altitude M, this Analogy is manifeſt, ... K . L :: □K,M . □L,M.
46. Therefore from 44° and 45°, (*per prop.* 11. *Elem.* 5.) . . . . . □K,M . □L,S :: □K,M . □L,M.
47. Therefore from 46°, (*per prop.* 14. *Elem.* 5.) □L,S = □L,M.
48. And from 47°, (*per prop.* 14. *Elem.* 6.) . L . M :: L . S.
49. Therefore from 48°, (*per prop.* 14. *Elem.* 5.) M = S.
50. Again, by *Conſtr* in 31°, ... K . C :: EG . R.
51. And by *Conſtr.* in 36°, ... K . C :: C . M.
52. Therefore from 50° and 51°, (*per prop.* 11. *Elem.* 5.) . . . . . EG . R :: C . M.
53. Therefore alternly, ... EG . C :: R . M.
54. Therefore from 38° and 53°, by Diviſion of Reaſon, ... EG—C . C :: R—M . M.
55. And becauſe by *Conſtr.* in 29°, . . . K = EG — C.
56. Therefore from 54° and 55°, ... K . C :: R—M . M.
57. But by *Conſtr.* in 36°, ... K . C :: C . M.
58. Therefore from 56° and 57°, (*per prop.* 11. *Elem* 5.) . . . . . R—M . M :: C . M.
59. Therefore from 58°, (*per prop.* 14. *Elem.* 5.) R — M = C.
60. But it hath been proved in 49°, that . M = S.
61. Therefore from 59° and 60°, (*per Ax.* 6. *Chap.* 2.) . . . . . R — S = C. Which was to be Dem.

Secondly, that the difference of the extremes Q and T is equal to the given difference B, I demonſtrate thus:

62. . . . *Req. demonſtr.* . . . . . Q — T = B.

Demonſtr.

Demonſtration.

63. By *Conſtr.* in 33°, ... Q = EG + R.
64. And by *Conſtr.* in 34°, ... T = S — GI.
65. And from 61°, ... R ⊏ S.
66. Therefore from 63°, 64° and 65°, . . Q ⊏ T; and EG + R ⊏ S — GI.
67. And by ſubtracting the Equation in 64° from the Equation in 63°, ... Q — T = EG + R + GI — S.
68. But by *Conſtr.* in 28°, ... EI = EG + GI.
69. Therefore from 67° and 68°, (*per Ax.* 6. *Chap.* 2.) . . . . . Q — T = EI + R — S.
70. Again, by *Conſtr.* in 27°, ... B — C = EI.
71. And it hath been proved in 61°, that . . C = R — S.
72. Therefor the ſumm of the Equations in 70° and 71° gives . . . . . B = EI + R — S.
73. Therefore from 69° and 72°, (*per Ax.* 1. *Chap.* 2.) . . . . . Q — T = B. Which was to be Dem.

Thirdly and laſtly, that Q, R, S, T are in continual proportion I demonſtrate thus:

74. . . . *Req. demonſtr.* . . . . . Q . R :: R . S :: S . T.

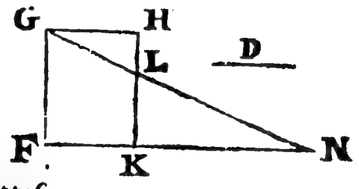
Demonſtration.

75. By *Conſtr.* in 31°, ... K . C :: EG . R.
76. And by *Conſtr.* in 29° and 33°, . . . EG — C = K; and Q — R = EG.
77. And it hath been proved in 61°, that . . R — S = C.
78. Therefore from 75°, 76° and 77°, by exchanging equal right lines, ... Q—R—R—S . R—S :: Q—R . R.
79. Therefore from the laſt preceding Analogy, by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. Q, R, S are ÷÷, *viz.* . . . . . Q . R :: R . S.
80. Again, by *Conſtr.* in 32°, ... L . GI :: C . S.
81. And by *Conſtr.* in 30° and 34°, . . . C — GI = L; and S — T = GI.
82. And it hath been proved in 61°, that . . R — S = C.
83. Therefore from 80°, 81° and 82°, by exchanging equal right lines, ... R—S—S—T . S—T :: R—S . S.
84. Therefore from the laſt preceding Analogy by the *Theor.* in 14° of *Probl.* 6. *Chap.* 7. R, S, T are ÷÷, *viz.* . . . . . R . S :: S . T.
85. Wherefore from 79° and 84°, (*per prop.* 11. *Elem.* 5.) . . . . . Q . R :: R . S :: S . T.

Which was to be demonſtrated in the laſt place. Therefore the Problem is ſatisfied.

Probl. X.

A Rectangle FGHK being given by Poſition, to draw a right line GN from G one of the angles oppoſite to the Baſe FK, to cut the Baſe produced, ſuppoſe in N, ſo as to make the Triangle KLN (lying without the Rectangle) equal to a given Space; ſuppoſe the Square of the right line D.



FK = 12 | KL = 10
FG = 15 | GN = 39
□D = 120 | LN = 26
KN = 24 | GL = 13

Suppoſ.

1. FGHK is a □ given.
2. *b* = FK or GH is given.
3. *c* = FG or KH is given.
4. *d* = D the ſide of a Square given.

Req.

Req. to find

5. KN a right line to be added to FK, ſo, as FKN may be a ſtrait line, and that GN being drawn it may make △KLN = □D.

Reſolution.

6. Put a for the deſired increaſe of the Baſe FK, *viz.* } $a = KN.$

7. Then becauſe △NKL and △NFG are equiangular, theſe ſides are Proportionals, (*per prop.* 4. *Elem.* 6.) *viz.* . . . . . } FN . FG :: KN . KL.

8. That is, in the letters belonging to the Reſolution, } $a+b \;.\; c :: a \;.\; \frac{ca}{a+b}$.

9. And becauſe (*per prop.* 41. *Elem.* 1.) . . . } □KN, KL = 2△KLN.

10. Therefore from 8°, 9°, 4°, 5° and 6°, . . . } $a \times \frac{ca}{a+b} = 2dd = 2\triangle KLN.$

11. Now to avoid an Equation between Solids, let it be made as c to d, ſo $2d$ to a fourth Proportional, call it t, therefore . . . . . } $c \;.\; d :: 2d \;.\; t.$

12. Whence, (*per prop.* 16. *Elem.* 6.) . . . } $ct = 2dd.$

13. Therefore from 10° and 12°, (*per Ax.* 1. *Ch.* 2) } $a \times \frac{ca}{a+b} = ct.$

14. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . } $a \;.\; t :: c \;.\; \frac{ca}{a+b}$.

15. And by drawing $a+b$ as a common Factor into each of the two latter Terms of the laſt Analogy, this ariſeth, ... } $a \;.\; t :: ca+cb \;.\; ca.$

16. And by caſting away the common Factor c, this Analogy ariſeth, ... } $a \;.\; t :: a+b \;.\; a.$

17. And from the laſt Analogy, by Diviſion of Reaſon, ... } $a-t \;.\; t :: b \;.\; a.$

18. Therefore, by comparing the Rectangle of the extremes to the Rectangle of the means, . . } $aa - ta = tb.$

19. Which Equation may be reſolved into this Analogy, ... } $a-t \;.\; \sqrt{tb} :: \sqrt{tb} \;.\; a.$

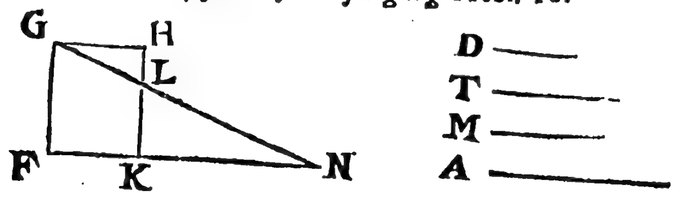
But the laſt Analogy doth manifeſtly conſiſt of three Proportionals, whereof the mean, to wit, $\sqrt{tb}$ is given, as alſo t the difference of the extremes a and $a-t$; therefore the extremes ſeverally, the greater whereof is the deſired increaſe KN, ſhall be given alſo, *per Probl.* 12. *Chap.* 5. and the Theorem in 24° of the ſame *Probl.* gives this following

CANON.

20. . . . . $\frac{1}{2}t + \sqrt{\frac{1}{4}tt + tb}$: = KN. That is, in words,

Let it be made as FG the altitude of the given Rectangle, to D the ſide of the given Square; ſo 2D the double of the ſame ſide, to a fourth Proportional, which may be called T. Then to the Square of half that fourth Proportional T, add the Rectangle of T into FK the Baſe of the given Rectangle, and extract the ſquare Root of the ſumm. Laſtly, that ſquare Root added to half T, gives KN the deſired increaſe of the Baſe.

The Compoſition of the foregoing Probl. 10.



Suppoſ.

21. FGHK is a □ given.

22. D is the ſide of a Square given.

Req. to find

23. KN a right line to be added to FK, ſo, as FKN may be a ſtrait line, and that GN being drawn, it may make △KLN = □D.

Conſtr.

Conſtruction.

24. By *Probl.* 8. *Chap.* 5. let it be made as FG to D, ſo 2D to a fourth proportional line, ſuppoſe it be found T; therefore, ... } FG . D :: 2D . T.

25. By *Probl.* 9. *Chap.* 5. find a mean Proportional, as M, between T and FK, therefore . . . } T . M :: M . FK.

26. Making M to be the mean of three Proportionals, and T the difference of the extremes, find the extremes ſeverally, (*per Probl.* 12. *Chap.* 5.) the greater whereof ſuppoſe to be A, then the leſſer ſhall be A — T, therefore . . . . . } A — T . M :: M . A.

That is, in 19° the laſt ſtep of the Reſolution, . . } $a-t \;.\; \sqrt{tb} :: \sqrt{tb} \;.\; a.$

27. Produce FK to ſuch a point N, that KN may be equal to the right line A found out in 26°, and draw GN; ſo ſhall △KLN be equal to the Square of the given line D, as was required. But that △KLN = □D, the following Demonſtration, form'd out of the preceding Reſolution by a repetition of the ſteps thereof in a backward (not direct) order will make manifeſt.

28. . . . *Req. demonſtr.* . . . . . △KLN = □D.

Demonſtration.

29. By *Conſtr.* in 26° and 27°, . . . } KN — T . M :: M . KN.
That is, in 19° the laſt ſtep of the Reſolution, ... } $a-t \;.\; \sqrt{tb} :: \sqrt{tb} \;.\; a.$

30. Therefore from 29°, (*per prop.* 17. *Elem.* 6.) . . . . . } □KN — □T,KN = □M.

31. Likewiſe from the *Conſtr.* in 25°, ... } □T,FK = □M.

32. Therefore from 30° and 31°, (*per Ax* 1. *Chap.* 2.) . . . . . } □KN — □T,KN = □T,FK.
That is, in 18°, ... } $aa - ta = tb.$

33. Therefore from 32°, (*per prop.* 14. *Elem.* 6.) . . . . . } KN — T . T :: FK . KN.
That is, in 17°, ... } $a-t \;.\; t :: b \;.\; a.$

34. And from 33°, by Compoſition of Reaſon, ... } KN . T :: KN+FK . KN.
That is, in 16°, ... } $a \;.\; t :: a+b \;.\; a.$

35. And from 34°, by taking in the the common altitude FG, . . } KN . T :: □FG,KN+□FG,FK . □FG,KN.
That is, in 15°, ... } $a \;.\; t :: ca + cb \;.\; ca.$

36. And becauſe △NKL and △NFG are equiangular, therefore (*per prop.* 4. *Elem.* 6.) . . . . . } FN . FG :: KN . KL.

37. And conſequently, (*per prop.* 16. *Elem.* 6.) . . . . . } □FN,KL = □FG,KN.

38. Therefore from 35° and 37°, by exchanging equal Rectangles, } KN . T :: □FG,KN+□FG,FK . □FN,KL.

39. And from 38°, by rejecting the the common altitude KN + FK, that is, FN, ... } KN . T :: FG . KL.
That is, in 14°, ... } $a \;.\; t :: c \;.\; \frac{ca}{a+b}$.

40. Therefore from 39°, (*per prop.* 16. *Elem.* 6.) . . . . . } □KN,KL = □FG,T.
That is, in 13°, ... } $a \times \frac{cn}{a+b} = ct.$

41. And becauſe from the *Conſtr.* in 24°, (*per prop.* 16. *Elem.* 6.) . . } . . 2□D = □FG,T.

42. Therefore from 40° and 41°, (*per Ax* 1. *Chap.* 2.) . . . } □KN,KL = 2□D.
That is, in 10°, ... } $a \times \frac{ca}{a+b} = 2dd.$

43. But (*per prop.* 41. *Elem.* 1.) . } □KN,KL = 2△KLN.

44. There-

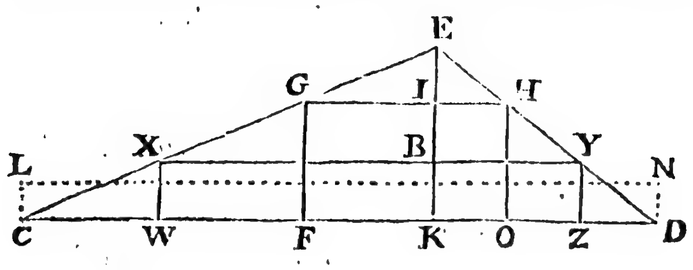
44. Therefore from 42° and 43°, *(per Ax. 1. Chap. 2.)* . ∴ 2 △ KLN = 2 □ D.
45. Therefore from 44°, *(per Ax. 21. Chap. 2.)* . . ∴ △ KLN = □ D.

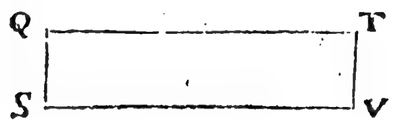
Which was to be demonſtr. Therefore that is done which the Problem required.

Probl. XI.

In a given Triangle to inſcribe a Rectangle equal to a given Rectangle. But the right line ariſing by the Application of the given Rectangle to the Baſe of the given Triangle, muſt not exceed a quarter of the Perpendicular falling upon that Baſe; and conſequently the double of the Rectangle muſt not be greater than the Triangle.

Note. By the Baſe of the Triangle given in this Problem, is meant ſuch a ſide as hath not an obtuſe angle at either of its ends within the Triangle; for 'tis eaſie to conceive, that if the Triangle be obtuſangled at the Baſe, a Rectangle cannot be inſcribed within the Triangle, ſo, as that the Baſe of a Rectangle may be a ſegment of the Baſe of the Triangle, and all the angular points of the Rectangle lye in the ſides of the Triangle.





Suppoſ.

1. CDE is a △ given.
2. b = CD the Baſe is given, and neither < C nor < D is obtuſe.
3. p = EK the Perpendicular is given.
4. □ ST, and the ſides thereof, to wit, SV and SQ are given.
5. $r = \frac{\square ST}{CD}$ = CL or DN is given; whence, □ CN = □ ST.

Req. to inſcribe

6. FGHO a □ within the △ CDE, with condition that
7. □ FGHO may be equal to □ ST.

Reſolution.

8. Put a for the altitude of the Rectangle required, *viz.* . . . . . } a = FG = KI = OH.
9. Which altitude ſubtracted from the Perpendicular EK, leaves EI, therefore from 3° and 8°, } $p - a$ = EI.
10. It is manifeſt by the *Lemma* prefixt before *Probl.* 11. *Chap.* 7. that . . . . . } EK . EI :: CD . GH.
11. Therefore (from 3°, 9° and 2°,) in the letters belonging to the Reſolution, ... } $p \,.\, p-a :: b \,.\, \frac{bp-ba}{p}$.
12. And becauſe according to the import of the Problem, ... } □ FG,GH = □ ST, or □ CD,CL.
13. Therefore (from 8°, 11°, 2° and 5°,) in the letters of the Reſolution, ... } $a \times \frac{bp-ba}{p} = br$.
14. Which laſt Equation is reducible to this Analogy, *(per prop. 14. Elem. 6.)* . . . . } $a \,.\, r :: b \,.\, \frac{bp-ba}{p}$.
15. Therefore from 11° and 14°, *(per prop. 11. Elem. 5.)* . . . . . } $p \,.\, p-a :: a \,.\, r$.

16. Which

16. Which laſt Analogy gives this Equation, *(per prop. 16. Elem. 6.)* . . . . . } $pa - aa = pr$.
17. And that Equation may be reſolved into theſe Proportionals, *(per prop. 14. Elem. 6.)* . . . . . } $p - a \,.\, \sqrt{pr} :: \sqrt{pr} \,.\, a$.

Of which three Proportionals the mean, to wit, $\sqrt{pr}$ is given, as alſo p the ſumm of the extremes, therefore the extremes ſeverally, (to wit, KI and EI, or EB and KB, the ſegments of the Perpendicular EK,) ſhall be given alſo, (by *Probl.* 13. *Chap.* 5.) and the Theorem in 21° of the ſame *Probl.* 13. gives this following

CANON.

18. . . . . } $\frac{1}{2}p + \sqrt{:\frac{1}{4}pp - pr:}$ = KI (= EB,) } *viz.* in words,
} $\frac{1}{2}p - \sqrt{:\frac{1}{4}pp - pr:}$ = KB (= EI,) }

Firſt, let it be made as the Baſe of the given Triangle, to one of the ſides of the given Rectangle; ſo the other ſide of the ſame Rectangle, to a fourth Proportional, which may be called r. Secondly, from the Square of half the Perpendicular falling upon the ſaid Baſe, ſubtract the Rectangle made of the ſaid Perpendicular and fourth Proportional r. Thirdly, extract the ſquare Root of the remainder. Fourthly, add and ſubtract the ſaid ſquare Root to and from the ſaid half Perpendicular; the ſumm and remainder ſhall be ſegments of the Perpendicular, either of which may be taken for the altitude of the Rectangle required to be inſcribed. Laſtly, the Baſe of the Rectangle ſought is equal to the right line ariſing by the Application of the given Rectangle to the altitude before found.

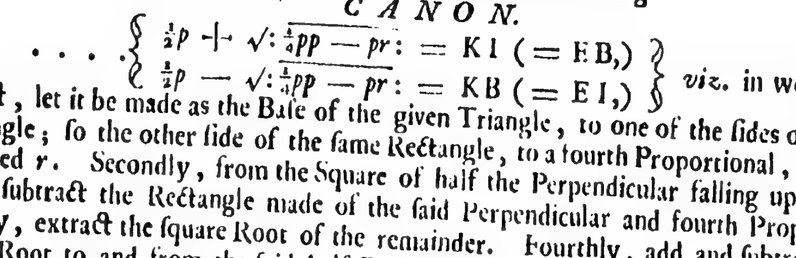
This Canon may be propounded in the form of a Theorem, which may eaſily be demonſtrated by a repetition of the ſteps of the preceding Reſolution in a direct (not retrograde) order; but taking the truth of the Canon for granted, I ſhall proceed to the Compoſition of the Problem, for the effecting whereof, 'tis neceſſary that the given quantities be qualified according to the tenour of this

Determination.

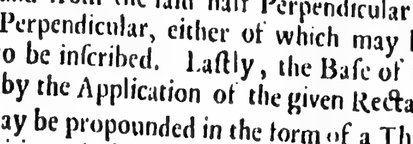
19. The right line ariſing by the Application of the given Rectangle, to the Baſe of the given Triangle, muſt not exceed a quarter of the Perpendicular falling upon that Baſe; and conſequently, the double of the Rectangle muſt not be greater than the Triangle.

Which Determination ſhews it ſelf openly in the Canon, where it appears, that to the end there may be a poſſibility of ſubtracting pr from $\frac{1}{4}pp$, 'tis neceſſary that . . . } pr not ⊏ $\frac{1}{4}pp$.
And conſequently, by dividing each part by p, ... } r not ⊏ $\frac{1}{4}p$.
And by doubling each part, ... } $2r$ not ⊏ $\frac{1}{2}p$.
And by drawing b into each part, ... } $2rb$ not ⊏ $\frac{1}{2}pb$.
But from 2° and 5°, ... } $2rb$ = 2 □ ST.
And by *prop.* 41. *Elem.* 1. . . . . } $\frac{1}{2}pb$ = △ CDE.
Therefore from the three laſt preceding ſteps, by exchanging equal quantities, ... } 2 □ ST not ⊏ △ CDE.
Therefore from the premiſſes, both the riſe and truth of the Determination are manifeſt.

The Compoſition of the foregoing Probl. 11.



M



Suppoſ.

20. CDE is a △ given.
21. CD the Baſe is given, and neither < C nor < D is obtuſe.

X x

22. EK

22. EK the Perpendicular is given.
23. □ST, and the ſides thereof, to wit, SV and SQ are given.
24. $CL = DN = \frac{\square ST}{CD}$ is given, *(per Probl. 8. Chap 5.)*
25. CL not $\sqsubset \frac{1}{4}$ EK. *(Determination.)*

Req. to inſcribe

26. □FGHO in △CDE, ſo, that □FGHO = □ST.

Conſtruction.

27. By *Probl. 9. Chap. 5.* find a mean Proportional, as M, between EK and CL, therefore,
EK . M :: M . CL.
28. By *Probl. 14. Chap. 5.* cut EK into two ſuch parts in I, that M (before found) may be a mean Proportional between the parts, which effection is poſſible if M be not greater than $\frac{1}{2}$ EK; but that M is greater than $\frac{1}{2}$ EK, I prove thus;

By the Determination in 25°, ... } CL not $\sqsubset \frac{1}{4}$EK.
Therefore by drawing EK into each part, ... } □EK,CL not $\sqsubset \frac{1}{4}$□EK.
But from the *Conſtr.* in 27°, *(per prop. 17. Elem 6.)* . . } □EK,CL = □M.
Therefore from the two laſt preceding ſteps, *(per Ax. 4. Chap. 2.)* . . . . . } □M not $\sqsubset \frac{1}{4}$□EK.
Therefore by extracting the ſquare Root out of each part, . } M not $\sqsubset \frac{1}{2}$EK.

Which was to be proved. Therefore 'tis poſſible to cut EK into two ſuch parts, that M ſhall be a mean Proportional between them; ſuppoſe then it be done, *(per Probl. 14. Chap. 5.)* and that the parts are EI and KI, therefore EI + KI = EK. Alſo,
EI (or EK — KI) . M :: M . KI.
That is, in 17°, ... $p - a \;.\; \sqrt{pr} :: \sqrt{pr} \;.\; a$.

29. Then ſet either of the ſaid parts of EK, ſuppoſe the greater part, from K to I; and by the point I, draw GIH parallel to CD. Laſtly, from the points G and H let fall GF and HO Perpendiculars to the Baſe CD; ſo ſhall FGHO be the incribed Rectangle required. But to make it manifeſt that the ſaid Rectangle will ſatisfie the Problem, two things are to be proved, *viz.* Firſt, that all the angles of the quadrilateral Figure FGHO are right angles; and then that the ſaid Rectangle is equal to the given Rectangle SQTV.
30. . . . *Req. demonſtr.* . . . . . FGHO is a □.

Demonſtration.

31. By *Conſtr.* in 29°, ... } GH ∥ FO.
32. Alſo by *Conſtr* in 29°, ... } GF and HO are ⊥ FO.
33. Therefore, *(per defin. 10. Elem. 1.)* . . . . . } ∠GFO = ∠HOF.
34. Therefore from 31°, 32°, 33°, *(per prop. 29. Elem. 1.)* } FGHO is □.

Which was to be Demonſtr.

It remains to prove that □FG,GH (that is, □FH) = □ST; but that equality will be manifeſt by the following Demonſtration, form'd out of the preceding Reſolution by a retrograde repetition of the ſteps thereof.

35. . . . *Req. demonſtr.* . . . . . □FG,GH (that is, □FH) = □ST.

Demonſtration.

36. By *Conſtr.* in 28°, ... } EK — KI . M :: M . KI.
That is, in 17°, ... } $p - a \;.\; \sqrt{pr} :: \sqrt{pr} \;.\; a$.
37. And from 22°, 29° and 34°, ... } FG = KI.
38. Therefore from 36° and 37°, ... } EK — FG . M :: M . FG.
39. And from 38°, *(per prop. 17. Elem. 6.)* . } □EK,FG, — □FG = □M.
40. Likewiſe from the *Conſtr.* in 27°, ... } . . . . □EK,CL = □M.
41. Therefore from 39° and 40°, *(per Ax. 1. Chap. 2.)* . . . . . } □EK,FG, — □FG = □EK,CL.
That is, in 16°, ... } $pa - aa = pr$.
42. Therefore from 41°, *(per prop. 14. Elem. 6.)* } EK . EK — FG :: FG . CL.
That is, in 15°, ... } $p \;.\; p - a :: a \;.\; r$.
43. By *Conſtr.* in 29°, ... } GH ∥ CD.

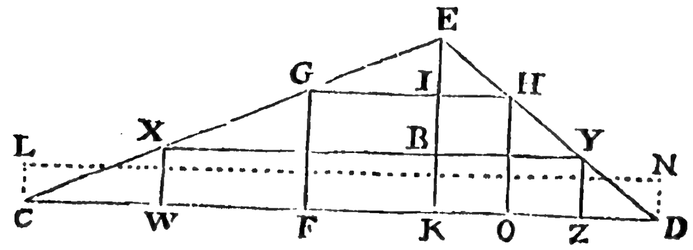
44. There-

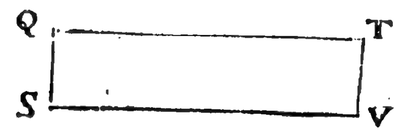
44. Therefore by the *Lemma* prefixt before *Probl. 11. Chap. 7.* . . . . . } EK . EK — FG :: CD . GH.
That is, in 11°, ... } EI . $p \;.\; p - a :: b \;.\; \frac{bp - ba}{p}$.
45. Therefore from 42° and 44°, *(per prop. 11. Elem. 5.)* . . . . . } FG . CL :: CD . GH.
46. And from 45°, *(per prop. 16. Elem. 6.)* . } □FG,GH = □CD,CL.
47. But from the *Conſtr.* in 24°, ... } . . . □ST = □CD,CL.
48. Therefore from 46° and 47°, *(per Ax. 1. Chap. 2.)* . . . . . } □FG,GH (that is, □FH) = □ST.

Which was to be Demonſtr. Therefore that is done which the Problem required.

49. *Note.* If KB be made equal to EI, then ſhall EB be equal to IK, (by reaſon of the common interſegment IB,) and conſequently EK is cut in B as well as in I, according to the import of the preceding Conſtruction in 28°. Therefore if by the point B a parallel be drawn to the Baſe CD, as XBY, and from the points X and Y, perpendiculars be let fall upon CD, as XW and YZ, the inſcribed □WY, that is, WXYZ ſhall be alſo equal to the given Rectangle ST, that is, SQTV, and the Demonſtration may be formed as before, by taking KB or WX inſtead of KI. So two Rectangles are inſcribed in the given △CDE, each of which is equal to the given Rectangle SQTV.

Examples in Numbers to illuſtrate the preceding Reſolution of Probl. 11.





Suppoſ.

50. CD = 168 the Baſe
51. CE = 117
52. DE = 75 } the leggs } of △CDE are given ſeverally.
53. SV = 84
54. SQ = 20 } the ſides of □ST, therefore □ST = 1680.
55. □FH = □ST = 1680.
56. EK ⊥ CD.

Req. to find in Numbers,

57. FG or HO,
58. GH or FO, } the ſides of □FH.

Solution Arithmetical.

59. EK = 45, found out by the three ſides of △CDE given in 50°, 51°, 52°, by the help of *Theor. 4.* in 68° of *Probl. 8. Chap. 8.*
60. KI = 30
61. IE = 15 } found out by the Canon in 18° of this Problem.
62. FG = 30 = KI, found out in 60°.
63. GH = 56 = FO, given from 55° and 62°. For $\frac{1680}{30} = 56$.

The Proof.

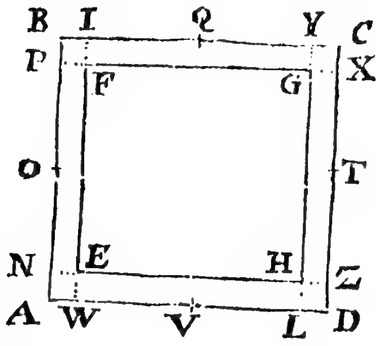
64. □FG,GH = 1680 = □SV,SQ, (= □ST.) Alſo,
65. . . { 15 (or 45 — 30) . 56 :: 45 . 168; that is, EI (or EK — KI) . GH :: EK . CD.
66. Therefore by the converſe of the *Lemma* prefixt before *Probl. 11. Chap. 7.* GH ∥ CD. Alſo G and H are in CE and DE.

Another Example.

67. Again, the ſame △CDE and its ſides being given in numbers as before in 50°, 51° and 54°, you will find (by the like Operation as in *Example* 1.) XW = 15 = YZ, and XY = 112 = WZ; whence the Area of □XZ is 1680, which is the ſame with the Area of □ST preſcribed in *Example* 1. And that the Rectangle XZ or WXYZ is inſcribed in △CDE, may be proved in like manner as before in 65° and 66°.

Probl. XII.

Within a given Rectangle to make a Rectangle, with this condition, that there may be an equal parallel diſtance between their ſides; and that the Space lying between the ſides of both the Rectangles may be to the inſcribed Rectangle in a given Reaſon.



BC = 440
BA = 400
R = 2
S = 9
FG = 400
FE = 360
FI = 20 = GX

Suppoſ.

1. ABCD is a □ given.
2. b = BC = AD is given.
3. c = BA = CD is given.
4. $b \sqsubset c$.
5. $d = b - c$ = BC − BA is given.
6. r and s the Terms of a given Reaſon.

Req. to make

7. □EFGH within the □ABCD in ſuch manner, that
8. FI = GX = HL = EN. Alſo, that
9. □BC,BA − □FG,FE . □FG,FE :: r . s.

Prepar.

10. By viewing the Diagram, and reflecting upon what is given and required, it will be evident that — BC = FG + 2GX (2FI.)
11. Likewiſe, ... — BA = FE + 2GX (2FI.)
12. And by ſubtracting the Equation in 11° from that in 10°, this remains, *viz.* . . . . . — BC − BA = FG − FE (= d.)
13. Whence 'tis manifeſt that the difference between the length and breadth of the Rectangle required to be inſcribed is given; for 'tis equal to the difference between the length and breadth of the given □ABCD.

Reſolution.

14. Put a for the ſhorter ſide of the required Rectangle EFGH, *viz.* . . . . . — a = EF = HG.
15. Therefore from 5°, 13° and 14°, the longer ſide ſhall be . . . . . — $a + d$ (= FG.)
16. Therefore from 14° and 15°, the Area of □EFGH is equal to . . . . . — $aa + da$ (= □EFGH.)
17. And from 2° and 3° the Area of the given Rectangle is . . . . . — bc (= □ABCD.)
18. And by ſubtracting the Area in 16° from that in 17°, there will remain . . . . . — $bc - aa - da$ (= BFGCHDEA.)
19. Therefore from 9°, 18° and 16°, according to the tenour of the Problem, this Analogy ariſeth, *viz.* — $r . s :: bc - aa - da . aa + da$.

20. There-

20. Therefore from 19°, by Compoſition of Reaſon, — $r + s . s :: bc . aa + da$.
21. Now to avoid an Equation between Solids, let it be made as $r + s$ to s, ſo b to a fourth Proportional, call it f, therefore . . . . — $r + s . s :: b . f$.
22. Therefore from 20° and 21°, (*per prop.* 11. *El.* 5.) — $b . f :: bc . aa + da$.
23. And this Analogy, by reaſon of the common Factor c is evident, (*per prop.* 1. *Elem.* 6.) *viz.* — $b . f :: bc . fc$.
24. Therefore from 22° and 23°, (*per prop.* 11. *Elem.* 5.) . . . . . — $bc . aa + da :: bc . fc$.
25. Therefore from 24°, (*per prop.* 14. *Elem.* 5.) — $aa + da = fc$.
26. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . — $a . \sqrt{fc} :: \sqrt{fc} . a + d$.

Of which three Proportionals the mean, to wit, $\sqrt{fc}$ is given, as alſo d the difference of the extremes $a + d$ and a, therefore *per Probl.* 12. *Chap.* 5. the extremes ſhall be given ſeverally, (which are the ſides of the Rectangle required to be inſcribed;) and the Theorem in 24° of the ſaid *Probl.* 12. gives this following

CANON.

27 . . . . $\left\{\begin{array}{l}\sqrt{:\frac{1}{4}dd + fc:} - \frac{1}{2}d = \text{EF.} \\ \sqrt{:\frac{1}{4}dd + fc:} + \frac{1}{2}d = \text{FG.}\end{array}\right\}$ *viz.* in words,

Make $r + s$ the ſumm of the Terms of the given Reaſon the firſt of four Proportionals, s the latter of thoſe Terms the ſecond Proportional, BC or AD the longer ſide of the given Rectangle the third Proportional, and to thoſe three find a fourth, which may be called f. Then to the Square of half the difference between the length and breadth of the given Rectangle, add the Rectangle made of the ſaid fourth Proportional and the ſaid breadth. Then to and from the ſquare Root of that ſumm, add and ſubtract the ſaid half difference, ſo ſhall the ſumm and remainder made by that addition and ſubtraction be the deſired length and breadth of the Rectangle to be inſcribed; which length or breadth being ſubtracted from the length or breadth of the given Rectangle, the half of the remainder is the parallel diſtance between the ſides of both the ſaid Rectangles.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 12.

Suppoſ.

28. BC = 440
29. BA = 400
} the ſides of the given Rectangle ABCD.
30. R = 2
31. S = 9
} the Terms of the given Reaſon.

Req. to make

32. □EFGH within the □ABCD, in ſuch manner, that
33. FI = GX = HL = EN. Alſo,
34. □BC,BA − □FG,FE . □FG,FE :: R . S :: 2 . 9.

Solution Arithmetical.

35. □BC,BA = 176000, from 28° and 29°.
36. FG . . . . = . . 400,
37. FE . . . . = . . 360,
} found out by the Canon in 27°.
38. □FG,FE = 144000, from 36° and 37°.
39. □BC,BA − □FG,FE = 32000, from 35° and 38°.

The Proof.

40. R . S :: □BC,BA − □FG,FE . □FG,FE.
41. 2 . 9 :: 32000 . 144000.
42. FI = 20 = GX = HL = EN the parallel diſtance.

Another way of reſolving the preceding Probl. 12.

43. The ſame things being given and required as before, let a be put for the ſide of a Square equal to the inſcribed Rectangle, therefore . . . . . — aa = □EFGH.
44. From 2° and 3° the Area of the given Rectangle is — bc.
45. Therefore the difference of thoſe Rectangles is — $bc - aa$.

46. There-

46. Therefore according to the tenour of the Problem this Analogy ariſeth, *viz.* . . . . . } $r \cdot s :: bc - aa \cdot aa.$

47. Whence, by Compoſition of Reaſon, this Analogy ariſeth, which gives the Area of the Rectangle to be inſcribed, ... } $r + s \cdot s :: bc \cdot aa.$

From the laſt Analogy ariſeth

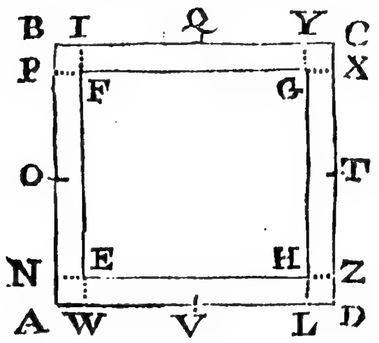
CANON 2.

As the ſumm of the Terms of the given Reaſon is to the latter Term, ſo is the Area of the given Rectangle to the Area of the inſcribed Rectangle; therefore the Area of the inſcribed Rectangle is given alſo. Then the Area of the inſcribed Rectangle being given, as alſo the difference of the ſides, (for this difference, as before hath been ſhewn in 13°, is equal to the difference of the ſides of the given Rectangle,) the ſides ſhall be given ſeverally by *Probl.* 1. of this Chapter. And laſtly, the length of the inſcribed Rectangle being ſubtracted from the length of the given Rectangle, or the breadth from the breadth, the half of the remainder is the parallel diſtance between the ſides of both the Rectangles.

This Canon may be exemplified by the numbers given in the preceding *Examp.* 1. And in regard the Compoſition of this Problem according to either of the ſaid ways of Reſolution will not be difficult to him that underſtands the preceding Problems of this Chapter, I ſhall wave the Compoſition, and leave it as an exerciſe to the induſtrious Learner.

Probl. XIII.

A Nobleman having made choice of a plot of ground for the making of a Garden of pleaſure, gives direction to a Surveyor to trace out a Rectangle, or long-Square, whoſe length and breadth ſhall be equal to two given right lines B C and B A. Alſo to make another long-Square within the former, in ſuch manner, that there may be an equal parallel diſtance between the ſides of both the ſaid long-Squares. Moreover, the Nobleman's deſign is, that the ſpace lying between the ſides of both the long-Squares ſhall be ſunk perpendicularly, to make a Mote or Ditch whoſe depth ſhall be equal to a given right line S, and the breadth thereof ſuch, that the earth digged out of the intended Ditch being layd upon the ſaid interiour long-Square as a Baſe, may be capable of raiſing a rectangular Mount whoſe altitude ſhall be equal to a given right line R. The Queſtion is, to find out the length and breadth of the interiour long-Square, as alſo the breadth of the Ditch, that is, the parallel diſtance between the ſides of both the long-Squares.



R ——

S ——————

Suppoſ.

1. B C = A D, } the ſides of □ A B C D are ſeverally given.
2. B A = C D, }
3. R = a right line given for the height of the deſired Mount.
4. S = a right line given for the depth of the deſired Mote or Ditch.

Req. to find

5. F G, or E H, the length of the interiour □ E F G H.
4. E F, or H G, the breadth of the ſaid □ E F G H.

7. F I =

7. F I = G X = H L = E N the parallel diſtance.
8. R × □ E F G H = S × Plane B F G C D H E A.

Conſtruction.

9. By the preceding *Probl.* 12. let a Rectangle or long-Square be made within the given □ A B C D, in ſuch manner, that there may be an equal parallel diſtance between their ſides, and that the Space lying between the ſides of both Rectangles may have ſuch proportion to the inſcribed Rectangle, as the given right line R, (preſcribed for the height of the Mount,) hath to the given right line S, (preſcribed for the depth of the Ditch.) Now ſuppoſe that by the ſaid 12th Problem the □ E F G H is ſo made within the □ A B C D, that the ſides of the one keep an equal parallel diſtance to the ſides of the other, *viz.* F I = G X = H L = E N; and that as R is to S, ſo the interval or Plane B F G C D H E A, to the □ E F G H. Then it will be manifeſt (*per prop.* 34. *Elem* 11.) that R × □ E F G H (which is equal to the Solidity of the Mount,) is equal to S × Plane B F G C D H E A, (which is equal to the ſolidity of the Ditch;) as was required.

The quantities of the length and breadth of the inſcribed Rectangle (or Baſe of the Mount,) as alſo of the parallel diſtance (or breadth of the Ditch) may be found out in numbers by either of the Canons of the preceding *Probl.* 12. and for the greater evidence, I ſhall here add

An Example in Numbers, to illuſtrate the preceding Conſtruction of Probl. 13.

Suppoſ.

10. B C = 440 } the ſides of the given □ A B C D.
11. B A = 400 }
12. R = 2 } the given altitude of the Mount to be raiſed perpendicularly upon □ E F G H.
13. S = 9 } the given depth of the Ditch B F G C D H E A.

Req. to find out in Numbers,

14. F G and F E the ſides of □ E F G H. Alſo,
15. F I = G X = H L = E N the parallel diſtance; with condition alſo, that
16. R × □ F G, F E may be equal to S × Plane B F G C D H E A.

Solution Arithmetical.

17. F G = 400 }
18. F E = 360 } found out by the quantities given in 10°, 11°, 12°, 13°, according
19. F I = 20 } to the preceding Conſtruction in 9° of this *Probl.* 13.

The Proof.

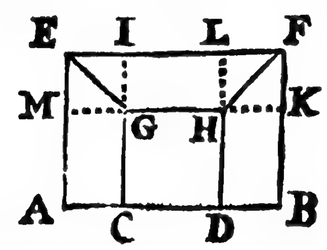
20. □ F G, F E = 144000, the Area of □ E F G H, *viz.* the Baſe of the Mount.
21. R × □ F G, F E = 288000, the Solid content of the Mount.
22. □ B C, B A − □ F G, F E = 32000, the Area of B F G C D H E A.
23. S × Area of B F G C D H E A = 288000, the Solidity of the Ditch.
24. R × □ F G, F E = S × Plane B F G C D H E A = 288000; as was required.

Probl. XIV.

Within a given Rectangle A E F B to make a Rectangle C G H D, with this condition, that after the right lines E G and H F are drawn, the Spaces C G H D, E G H F, H D B F and G C A E may be equal to one another, and conſequently every one of them equal to a fourth part of the given Rectangle A E F B.

Suppoſ.

1. A E F B is a □ given.
2. I E = L F = A C = D B.
3. E M = F K = H L = I G.
4. I L = G H = C D.
5. *g* = A B = E F is given.
6. *f* = A E = B F is given.



Req.

Req. to make

7. $\square CGHD = EGHF = HDBF = GCAE = \frac{1}{4}\square AEFB$.

Reſolution.

8. Put . . . . . $a = CD = GH = IL$.
9. Then becauſe $EF - IL = IE + LF$, 'tis manifeſt from 2°, 5° and 8°, that . . . $g - a = IE + LF$.
10. And becauſe by *Suppoſ* in 2° $IE = LF$, the half of the Equation in 9° gives . . $\frac{1}{2}g - \frac{1}{2}a = IE = LF$.
11. And the ſumm of the Equations in 8° and 10° gives . . . . . $\frac{1}{2}g + \frac{1}{2}a = IF = EL$.
12. Then ſuppoſing $\square CGHD$ to be equal to $\frac{1}{4}\square AEFB$, that is, $\frac{1}{4}fg$; let $\frac{1}{4}fg$ be divided by a, that is, GH, and the Quotient gives . $\frac{\frac{1}{4}fg}{a} = HD = BK$.
13. And becauſe $BF - BK = KF$, by ſubtracting $\frac{\frac{1}{4}fg}{a}$ from f, that is, BK from BF, there will remain . . . . . $f - \frac{\frac{1}{4}fg}{a} = KF = LH$.
14. Now the Problem requires that . . . $\square IF,KF$ (that is, $EGHF$) $= \frac{1}{4}\square AE,EF$.
15. That is, in the letters belonging to the Reſolution, (as appears by the 11th and 13th ſteps,) . . . . . $\frac{1}{2}g + \frac{1}{2}a \times f - \frac{\frac{1}{4}fg}{a} = \frac{1}{4}fg$.
16. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $\frac{1}{2}g + \frac{1}{2}a \,.\, \frac{1}{2}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}$.
17. And by doubling the two firſt Terms of that Analogy, this ariſeth, *viz.* . . . $g + a \,.\, \frac{1}{2}g :: f \,.\, f - \frac{\frac{1}{4}fg}{a}$.
18. Whence by Converſion of Reaſon, . . $g + a \,.\, \frac{1}{2}g + a :: f \,.\, \frac{\frac{1}{4}fg}{a}$.
19. And by drawing a into each of the two latter Terms of the laſt preceding Analogy, $g + a \,.\, \frac{1}{2}g + a :: fa \,.\, \frac{1}{4}fg$.
20. And by dividing each of the two latter Terms of the Analogy in 19° by f, this ariſeth, *viz.* . . . . . $g + a \,.\, \frac{1}{2}g + a :: a \,.\, \frac{1}{4}g$.
21. Whence, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation ariſeth, *viz.* . . . . . $aa + \frac{1}{2}ga = \frac{1}{4}gg + \frac{1}{4}ga$.
22. And by ſubtracting $\frac{1}{4}ga$ from each part of the Equation in 21°, this ariſeth, *viz.* . $aa + \frac{1}{4}ga = \frac{1}{4}gg$.
23. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $a + \frac{1}{4}g \,.\, \frac{1}{2}g :: \frac{1}{2}g \,.\, a$.
24. But of thoſe three continual Proportionals, the mean, to wit, $\frac{1}{2}g$ is given, as alſo $\frac{1}{4}g$ the difference of the extremes $a + \frac{1}{4}g$ and a, therefore the extremes ſhall be given ſeverally, (*per Probl.* 12. *Chap.* 5.) the leſſer of which ſhall be equal to the deſired line CD, (= GH = IL,) repreſented by a in the precedent Reſolution of this Problem: And the Theorem in 24° of *Probl.* 12. *Chap.* 5. gives this following

CANON.

25. . . . . $a = \frac{1}{2}\sqrt{\frac{1}{64}gg} - \frac{1}{8}g = CD = GH = IL$

That is, in words,

To the Square of one eighth part of the Baſe (that is, either of the ſides) of the given Rectangle, add a quarter of the Square of the ſame Baſe, and from the ſquare Root of the ſumm ſubtract one eighth part of the ſaid Baſe; the remainder ſhall be that ſide of the required Rectangle which is a ſegment of the Baſe of the given Rectangle. Whence the reſt of the lines in the Diagram belonging to this *Probl.* 14. ſhall be given alſo.

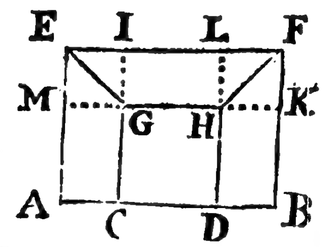
26. It is evident, that no Term of any Analogy or Equation in the foregoing Reſolution exceeds the dimenſions of a Square, and therefore the forming of the Compoſition of this Problem by a retrograde repetition of the ſteps of the Reſolution will not be difficult to him that underſtands the Reſolutions and Compoſitions of the precedent Problems of this

this Chapter; waving therefore the Geometrical Effection and Demonſtration, I ſhall apply the Canon before expreſt to the Arithmetical Solution of the Problem propounded.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 14.

Suppoſ.

27. AEFB is a $\square$, whoſe Baſe is AB, and altitude AE.
28. $AB = 10 = EF$ is given.
29. $AE = 6 = BF$ is given.
30. $EM = FK = HL = IG$.
31. $IE = LF = AC = DB$.



Req. to find in numbers,

32. The quantities of the lines CD, (= IL,) IE, (= LF,) HD, (= BK,) FK, (= HL,) with this condition, that the Area of every one of theſe four Spaces, to wit, $\square CGHD$, $EGHF$, $HDBF$ and $GCAE$ may be equal to a quarter of the Area of the given Rectangle AEFB, *viz.*

$\square CGHD = EGHF = HDBF = GCAE = 15 = \frac{1}{4}\square AEFB$.

Solution Arithmetical.

33. From 28°, by the Canon in 25°, you will find . . . $\sqrt{\frac{225}{16}} - \frac{1}{4} = CD = GH = IL$.
34. And by ſubtracting $\sqrt{\frac{225}{16}} - \frac{1}{4}$ from 10, that is, IL from EF, there will remain . . . $\frac{41}{4} - \sqrt{\frac{225}{16}} = IE + LF$.
35. And becauſe $IE = LF$, the half of $\frac{41}{4} - \sqrt{\frac{225}{16}}$ gives . . . . . $\frac{41}{8} - \sqrt{\frac{225}{64}} = IE = LF = KH$.
36. And the ſumm of the numbers in 33° and 35° makes . . . . . $\frac{21}{8} + \sqrt{\frac{225}{64}} = IF = EL$.
37. Then by dividing 15 the Area of $\square CGHD$, that is, $\frac{1}{4}$ of 60 the Area of $\square AEFB$, by $\sqrt{\frac{225}{16}} - \frac{1}{4}$, that is, CD, the Quotient gives . $\frac{1}{4} + \sqrt{\frac{1225}{400}} = DH = BK$.
38. And by ſubtracting $\frac{1}{4} + \sqrt{\frac{1225}{400}}$ from 6, that is, BK from BF, there will remain . . . $\frac{23}{4} - \sqrt{\frac{1225}{400}} = FK = HL$.

So in the ſix laſt preceding ſteps the quantities of all the lines ſought by *Probl.* 14. are found out in numbers; but that they will ſatisfie the condition preſcribed in 32°, will be evident by

The Proof.

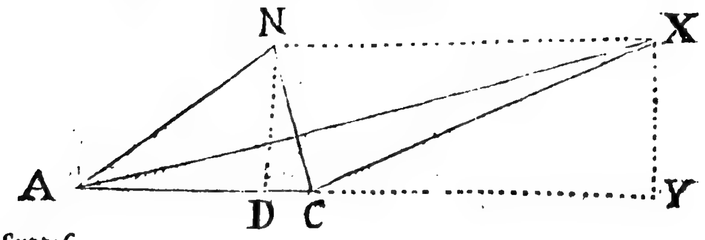
39. The Product of . . . . . $\overline{-\frac{1}{4} + \sqrt{\frac{225}{16}}}$ into $\overline{\frac{1}{4} + \sqrt{\frac{1225}{400}}} = 15$.
That is, ... CD into DH $= CGHD$.
40. The Product of . . . . . $\overline{\frac{21}{8} + \sqrt{\frac{225}{64}}}$ into $\overline{\frac{21}{4} - \sqrt{\frac{1225}{400}}} = 15$.
That is, ... IF into FK $= EGHF$.
41. The Product of . . . . . $\overline{\frac{1}{4} + \sqrt{\frac{1225}{400}}}$ into $\overline{\frac{41}{8} - \sqrt{\frac{225}{64}}}$
That is, ... BK into KH $= \square HDBK$.
42. The Product of . . . . . $\overline{\frac{21}{8} - \sqrt{\frac{1225}{1600}}}$ into $\overline{\frac{41}{8} - \sqrt{\frac{225}{64}}}$
That is, ... $\frac{1}{2}KF$ into KH $= \triangle HKF$.
43. The ſumm of the Products in 41° and 42° makes . . . . . $15 = \square HDBK + \triangle HKF = HDBF$.
44. And from 27°, 30°, 31° and 43°, ... $GCAE = HDBF = 15$.
45. Therefore from 39°, 40°, 43° and 44°, 'tis evident that
$\square CGHD = EGHF = HDBF = GCAE = 15 = \frac{1}{4}\square AE,FB$.

Which was to be done. All which Calculations will be evident to him that underſtands the Arithmetick of Surd-numbers, handled at large in *Chap.* 9. *Book* II. of this Treatiſe.

Probl. XV.

The Baſe, Perpendicular and Proportion of the leggs of a plain Triangle being ſeverally given, to find out the Triangle. But the given lines muſt be ſubject to the Determination hereafter expreſt.

Note. There is more than enough given in this Problem, unleſs it requires a Triangle that hath either unequal acute angles, or elſe an obtuſe angle at the Baſe; in the firſt of thoſe Caſes the Perpendicular falls within the Triangle, in the latter without; but the following Reſolution may be applied to each Caſe.



Suppoſ.

1. △ ACN is acute-angled at the ends of the Baſe AC.
2. △ ACX is obtuſe-angled at C, the end of the Baſe AC.
3. b = AC the Baſe is given.
4. p = ND = XY the Perpendicular is given.
5. r and s are the given Terms of the Proportion of the leggs, *viz.*

$$r \,.\, s :: AN \,.\, NC :: AX \,.\, XC.$$

6. $r \sqsubset s$.

Req. to find the Triangle.

Reſolution.

7. Put a for the diſtance from the foot of the Perpendicular to the remoter end of the Baſe, *viz.* ſuppoſe . . . . a = DA or YA.
8. Therefore from 3° and 7°, the diſtance from the foot of the Perpendicular to the nearer end of the Baſe is . . . $b-a$, or $a-b$; viz. DC or YC.
9. The Square of which diſtance is . . $aa-2ba+bb$ (= □ DC or □ YC.)
10. The Square of the diſtance in 7° is aa (= □ DA or □ YA.)
11. The Square of the given Perpendicular in 4° is . . . . . pp (= □ ND = □ XY.)
12. By *prop.* 47. *Elem.* 1. . . . . □ DC + □ ND = □ CN.
13. Likewiſe, ... □ YC + □ YX (□ ND) = □ CX.
14. Therefore from 9°, 11°, 12° and 13°, the Square of the leſſer legg, in the letters of the Reſolution, is . . . $aa-2ba+bb+pp$ (= □CN or □CX.)
15. Again, by *prop.* 47. *Elem.* 1. . . □ DA + □ ND = □ AN.
16. Likewiſe, ... □ YA + □ YX (□ ND) = □ AX.
17. Therefore from 10°, 11°, 15° and 16°, the Square of the greater legg is $aa+pp$ (= □ AN or □ AX.)
18. And conſequently the greater legg is $\sqrt{:aa+pp:}$ (= AN or AX.)
19. And from 14°, the leſſer legg is . $\sqrt{:aa-2ba+bb+pp:}$ (= CN or CX.)
20. Therefore from 5°, 18° and 19°, according to the tenour of the Problem, $r \,.\, s :: \sqrt{:aa+pp:} \,.\, \sqrt{:aa-2ba+bb+pp:}$
21. Therefore from 20°, (*per prop.* 22. *Elem.* 6.) . . . . . $rr \,.\, ss :: aa+pp \,.\, aa-2ba+bb+pp.$
22. Now in order to find out an Equation wherein the higheſt Power of the line a ſought may not exceed a Square; to r and s find a third Proportional, which may be called t, therefore, . . . $r \,.\, s :: s \,.\, t.$

23. There-

23. Therefore from 22°, (*per Coroll. prop.* 20. *Elem.* 6.) . . . . . $rr \,.\, ss :: r \,.\, t.$
24. Therefore from 21° and 23°, (*per prop.* 11. *Elem.* 5.) . . . . . $r \,.\, t :: aa+pp \,.\, aa-2ba+bb+pp.$
25. Therefore from 24°, by Converſion of Reaſon, ... $r \,.\, r-t :: aa+pp \,.\, 2ba-bb.$
26. Therefore inverſly, ... $r-t \,.\, r :: 2ba-bb \,.\, aa+pp.$
27. Let it be made as $r-t$ to r, ſo b to a fourth Proportional, which may be called f, therefore, ... $r-t \,.\, r :: b \,.\, f.$
28. Therefore from 26° and 27°, (*per prop.* 11. *Elem.* 5.) . . . . . $b \,.\, f :: 2ba-bb \,.\, aa+pp.$
29. And by drawing $2a-b$ as a common Factor into b and f ſeverally, this Analogy is manifeſt, (*per prop.* 1. *El.* 6.) $b \,.\, f :: 2ba-bb \,.\, 2fa-fb.$
30. Therefore from 28° and 29°, (*per prop.* 11. *Elem.* 5.) . . . . . $2ba-bb \,.\, aa+pp :: 2ba-bb \,.\, 2fa-fb.$
31. Therefore from 30°, (*per prop.* 14. *Elem.* 5.) this Equation ariſeth, . . $2fa-fb = aa+pp.$
32. Whence, by adding fb to each part, $2fa = aa+pp+fb.$
33. And by ſubtracting aa from each part of the laſt Equation, this ariſeth, . . $2fa-aa = pp+fb.$
34. Which laſt preceding Equation may be converted into this Analogy, *viz.* $2f-a \,.\, \sqrt{:pp+fb:} :: \sqrt{:pp+fb:} \,.\, a.$
35. But that Analogy doth manifeſtly conſiſt of three continual Proportionals, whereof the mean, to wit, $\sqrt{:pp+fb:}$ is given, as alſo $2f$ the ſumm of the extremes $2f-a$ and a; therefore the extremes ſhall be given ſeverally, (by *Probl.* 13. *Chap.* 5.) either of which may be taken for the line a ſought, *viz.*

$$a = f + \sqrt{:ff-pp-fb:} = YA;$$
$$\text{Or, } a = f - \sqrt{:ff-pp-fb:} = DA.$$

36. From 34° and 35° 'tis eaſie to perceive that $\sqrt{:pp+fb:}$ cannot be greater than f; for the mean of three Proportionals never exceeds half the ſumm of the extremes, (as hath been ſhewn in 20° of *Probl.* 13. *Chap.* 5.) But the ſaid $\sqrt{:pp+fb:}$ may ſometimes be equal to, and ſometimes leſs than f; to the end therefore there may be a poſſibility of finding out a Triangle to ſatisfie the Problem propounded, the given lines muſt be ſubject to this following

Determination, ... $\sqrt{:pp+fb:}$ not $\sqsubset f$.

That is, in words,

Firſt, if it be made as r to s, ſo s to a third Proportional t. Secondly, as the exceſs of r above t, to r; ſo the given Baſe b to a fourth Proportional f Then the ſide of a Square equal to the ſumm of the Square of the given Perpendicular p and the Rectangle of f into b, muſt not be greater than f; for when the ſaid ſide happens to be greater than f, 'tis impoſſible to find a Triangle qualified as the Problem requires, by the help of the given lines r, s, b and p.

This Determination is diſcovered by the three Proportionals in 34°, which are rightly inferr'd from the preceding Reſolution; and ſince the Reſolution is clearly Geometrical as well as Arithmetical, I ſhall take the truth of the Determination for granted.

37. It hath before been declared in 35°, that the diſtance ſought, which is repreſented by a in the Reſolution, may be either of the two right lines or extreme Proportionals found out in the ſaid 35th ſtep; which two right lines will be equal to one another when $\sqrt{:pp+fb:} = f$, for then each of thoſe lines will be equal to f; (as is evident by the Equations in 35°;) in which Caſe, there can but one Triangle be found out to ſolve the Problem, and that Triangle will always be obtuſe-angled at the Baſe. But when it happens that $\sqrt{:pp+fb:} \sqsupset f$, then the ſaid extreme Proportionals, (to wit, the values of a in 35°,) will be unequal between themſelves, and in this Caſe the Problem propounded may be ſolved by either of thoſe two right lines, or extreme Proportionals,

viz. two different Triangles may be found out wherein theſe three things will be common, to wit, the Baſe, the Perpendicular, and the Proportion of the leggs; of which Triangles, that which is formed by the help of the greater of the ſaid two right lines, (or extreme Proportionals,) will always be obtuſe-angled at the Baſe; but the other Triangle form'd by the help of the leſſer of the ſaid two right lines will ſometimes be obtuſe-angled at the Baſe, ſometimes acute-angled, and ſometimes right angled. Now to diſcover which of thoſe three kinds of Triangles will happen, I ſhall give three Rules, which preſuppoſe the quantities of the given lines to be expreſt by Numbers.

Rule I.

38. If $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$; then the leſſer value of a in 35°, (that is, $f - \sqrt{:ff - pp - fb:}$) is greater than the Baſe b, and conſequently the Triangle form'd by the help of the ſaid leſſer value ſhall be obtuſe-angled at the Baſe.

Rule II.

39. If $\frac{pp}{b} + b \sqsupset f$; then the leſſer value of a in 35° is leſs than the Baſe, and conſequently the Triangle form'd by the help of the ſaid leſſer value ſhall be acute-angled at the Baſe.

Rule III.

40. If $\frac{pp}{b} + b = f$; then the leſſer value of a in 35° is equal to the Baſe, and conſequently, the Triangle form'd by the help of the ſaid leſſer value ſhall be right-angled at the Baſe.

The truth of *Rule* 1. may be demonſtrated thus;

41. . . . *Suppoſ.* in *Rule* 1. . . $\frac{pp}{b} + b \sqsubset f$; but $\frac{pp}{f} + b$ not $\sqsubset f$.

42. . . . *Req. demonſtr.* . . . $f - \sqrt{:ff - pp - fb:} \sqsubset b$; as is affirmed in *Rule* 1.

Demonſtration.

43. By *Suppoſ.* in 41°, ... $\therefore \frac{pp}{b} + b \sqsubset f$.

44. Therefore by multiplying each part by b, . $\therefore pp + bb \sqsubset fb$.

45. And by adding ff to each part in 44°, . . . $\therefore ff + pp + bb \sqsubset ff + fb$.

46. And by ſubtracting pp from each part in 45°, (which the *Suppoſition* in 41°, or the *Determination* in 30° ſhews to be poſſible,) it follows that $ff + bb \sqsubset ff + fb - pp$.

47. Likewiſe by ſubtracting $2fb$ from each part in 46°, ... $ff + bb - 2fb \sqsubset ff - pp - fb$.

48. And by extracting the ſquare Root out of each part in 47°, ... $f - b \sqsubset \sqrt{:ff - pp - fb:}$

49. And by adding b to each part in 48°, . . . $\therefore f \sqsubset \sqrt{:ff - pp - fb:} + b$.

50. Wherefore by ſubtracting $\sqrt{:ff - pp - fb:}$ from each part in 49°, ... $f - \sqrt{:ff - pp - fb:} \sqsubset b$.

Which was to be Demonſtr.

After the ſame manner the truth of the preceding ſecond and third Rules may be demonſtrated; and from the premiſſes the following Canon is deducible, for the Arithmetical Solution of the Problem propounded.

CANON.

51. Let it be made as r the greater Term of the given Reaſon, (or Proportion,) to s the leſſer; ſo the ſame s to a third Proportional, which may be called t. Let it alſo be made as the exceſs of r above t to r, ſo the given Baſe AC to a fourth Proportional, which may be called f. Then from the Square of f ſubtract the Square of the given Perpendicular ND (= XY) together with the Rectangle made of f into the Baſe AC, and out of the remainder, if any happen, extract the ſquare Root. That done, add the the ſaid ſquare Root to f before found, and the ſumm ſhall be the diſtance from the foot of the Perpendicular falling without the Triangle upon the Baſe continued to the remoter end of the Baſe; which diſtance we may ſuppoſe to be AY in the following *Fig.* 1, 2, and 3.

and 3. Whence CY − AY − AC is given; and conſequently, (*per prop.* 47. *Elem.* 1.) $CX - \sqrt{:\square CY + \square XY:}$ is given, and $AX = \sqrt{:\square AY + \square YX:}$ is given alſo. Therefore $\triangle ACX$, whoſe angle ACX is obtuſe, is given, which I ſhall call the firſt of the two Triangles that will ſolve the Problem propounned.

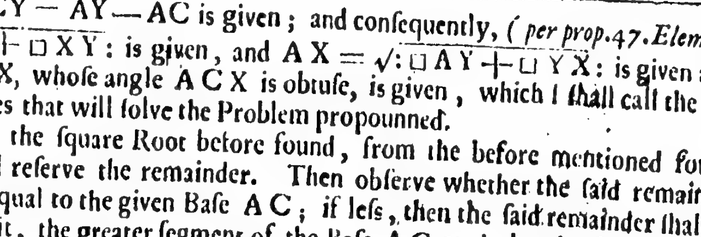
Again, ſubtract the ſquare Root before found, from the before mentioned fourth Proportional f, and reſerve the remainder. Then obſerve whether the ſaid remainder be leſs, greater, or equal to the given Baſe AC; if leſs, then the ſaid remainder ſhall be equal to AD, to wit, the greater ſegment of the Baſe AC made by the falling of the Perpendicular ND within the Triangle ANC in *Fig.* 1. Whence $AN = \sqrt{:\square AD + \square ND:}$ is given. Likewiſe $CN = \sqrt{:\square CD + \square DN:}$ is given, and therefore $\triangle ACN$ acute-angled at A and C is given, which I call the latter of the two Triangles that will ſolve the Problem. But if the remainder before reſerved happens to be greater than the given Baſe AC, then the ſaid remainder ſhall be the diſtance from the foot of the Perpendicular falling without the Triangle to the remoter end of the Baſe, which diſtance we may ſuppoſe to be AD in *Fig.* 2. whence CD = AD − AC is given, and conſequently, (*per prop.* 47. *Elem.* 1.) $CN = \sqrt{:\square CD + \square DN:}$ is given. Likewiſe $AN = \sqrt{:\square AD + \square DN:}$ is given; and therefore in *Fig.* 2. $\triangle ACN$ obtuſe-angled at C is given, which ſhall be the latter of two Triangles that will ſolve the Problem. But if the remainder before reſerved happens to be equal to the given Baſe AC, then the latter of two Triangles that will ſolve the Problem ſhall be right-angled at the Baſe, as the $\triangle ACN$ right-angled at C, in *Fig.* 3. and conſequently, $AN = \sqrt{:\square AC + \square CN:}$ (*per prop.* 47. *Elem.* 1.) is given. Therefore in *Fig.* 3. $\triangle ACN$ is given alſo.

Laſtly, when it happens that nothing remains after ſubtraction is made of the ſumm of the Square of the given Perpendicular ND and the Rectangle of the given Baſe AC into the fourth Proportional f, from the Square of the ſame f; then f it ſelf ſhall be the diſtance from the foot of the Perpendicular falling upon the Baſe continued to the remoter end of the ſaid Baſe; which diſtance we may ſuppoſe to be AD in $\triangle ADN$ in *Fig.* 4. Whence CD = AD − AC is given, and conſequently, (*per prop.* 47. *Elem.* 1.) $CN = \sqrt{:\square CD + \square DN:}$ is given: Likewiſe $AN = \sqrt{:\square AD + \square DN:}$ is given. Therefore in *Fig* 4. $\triangle ACN$ obtuſe-angled at C is given, which is the only Triangle in this Caſe that will ſolve the Problem.

TRIANGLES in Numbers, to illuſtrate the preceding Canon *of* Probl. 15.

FIG. 1.

52. In this *Fig.* 1. the Triangles ACX and ACN, the firſt of which is obtuſe-angled at the Baſe AC, and the latter acute-angled, have one common Baſe AC, alſo equal Perpendiculars ND and XY; and the leggs AX, CX of $\triangle ACX$ have the ſame Proportion one to the other, as the leggs AN, CN of $\triangle ACN$.



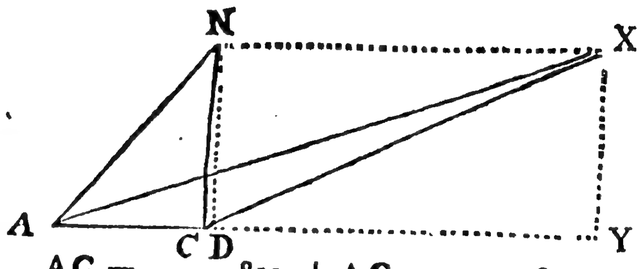
AC = 39	AC = 39
AN = 40	AX = $8\sqrt{153}$
CN = 25	CX = $5\sqrt{153}$
ND = 24	XY = 24
DA = 32	YA = 96
DC = 7	YC = 57

AN . CN :: AX . CX :: 8 . 5.

FIG. 2.

FIG. 2.

53. In this *Fig.* 2. the Triangles A C X and A C N, each of which is obtuſe-angled at the Baſe A C, have one common Baſe A C, alſo equal Perpendiculars N D and X Y, and the leggs A X, C X of △ A C X have the ſame Proportion one to the other, as the leggs A N, C N of △ A C N.

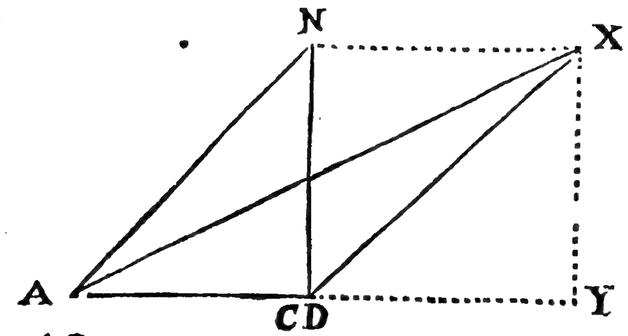


A C =	812	A C =	812
A N =	$4\sqrt{102544}$	A X =	$4\sqrt{570256}$
C N =	$3\sqrt{102544}$	C X =	$3\sqrt{570256}$
N D =	960	X Y =	960
D C =	36	Y C =	2052
D A =	848	Y A =	2864

A N . C N :: A X . C X :: 4 . 3.

FIG. 3.

54. In this *Fig.* 3. the Triangles A C X and A C N, the firſt of which is obtuſe-angled at the Baſe A C, and the latter right-angled, have one common Baſe A C, alſo equal Perpendiculars N D and X Y; and the leggs A X, C X of △ A C X have the ſame Proportion one to the other, as the leggs A N, C N of △ A C N.

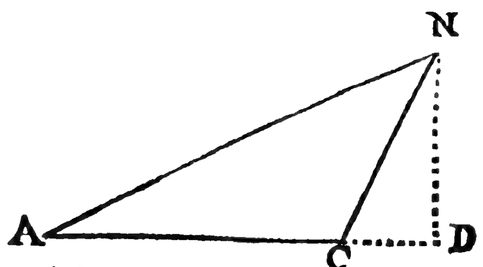


A C =	8	A C =	8
A N =	10	A X =	$5\sqrt{13}$
C N =	6	C X =	$3\sqrt{13}$
X Y =	6		
Y C =	9		
Y A =	17		

A N . C N :: A X . C X :: 5 . 3.

FIG. 4.

55. In this *Fig.* 4. the Triangle A C N obtuſe-angled at C cannot be matcht with any other Triangle that ſhall have its Baſe, Perpendicular, and Proportion of the leggs, equal to the Baſe A C, Perpendicular N D, and Proportion of the leggs A N, C N of the ſaid △ A C N.



A C =	30	N D =	20
A N =	$2\sqrt{500}$	D C =	10
C N =	$1\sqrt{500}$	A D =	40

A N . C N :: 2 . 1.

56. It is preſcribed by the preceding Determination in 36°, that $\sqrt{:pp+fb:}$ muſt not be greater than f, I ſhall therefore divide the Compoſition of this *Probl.* 15. into two Caſes, *viz.*

Caſe 1. when $\sqrt{:pp+fb:} \sqsupset f$. *Caſe* 2. when $\sqrt{:pp+fb:} = f$.

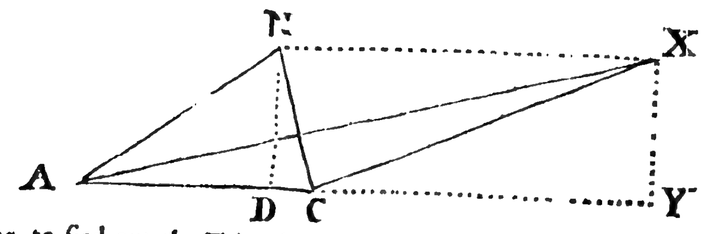
The Compoſition of Caſe 1. Probl. 15.

Suppoſe.

57. B a right line equal to the Baſe of a plain Triangle is given.
58. P a right line equal to the Perpendicular is given.

59. R and S

59. R and S two right lines expreſſing the Reaſon (or Proportion) of the leggs of the ſame Triangle, are given.
60. R ⊏ S.



Req. to find out the Triangle.

B
P
R
S
T
F
M
K
L

Conſtruction.

61. To the given lines R and S find a third Proportional, (by *Probl.* 7. *Chap.* 5.) ſuppoſe the line T, therefore } R . S :: S . T.
62. Alſo (by *Probl.* 8. *Chap.* 5.) let it be made as R — T to R; ſo the given Baſe B to a fourth Proportional line F, therefore } R — T . F :: B . F.
63. By *Probl.* 2. *Chap.* 5. find a right line M, ſuch that its Square may be equal to □ P +|- □ B, F, therefore } □ M = □ P +|- □ F, B.
64. By *Probl.* 14. *Chap.* 5. divide the double of F into two ſuch parts, that the line M may be a mean between them; which Effection is poſſible, for by *Suppoſ* in *Caſe* 1. (before expreſt in 56°,) the line M (that is, $\sqrt{:pp+fb:}$) is leſs than F; ſuppoſe then that 2 F is cut into two parts, whereof the greater is equal to the line K, and the leſſer equal to the line L; and that the line M is a mean Proportional between K and L, therefore theſe are Proportionals, *viz.*

$$\frac{2F-K}{L} \cdot M :: M \cdot K, \quad \| \quad \frac{2F-L}{K} \cdot M :: M \cdot L,$$

Each of which Analogies is correſpondent to that in the 34th ſtep of the preceding Reſolution, *viz.*

$$2f-a \cdot \sqrt{:pp+fb:} :: \sqrt{:pp+fb:} \cdot a.$$

Now by the help of the line K, found out as above, an obtuſe-angled plain Triangle to ſolve the Problem propounded may be made in manner following, *viz.*

65. Make A C = B, (the given Baſe.)
66. Produce A C to Y, ſo that A Y may be equal to K, which is greater than A C, as may be proved thus;

It is manifeſt that . . . . . } R ⊏ R — T.
Therefore from the Analogy in 62°, (*per Coroll.* of 14. *prop.* 5. *Elem.*) . . . . . } F ⊏ B.
But by *Conſtr.* in 65°, ... } A C = B.
Therefore (*per Ax.* 3. *Chap.* 2.) . . . . . } F ⊏ A C (or B.)
And becauſe the greateſt of three Proportionals is greater than half the ſumm of the extremes, therefore from 64°, . . } K ⊏ F.
Therefore (*per Ax.* 5. *Chap.* 2.) . . . . . } A Y or K ⊏ A C.

Which was to be demonſtr.

67. Make Y X ⊥ A Y, alſo make Y X = P, the given Perpendicular.
68. Laſtly, from A and C (the ends of the Baſe A C) draw the right lines A X and C X to meet with the top of the Perpendicular Y X in X, ſo the Triangle A C X obtuſangled at C, (for as before hath been proved in 66°, A Y ⊏ A C) ſhall be

one

one of the two Triangles which in *Caſe* 1. will ſatisfie the Problem; which I prove thus,

69. Firſt, by *Conſtruction* in 65° the Baſe AC is equal to the given Baſe B; ſecondly, by *Conſtr.* in 67° the line YX is perpendicular to AC continued, and equal to the given Perpendicular P. It remains only to prove that the greater legg AX hath ſuch proportion to the leſſer legg CX, as R to S; which Analogy will be made manifeſt by the following Demonſtration, formed out of the preceding Reſolution by a repetition of its ſteps in a retrograde order, *viz.* by returning backwards from the end to the beginning of the Reſolution.

70. . . . *Req. demonſtr.* . . . . . R . S :: AX . CX.

Demonſtration.

71. Foraſmuch as by *Conſtr.* in 64°, . . . } 2F — K . M :: M . K.
72. And by *Conſtr.* in 66°, ... } . . . . . AY = K.
73. Therefore from 71° and 72°, ... } 2F—AY . M :: M . AY.
That is, in 34°, (the laſt ſtep of the Reſolution,) } $2f-a \cdot \sqrt{:pp+fb:} :: \sqrt{:pp+fb:} \cdot a.$
74. But from 73°, (*per* 1. *prop.* 6. *Elem.*) } 2□F,AY—□AY = □M.
75. And by *Conſtr.* in 6°, ... } □P + □F,B = □M.
76. Therefore from 74° and 75°, (*per* 1. *Ax. Chap.* 2.) . . . . . } 2□F,AY—□AY = □P+□F,B.
77. And becauſe by *Conſtr.* in 67°, . . . } . . . . . YX = P.
78. And conſequently, ... } . . . . □YX = □P.
79. And by *Conſtr.* in 65°, ... } . . . . . AC = B.
80. Therefore out of 76°, 78° and 79°, . } 2□F,AY—□AY = □YX+□F,AC
That is, in 33°, ... } . . . $2fa - aa = pp + fb.$
81. And from 80°, by adding □AY to each part, } 2□F,AY=□AY+□YX+□F,AC.
That is, in 32°, ... } . . $fa = aa + pp + fb.$
82. And by ſubtracting □F, AC from each part of the Equation in 81°, ... } 2□F,AY—□F,AC=□AY+□YX.
That is, in 31°, ... } . . $2fa - fb = aa + pp.$

83. And this following Analogy is manifeſt, (*per prop* 7. *Elem.* 5.) for the firſt and third Proportionals are one and the ſame, and the ſecond and fourth equal one to the other, (as hath before been proved in 82°;)

$$\left.\begin{array}{r} 2\square AC,AY - \square AC \\ \square AY + \square YX \\ 2\square AC,AY - \square AC \\ 2\square F,AY - \square F,AC \end{array}\right\} :: \text{ that is, in 30°, } :: \left\{\begin{array}{l} 2ba - bb \\ aa + pp \\ 2ba - bb \\ 2fa - fb \end{array}\right.$$

84. And by reaſon of the common altitude 2AY — AC in the two latter Terms of the ſubſequent Analogy, it will be manifeſt (*per* 1. *prop.* 6. *Elem.*) that

$$\left.\begin{array}{r} AC \\ F \\ 2\square AC,AY - \square AC \\ 2\square F,AY - \square F,AC \end{array}\right\} :: \text{ that is, in 29°, } :: \left\{\begin{array}{l} b \\ f \\ 2ba - bb \\ 2fa - fb \end{array}\right.$$

85. And becauſe the two latter Terms of the Analogy in 84°, are the ſame, and in the ſame order with the two latter Terms of the Analogy in 83°, therefore from 83° and 84° (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} AC \\ F \\ 2\square AC,AY - \square AC \\ \square AY + \square YX \end{array}\right\} :: \text{ that is, in 28°, } :: \left\{\begin{array}{l} b \\ f \\ 2ba - bb \\ aa + pp \end{array}\right.$$

86. But from the *Conſtr.* in 62° and 65°, . . . AC . F :: R — T . R.
That is, in 27°, ... $b \cdot f :: r - t \cdot r.$

87. Therefore from 85° and 86°, (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} R - T \\ R \\ 2\square AC,AY - \square AC \\ \square AY + \square YX \end{array}\right\} :: \text{ that is, in 26°, } :: \left\{\begin{array}{l} r - t \\ r \\ 2ba - bb \\ aa + pp \end{array}\right.$$

88. And

88. And from 87°, by Reaſon Inverſe, theſe are Proportionals, *viz.*

$$\left.\begin{array}{r} R \\ R - T \\ \square AY + \square YX \\ 2\square AC,AY - \square AC \end{array}\right\} :: \text{ that is, in 25°, } :: \left\{\begin{array}{l} r \\ r - t \\ aa + pp \\ 2ba - bb \end{array}\right.$$

89. And from 88°, by Converſion of Reaſon, theſe are Proportionals, *viz.*

$$\left.\begin{array}{r} R \\ T \\ \square AY + \square YX \\ \square AY - 2\square AC,AY + \square AC + \square YX \end{array}\right\} :: \text{ which anſwer to thoſe in 24° in the Reſolution, } \textit{viz.}$$

$r \cdot t :: aa + pp \cdot aa - 2ba + bb + pp.$

90. And becauſe by *Conſtr.* in 61°, ... } R . S :: S . T.
That is, in 22°, ... } $r \cdot s :: s \cdot t.$
91. Therefore (*per Coroll.* of 20. *prop.* 6. *Elem.*) . . . } R . T :: □R . □S.
That is, in 23°, ... } $r \cdot t :: rr \cdot ss.$
92. Therefore from 89° and 91°, (*per* 11. *prop.* 5. *Elem.*) theſe ſhall be Proportionals, *viz.*

$$\left.\begin{array}{r} \square R \\ \square S \\ \square AY + \square YX \\ \square AY - 2\square AC,AY + \square AC + \square YX \end{array}\right\} :: \text{ which anſwer to thoſe in 21°, } \textit{viz.}$$

$rr \cdot ss :: aa + pp \cdot aa - 2ba + bb + pp.$

93. And becauſe by *Conſtr.* in 67°, ... } YX ⊥ AY.
94. Therefore (*per* 47. *prop.* 1. *Elem.*) . . . } □AX = □AY + □YX.
95. Likewiſe . . . . . } □CX = □CY + □YX.
96. Moreover by *Conſtr.* in 66°, and by the *Diagr.* } CY = AY — AC.
97. And conſequently, (*per* 5. *Theor.* 4. *Chap.*) } □CY=□AY—2□AC,AY+□AC.
That is, in 9°, ... } $\square:a-b:=aa-2ba+bb.$
98. Therefore if inſtead of □CY in 95°, we take that which in 97° is found equal to □CY, the Equation in 95° will be reduced to this, *viz.*

□CX = □AY — 2□AC,AY + □AC + □YX.

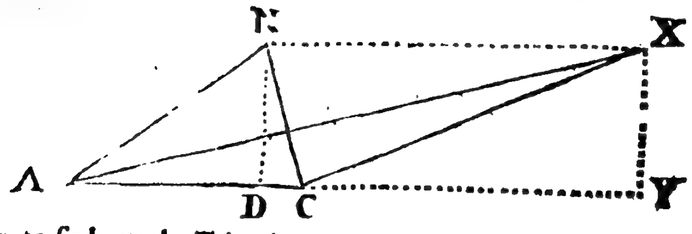
99. Likewiſe if inſtead of the third and fourth Proportionals in 92°, we take thoſe Squares which are found equal to them reſpectively in 94° and 98°, the Analogy in 92° will be reduced to this, *viz.*

□R . □S :: □AX . □CX.

100. Wherefore (*per prop.* 22. *Elem.* 6.)

R . S :: AX . CX. Which was to be Dem.

101. Another Triangle to ſolve the Problem in *Caſe* 1. before expreſt in 56°, may by the help of the leſſer Root L before found in 64°, be formed thus, *viz.* Let the lines before given and found out in 57°, 58°, 59°, 60°, 61°, 62°, 63°, 64°, together with the Diagram ſtanding between 60° and 61° be here repeated, then will the Conſtruction be as followeth.



Req. to find out the Triangle.

B ——————
P ——————
R ——————————
S ——————
T ——————
F ——————————
M ——————————
K ——————————————————
L ——————

Construction.

102. Make A C = B (the given Base.)

103. Upon A C, continued if need be, make A D = L, which lesser Root L, (as before hath been shewn,) will sometimes be greater than the Base; but supposing it be discovered (by *Rule* 2. in 39° of this *Probl.*) that L is lesser than B, or A C, cut off from A C a segment equal to L, as A D.

104. Make D N ⊥ A C in the point D, also make D N = P the given Perpendicular.

105. Lastly, from the ends of the Base A C draw the right lines A N and C N to meet with the top of the Perpendicular D N, in N; so the Triangle A C N acute-angled at A and C, (for by *Supposition* A D is lesser than A C,) will satisfie the Problem, as well as the △ A C X before found. For first, by *Construction* in 102° the Base A C is equal to the given Base B: Secondly, the Perpendicular D N (by *Constr.* in 104°) is equal to the given Perpendicular P; and by a repetition of the steps of the Resolution in a backward order, in like manner as before in the preceding Demonstration, saving that L must be used here instead of K, and N D instead of X Y; it may easily be proved that the leggs A N and C N are in the given Reason of R to S.

Moreover, when the lesser Root L is greater than the Base, the Triangle formed by the help of such lesser Root shall be obtuse-angled at the Base, and the Construction and Demonstration in every respect like to that by the greater Root.

But it must be remembred, that when the Perpendicular falls within the Triangle, then the Square of D C is equal to the Square of A C − A D; but when it falls without, then the Square of Y C is equal to the Square of Y A − A C: So that before the Construction and Demonstration by the lesser Root be entred upon, it will be requisite to find out the kind of the Triangle, by the help of the three preceding Rules in 38°, 39°, 40°; and when it happens that $r . s :: \sqrt{:bb + pp} : . p$, then 'tis evident (by 47. *prop.* 1. *Elem.*) that the Triangle formed by the lesser Root will be right-angled at the Base, and in such Case there is no need of further proof.

The Composition of Case 2. Probl. 15.

106. Which Case, in the letters belonging to the preceding Resolution presupposeth . . . . . } . $f = \sqrt{:pp + fb} := a$.

107. And consequently, by squaring each part, . . } . $ff = pp + fb = aa$.

108. That is, in the lines of the ensuing *Constr.* and *Diagr.* } □AD = □DN + □AD,AC.

Suppos.

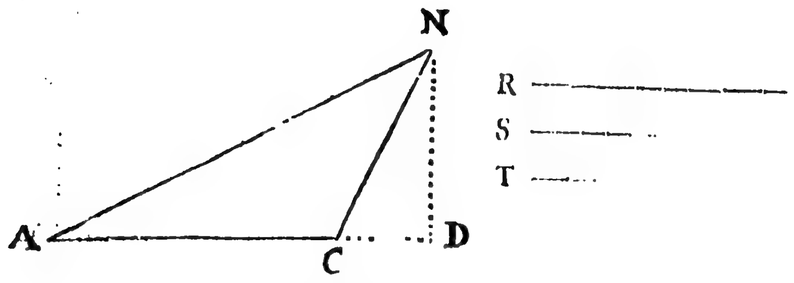
109. A C = the Base of a Triangle is given.

110. D N = the Perpendicular is given.

111. R and S two right lines expressing the Reason of the leggs are given.

112. R ⊏ S.

Req. to find the △.



Construction.

113. By *Probl.* 7. *Chap.* 5. let it be made as R to S, so S to a third Proportional, suppose it be found T, therefore,

R . S :: S . T.

114. By *Probl.* 8. *Chap.* 5. let it be made as R − T to R, so A C to a fourth Proportional, suppose it to be A D, therefore,

R − T . R :: . A C . A D.

Which fourth Proportional A D shall necessarily be greater than A C, because R is greater than R − T.

115. Make

115. Make D N ⊥ A D in the point D; then from A and C, the ends of the given Base A C, draw the right lines A N and C N to meet with the top of the Perpendicular D N in N; so shall A C N be the Triangle required. For first, the Base A C is equal to the given Base; also the Perpendicular N D is equal to the given Perpendicular. But that the leggs A N and C N are in the given Reason of R to S, it may easily be demonstrated by a backward repetition of the steps of the foregoing Resolution, in like manner as before in the Composition of *Case* 1; with this Caution, That as often as *a* is found in the Resolution, *f* must be taken instead of *a*, because in this second Case *f* is equal to *a*; for since by *Supposition* in 106°, $f = \sqrt{:pp + fb}$: it will be evident from 35°, that $f = a$. But in regard the Demonstration of this second Case differs not from that of the following *Probl.* 16. I shall wave it here.

COROLLARY.

116. From the premisses it follows, that the Perpendicular D N of the Triangle A C N formed in *Case* 2. (before expresst in 106°, 107° and 108°,) is a mean Proportional between A D and D C the distances from D the foot of the Perpendicular falling without the Triangle to the ends of the Base; and consequently, (*per prop.* 6. *Elem.* 6.) the Triangles A D N and C D N are equiangular. See the last preceding Diagram, and compare it with this following Demonstration.

117. . . . *Req. demonstr.* . . . . . { A D . D N :: D N . D C. Also, △ADN and △CDN are equiangular.

Demonstration.

118. By *Suppos.* in 108°, ... } □AD = □DN + □AD, AC.

119. Therefore by subtracting □ A D, A C from each part, ... } □AD − □AD, AC = □DN.

120. And from 119°, (*per prop.* 14. *Elem.* 6.) these are Proportionals, *viz.* . . . . } AD . DN :: DN . AD − AC.

121. But 'tis evident by the last preceding Diagram, that . . . . . } DC = AD − AC.

122. Therefore from 120° and 121°, . . } AD . DN :: DN DC.

123. Therefore from 122°, (*per prop.* 6. *Elem.* 6.) } △ADN and △CDN are equiangular. Which was to be Demonstr.

From the preceding *Corollary* and *Construction* of *Case* 2. the following *Probl.* 16. is deducible.

Probl. XVI.

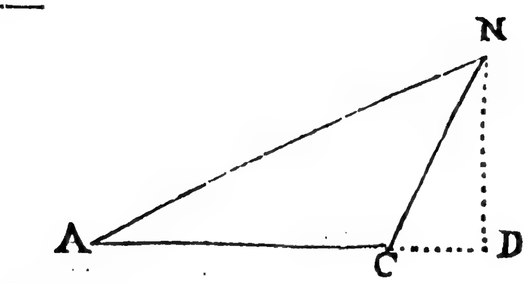
To find a plain Triangle obtuse-angled at the Base, and that the Base may be equal to a right line given. Also, that the Perpendicular falling upon the Base continued may be a mean Proportional between the distances from the foot of the Perpendicular to the ends of the Base: And that the leggs of the Triangle may be in a given Reason, suppose as R to S.

R
S
T

Suppos.

1. △ACN is obtuse-angled at C.
2. A C the Base is given.
3. A C D is a right line.
4. D N ⊥ A D.
5. A D . D N :: D N . D C.
6. R and S are right lines given.
7. R . S :: A N . C N.

Req. to make the △ A C N.



Construction.

8. Making R the first of three Proportionals, and S the second, find a third, (*per* 7. *Probl.* 5. *Chap.*) let it be T, therefore

R . S :: S . T.

9. Alſo making R — T the firſt of four Proportionals, R the ſecond, and the given Baſe A C the third, find a fourth, *(per 8. Probl. 5. Chap.)* ſuppoſe it be A D, therefore

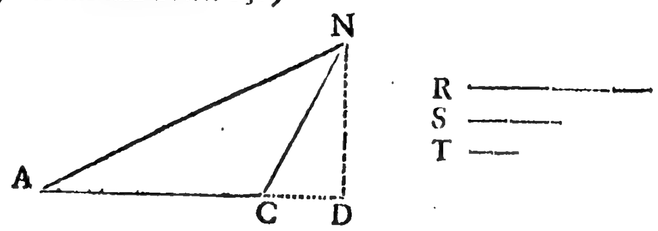
R — T . R :: A C . A D.

Which fourth Proportional A D ſhall neceſſarily be greater than A C, becauſe the ſecond Proportional R is greater than the firſt R — T.

10. Find a mean Proportional, as D N, between A D and D C, *(per 9. Probl. 5. Chap.)* therefore,

A D . D N :: D N . D C (= A D — A C.)

11. Make D N ⊥ A D in the point D; then draw the right lines A N and C N, ſo ſhall A C N be the Triangle required. Now we muſt ſhew that it will ſatisfie the Problem: Firſt then, A C the Baſe is equal to the right line preſcribed for the Baſe, and from the 9th ſtep it is leſs than A D; therefore the angle A C N is obtuſe: Secondly, the Perpendicular N D, (by *Conſtruction* in 10°,) is a mean Proportional between A D and D C, (to wit, the two diſtances from D the foot of the Perpendicular N D, to A and C the ends of the Baſe A C.) It remains only to prove, That the leggs A N and C N of the Triangle A C N, are in ſuch proportion one to the other as R to S; which Analogy I ſhall make manifeſt by the following Demonſtration, formed out of the Reſolution of the preceding *Probl.* 15. by a backward repetition of the ſteps of the ſaid Reſolution, in *Caſe* 2. (but reſpect muſt be had to the Caution given in 115° of the ſaid *Probl.* 15.)



R ——————
S ——————
T ——

12. . . . *Req. demonſtr.* . . . . . R . S :: A N . C N.

Demonſtration.

13. Foraſmuch as by *Conſtr.* in 10°, . . . ⟩ A D . D N :: D N . D C (A D — A C.)
14. Therefore *(per 17. prop. 6. Elem.)* . . ⟩ □ A D — □ A D, A C = □ D N.
15. And from 14°, by equal addition of □ A D, A C, ⟩ □ A D = □ D N + □ A D, A C.
16. And from 15°, by equal addition of □ A D, ⟩ 2 □ A D = □ A D + □ D N + □ A D, A C.
17. And from 16°, by equal ſubtraction of □ A D, A C, ... ⟩ 2 □ A D — □ A D, A C = □ A D + □ D N.
18. And becauſe in the following Analogy the firſt and third Terms are one and the ſame, and the ſecond and fourth equal one to the other, (as hath been proved in 17°,) therefore *(per 7. prop. 5. Elem.)*

$$\left\{\begin{array}{r} 2\square AC, AD - \square AC \\ \square AD + \square DN \\ 2\square AC, AD - \square AC \\ 2\square AD - \square AD, AC \end{array}\right. \begin{array}{c} . \\ :: \\ . \\ . \end{array} \left\}\begin{array}{l} \\ \\ \\ \end{array}\right. \text{Proportionals.}$$

19. And the ſubſequent Analogy, by reaſon of the common altitude 2 A D — A C in the two latter Terms, will be manifeſt, *(per 1. prop. 6. Elem.) viz.*

A C . A D :: 2 □ A C, A D — □ A C . 2 □ A D — □ A D, A C.

20. And becauſe the two latter Terms of the Analogy in 19° are the ſame and in the ſame order with the two latter Terms of the Analogy in 18°, therefore from 18° and 19°; *(per 11. prop. 5. Elem.)* theſe ſhall be Proportionals, *viz.*

A C . A D :: 2 □ A C, A D — □ A C . □ A D + □ D N.

21. But by *Conſtruction* in 9°,

A C . A D :: R — T . R.

22. Therefore from 20° and 21°, *(per 11. prop. 5. Elem.)*

R + T . R :: 2 □ A C, A D — □ A C . □ A D + □ D N.

23. And from 22°, by Reaſon inverſe,

R . R — T :: □ A D + □ D N . 2 □ A C, A D — □ A C.

24. And

24. And from 23°, by Converſion of Reaſon, theſe ſhall be Proportionals, *viz.*

$$\left\{\begin{array}{r} \square AD + \square DN - 2\square AC, AD \\ \square AD + \square DN \end{array}\right. \begin{array}{c} . \\ :: \end{array} \left\}\begin{array}{l} R \\ T \\ . \\ . \end{array}\right. \text{Proportionals.}$$

25. And becauſe by *Conſtr.* in 8°, ... ⟩ R . S :: S . T.
26. And conſequently, *(per Coroll. of 20. prop. 6. Elem.)* . . . . . ⟩ R . T :: □R . □S.
27. Therefore from 24° and 26°, *(per 11. prop. 5. Elem.)* theſe ſhall be Proportionals, *viz.*

$$\left\{\begin{array}{r} \square AD + \square DN - 2\square AG, AD \\ \square AD + \square DN + \square AC \end{array}\right. \begin{array}{c} . \\ :: \end{array} \left\}\begin{array}{l} \square R \\ \square S \\ . \\ . \end{array}\right. \text{Proportionals.}$$

28. And becauſe by *Conſtr.* in 11°, . . . ⟩ DN ⊥ AD.
29. Therefore, *per 47. prop. 1. Elem.* (reſpect being had to the Diagram,) . . . . ⟩ □ AN = □ AD + □ DN.
30. Likewiſe, ... ⟩ □ CN = □ DC + □ DN.
31. Again, by *Conſtr.* in 9°, and by view of the Diagram, ... ⟩ DC = AD — AC.
32. And conſequently, *(per Theor. 5. Chap. 4.)* ⟩ □DC = □AD + □AC — 2□AD, AC.
33. Therefore if inſtead of □DC in 30°, we ſet that which in 32° is found equal to □DC, the Equation in 30° will be reduced to this, *viz.*

□CN = □AD + □AC — 2□AD, AC + □DN.

34. Likewiſe, if inſtead of the third and fourth Proportionals in 27°, we take thoſe Squares which are found equal to them reſpectively in 29° and 33°, the Analogy in 27° will be reduced to this,

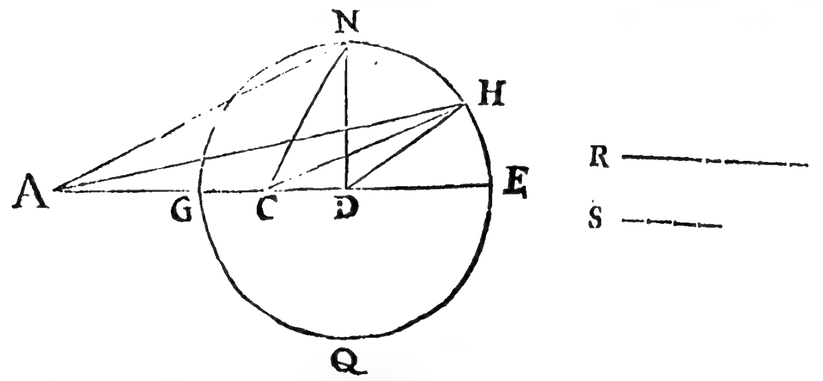
□R . □S :: □AN . □CN.

35. Wherefore . . . R . S :: AN . CN.

Which was to be demonſtrated. Therefore that is done which the Problem required.

Probl. XVII. (Probl. *Apollon. Pergæi.*)

Two points (A and C) being given in a Plane, to deſcribe a Circle in the ſame Plane, that two right lines drawn from thoſe points to concurr in any point of the Circumference may have a given Reaſon; ſuppoſe the greater line to the leſs, as R to S.



R ——————
S ——————

Conſtruction.

1. Upon the given line A C as a Baſe, to wit, the ſhorteſt diſtance between the given points A and C, make (by the preceding *Probl.* 16.) a Triangle A C N obtuſe-angled at C, and ſuch, that the Perpendicular N D falling upon A C produced, may be a mean Proportional between A D and D C; alſo, that the leggs A N and C N may be in the given Reaſon of R to S. Therefore by that Conſtruction theſe are Proportionals, *viz.*

A D . D N :: D N . D C.
R . S :: A N . C N.

2. Then from the Center D, at the diſtance of the Perpendicular D N, deſcribe the Circle D G N E Q, which ſhall neceſſarily cut A C; for by *Conſtruction* D N is a mean

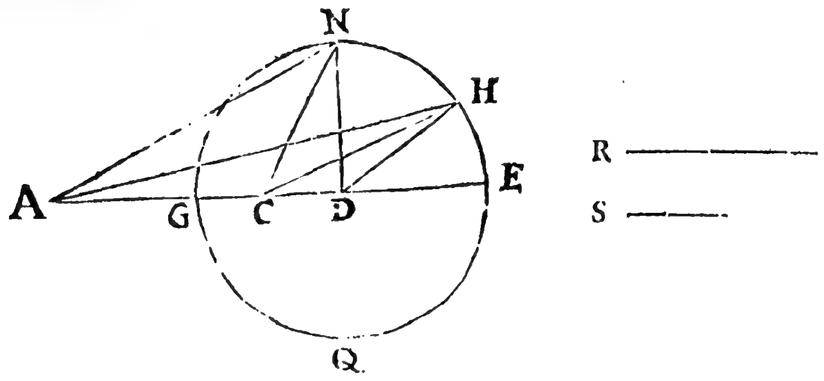
Propor-

Proportional between D A and D C, which D C being but part of D A is leſs than D A, therefore the mean, or Semidiameter D N or D G is leſs than D A, but greater than D C. Now I ſay the Circle D G N E Q is that which is required by the Problem, and therefore we muſt ſhew that if two right lines be drawn from the given points A and C to meet in any point of the Circumference of that Circle, thoſe right lines ſhall have ſuch proportion one to the other as the given lines R and S; the demonſtration whereof I ſhall divide into three Caſes, in regard there may be a threefold poſition of the point taken in the Circumference; for the point may be either E, or elſe G, to wit, the ends of that Diameter which lyes in the ſame ſtraight line with the given line A C; or laſtly, the point may be taken in any other part of the Circumference, as H; which Caſes I ſhall demonſtrate in their order.

Preparat.

3. Foraſmuch as in the Triangles A D N and C D N the angle at D is common, and the ſides about that angle are Proportionals, for by *Conſtruction* in 1° it hath been made, as A D . D N :: D N . D C, therefore *(per prop. 6. Elem. 6.)*
△ A D N and △ C D N are equiangular.
4. But we muſt enquire which angles in thoſe like Triangles are equal one to the other. Firſt then, becauſe the angle at D is common, the angle C N D in △ C D N muſt be equal either to the angle A N D, or to the angle N A D in △ A D N; but the angle C N D being but part of the angle A N D cannot be equal to it, therefore
∠C N D = ∠N A D. Alſo, ∠N C D = ∠A N D.
5. In like manner, becauſe by *Conſtr.* in 1° and 2°,
A D . D N (or D H) :: D N (or D H) . D C.
6. Therefore, *(per prop. 6. Elem. 6.)*
△ A D H and △ C D H are equiangular.
7. And for the like reaſon as before in 4°,
∠C H D = ∠H A D. Alſo, ∠H C D = ∠A H D.

Theſe things premiſed, I ſhall proceed to the Demonſtration of the three *CASES* before mentioned.



8. . . . I. *Req. Demonſtr.* . . . . . R . S :: A E . C E.

Demonſtration.

9. Becauſe by *Conſtr.* in 1°, ... A D . D N :: D N . D C.
10. Therefore by Compoſition of Reaſon, . . . AD+DN . DN :: DN+DC . DC.
11. And becauſe *(per defin. 15. Elem. 1.)* . . . . . DE = DN.
12. Therefore from 10° and 11°, ... AD+DE . DN :: DE+DC . DC.
13. That is, as is evident by the *Diagram*, . AE . DN :: CE . DC.
14. Therefore alternly, ... AE . CE :: DN . DC.
15. Again, it hath been proved in 3°, that . △ADN and △CDN are equiangular.
16. And in 4°, that . . . . . ∠NCD = ∠AND.
17. Therefore from 15° and 16°, *(per prop. 4. Elem. 6.)* . . . . . AN . DN :: CN . DC.
18. Therefore alternly, ... AN . CN :: DN . DC.
19. But by *Conſtr.* in 1°, ... AN . CN :: R . S.
20. Therefore from 18° and 19°, *(per 11. prop. 5. Elem.)* . . . . . R . S :: DN . DC.

21. But

21. But it hath been proved in 14°, that . AE . CE :: DN . DC.
22. Therefore from 20° and 21°, *(per 11. prop. 5. Elem.)* . . . . . R . S :: AE . CE.
Which was to be demonſtr.
23. . . . II. *Req. Demonſtr.* . . . . . R . S :: AG . CG.

Demonſtration.

24. Foraſmuch as by *Conſtr.* in 1°, ... AD . DN :: DN . DC.
25. Therefore by Diviſion of Reaſon, . . . AD−DN . DN :: DN−DC . DC.
26. And becauſe *(per defin. 15. Elem. 1.)* . . . . . DG = DN.
27. Therefore from 25° and 26°, ... AD−DG . DN :: DG−DC . DC.
28. That is, (as is evident by the *Diagram,*) AG . DN :: CG . DC.
29. Therefore alternly, ... AG . CG :: DN . DC.
30. But before in 20°, it hath been proved that . R . S :: DN . DC.
31. Therefore from 29° and 30°, *(per 11. prop. 5. Elem.)* . . . . . R . S :: AG . CG.
Which was to be demonſtrated in the ſecond place.
32. . . . III. *Req. Demonſtr.* . . . . . R . S :: AH . CH.

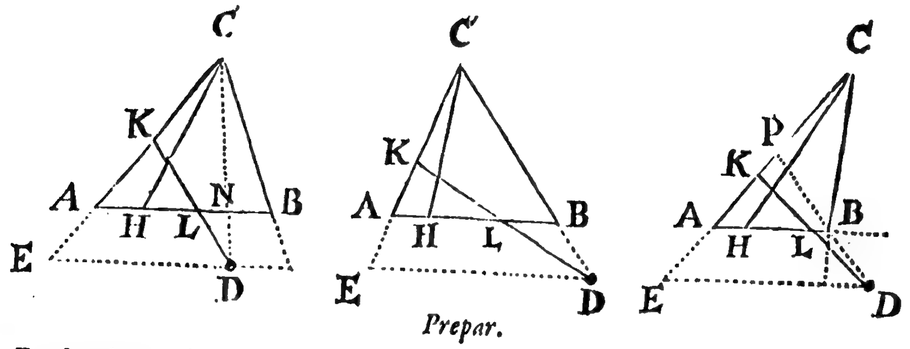
Demonſtration.

33. It hath before been proved in 6°, that . △ADH and △CDH are equiangular.
34. And in 7°, that . . . . . ∠HCD = ∠AHD.
35. Therefore from 33° and 34°, *(per 4. prop. 6. Elem.)* . . . . . CH . CD :: AH . HD.
36. And alternly, ... CH . AH :: CD . HD(or DN.)
37. Therefore inverſly, ... AH . CH :: DN . CD.
38. But it hath been proved in 20°, that . R . S :: DN . CD.
39. Therefore from 37° and 38°, *(per 11. prop. 5. Elem.)* . . . . . R . S :: AH . CH.
Which was to be demonſtrated in the laſt place. Therefore that is done which the Problem required.

Probl. XVIII.

To divide a given Triangle A B C into two parts which ſhall be in a given Reaſon, ſuppoſe as A H to H B, by a right line D K drawn from a given point D without the Triangle.

FIG. 1. *FIG.* 2. *FIG.* 3.



Prepar.

1. By the given point D draw D E parallel to the Baſe A B, and continue the leggs C A, C B beneath the Baſe; then the point D will either lye between the Increaſes of the leggs, as in *Fig.* 1. or elſe in one of the ſaid Increaſes, as in *Fig.* 2. or laſtly, between the Increaſes of the Baſe and one of the leggs, as in *Fig.* 3.
2. Divide the Baſe A B in H in the given Reaſon, and draw C H, therefore *(per prop. 1. Elem. 6.)* △ A C H . △ H C B :: A H . H B.
3. In *Fig.* 1. let a line be drawn from the given point D to C the angle oppoſite to the Baſe A B, as the line D C, which will either cut the Baſe A B in the point H, in which the

the Baſe is divided in the given Reaſon, in which Caſe the Problem is evidently ſatisfied; or elſe in ſome other point N, and then the point H will either lye between N and A, or between N and B; if H lye between N and A, then the deſired line of partition to be drawn from D will cut AB and AC; but if H lye between N and B, then the ſaid line of partition will cut AB and BC.

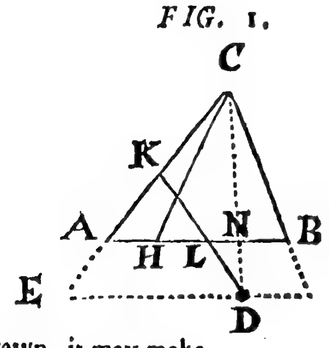
4. In *Fig.* 2. where the given point D lyes in CB increaſed, 'tis evident that the line of partition to be drawn from D ſhall neceſſarily cut AB and AC.

5. In *Fig.* 3. the line of partition to be drawn from D will ſometimes cut AB and AC, ſometimes it may paſs by the angular point B and cut AC only, and ſometimes it will cut BC and AC; but which of theſe lines will happen to be cut when the given point D is poſited according to the Definition in 1°, relating to *Fig.* 3. may be diſcovered by the Rule hereafter given in 34° of this Problem.

6. In every one of thoſe three Caſes before defined in 1°, which may happen by the various poſition of the given point D, the Reſolution of the Problem propos'd will be one and the ſame. Suppoſing then it be diſcovered, that the line of partition to be drawn from D muſt cut AB and AC in each of the three preceding Figures, the Scope of the Reſolution is to find a point in AC, as K, to which a right line being drawn from D, as DK, this line DK may cut the Baſe AB between H and N in *Fig.* 1. or between H and B in *Fig.* 2. likewiſe in *Fig.* 3. between H and B, (or elſe paſs by the angular point B,) ſo as to make the Triangle AKL equal to the Triangle ACH, whence it evidently follows that LKCB = △HCB, and △AKL . LKCB :: AH . HB. Theſe things premiſed, the Reſolution of the Problem propounded may be formed in manner following.

Suppoſ.

FIG. 1.



7. ABC is a △ given in *Fig.* 1.
8. D is a point given without the △ ABC.
9. AH and HB are in a given Reaſon.
10. b = AC is given.
11. c = AH is given.
12. g = AE is given.
13. h = ED (‖ AB) is given.

Req. to find

14. AK ſuch a ſegment of AC, that DK being drawn, it may make
15. △ALK . LKCB :: AH . HB :: △ACH . △HCB.

Reſolution.

16. Suppoſe that done which is required, and put . . . a = AK.
17. Then by conſidering well what is required, and by viewing *Fig.* 1. it will appear that △ALK = △ACH, and that ∠CAL is common to both Triangles; therefore (*per prop.* 15. *Elem.* 6.) . . . AK . AH :: AC . AL.
18. That is, in the letters of the Reſolution, . . . a . c :: b . $\frac{bc}{a}$.
19. And becauſe △AKL and △EKD are equiangular, (for by *Conſtr.* in 1° ED ‖ AL,) therefore (*per prop.* 4. *Elem.* 6.) . . . . . EK . AK :: ED . AL.
20. That is, in the letters of the Reſolution, . . . $a+g$. a :: h . $\frac{ha}{a+g}$.
21. And becauſe the fourth Proportional in 17° is the ſame with the fourth in 19°, therefore the fourth Proportionals in 18° and 20° ſhall alſo be equal to one another, *viz.* . . . . . $\frac{bc}{a} = \frac{ha}{a+g}$.
22. Now to avoid an Equation between Solids, let it be made as h to b, ſo c to a fourth Proportional, which may be called m; therefore, ... h . b :: c . m.
23. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . $hm = bc$.

24. There-

24. Therefore from 21° and 23°, by exchanging equal quantities, ... $\frac{hm}{a} = \frac{ha}{a+g} = \frac{bc}{a}$.
25. Whence 'tis eaſie to inferr that theſe are Proportionals, *viz.* . . . . . $a . m :: h . \frac{ha}{a+g} (= \frac{bc}{a})$
26. But it hath been ſhewn in 20°, that . . . $a+g . a :: h . \frac{ha}{a+g}$.
27. Therefore from 25° and 26°, (*per prop.* 11. *Elem.* 5.) . . . . . $a . m :: a+g . a$.
28. And from 27°, by comparing the Rectangle of the extremes to the Rectangle of the means, . . $aa = ma + mg$.
29. And from 28°, by ſubtracting ma from each part, ... $aa - ma = mg$.
30. Which laſt Equation may be reſolved into theſe Proportionals, *viz* . . . . . $a - m . \sqrt{mg} :: \sqrt{mg} . a$.
31. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as alſo m the difference of the extremes a and $a - m$; therefore the extremes ſhall be given ſeverally, (*per Probl.* 12. *Chap.* 5.) the greater whereof is equal to the deſired line AK, which (by the Theorem in 24° of the ſaid *Probl.* 12. *Chap.* 5) will be found equal to this right line, (or number,) *viz.*

$$\tfrac{1}{2}m + \sqrt{:\tfrac{1}{4}mm + mg:} = \text{AK} = a.$$

From which Equation and premiſſes we may deduce this following

CANON.

32. Let it be made as ED to AC, ſo AH to a fourth Proportional, which may be called M; then to the half of M add the ſquare Root of the ſumm of the Square of half M and the Rectangle of M into AE, ſo ſhall the ſumm of that addition be the value of AK ſought.

33. This Canon ſerves to find out the value of the line AK in every one of the three preceding Figures, and when the given point D is poſited according to the Definition of the firſt and ſecond Caſes in 1°, as in *Fig.* 1, and 2. it is eaſie to diſcover from what hath been ſaid in 3° and 4°, which of the ſides of the given Triangle ABC will be cut by the line of partition to be drawn from D. But when the point D is poſited according to the Definition of the third Caſe in 1°, as in *Fig.* 3. then it may be doubtfull which of the ſides are to be cut; to remove therefore this ambiguity, obſerve the following Directions, *viz.* Firſt, draw a right line from the given point D (in *Fig.* 3.) to paſs by the angular point B, as DBP; then is EABD a Trapezium, having (by *Conſtruction*) two parallel ſides AB and ED, and the other two ſides EA and DB which are not Parallels, being continued will meet in ſome point in AC, as in P, for (by *Conſtruction*) EAC is a right line. Now if AB, ED and EA be ſeverally given in numbers, the line AP ſhall be alſo given in number, for putting g = AE, and h = ED, (as before in the Reſolution,) alſo k = AB; the line AP (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gk}{h-k}$. It is alſo manifeſt, that if a right line be drawn from any point in AC, between P and C, to the given point D, the line ſo drawn muſt neceſſarily cut BC, for the line PBD is ſuppoſed to paſs by the angular point B; but if a right line be drawn from any point in AC between P and A to the point D, the line ſo drawn will evidently cut AB. From the premiſſes therefore we may inferr this following

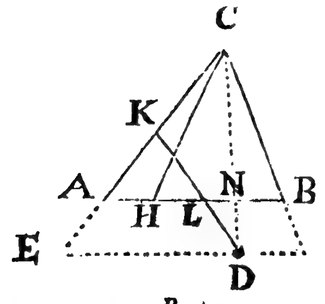
RULE.

34. If $\frac{1}{2}m + \sqrt{:\frac{1}{4}mm + gm:}$ the value of AK, be not greater than $\frac{gk}{h-k}$ the value of AP in *Fig.* 3. then the line of partition to be drawn from the given point D, ſhall either paſs by the angular point B, as the line DBP, or elſe cut AB in ſome point between B and A, and AC in ſome point between P and A. But if the ſaid value of AK be greater than the ſaid value of AP, then the line of partition will cut BC and AC, and in this latter Caſe a Parallel is to be drawn by the point D to BC, which is to be eſteemed the Baſe, and then the given lines being

reſpectively changed, the line found out by the Canon muſt be ſet from C towards A.

35. The Problem propounded needs not any Determination to be annexed to it, either to limit the quantities of the given lines, or the poſition of the given point without the given Triangle. But becauſe from what hath been ſaid in 1°, 2°, 3°, 4° and 5°, it evidently appears, that in every one of the three preceding Figures the given ſide AC muſt be greater than the quantity of the ſought line AK, except only in one Caſe in *Fig.* 1. when H lyes in the line DC, and is the ſame with the point N, for then AK is equal to AC; it will be requiſite to prove, that in all other caſes the ſide AC is greater than that right line which the Canon finds out for the value of AK, *viz.* in the letters of the Reſolution, that $b \sqsubset \frac{1}{2}m + \sqrt{:\frac{1}{4}mm + gm}$: The truth hereof I ſhall firſt demonſtrate in this following

FIG. 1.



M ——

Q —

Prepar.

36. Let it be made as ED to AH, ſo AC to a fourth proportional line, which may be called M, therefore, } ED . AH :: AC . M.

37. Let it alſo be made as ED to AH, ſo AE to a fourth proportional line, Q, therefore, . . . } ED . AH :: AE . Q.

38. Then from thoſe Analogies it follows (*per prop.* 11. *Elem.* 5.) that . . . . . } AE . Q :: AC . M.

39. . . . { *Reg. demonſtr.* . . . . . AC $\sqsubset \frac{1}{2}$M + $\sqrt{:\square$AE,M + $\frac{1}{4}\square$M:
That is, in the letters of the Reſolution, $b \sqsubset \frac{1}{2}m + \sqrt{: gm + \frac{1}{4}mm}$:

Demonſtration.

40. By *Conſtr.* in 1°, AN ‖ ED, therefore in *Fig.* 1. △ ACN and △ ECD are equiangular, and conſequently, (*per prop.* 4. *Elem.* 6.) . . . . . } EC . ED :: AC . AN.

41. Therefore from 40°, (*per prop.* 16. *Elem* 6.) . . . . . } $\square$EC,AN = $\square$ED,AC.

42. By *Suppoſ.* in *Fig.* 1. . . . . } AN $\sqsubset$ AH.

43. Therefore by drawing EC into each part, ... } $\square$EC,AN $\sqsubset$ $\square$EC,AH.

44. Therefore from 41° and 43° (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$EC,AH.

45. And becauſe, (as is evident by *Fig.* 1.) } AC + AE = EC.

46. Therefore from 45°, by drawing AH into each part, ... } $\square$AC,AH + $\square$AE,AH = $\square$EC,AH.

47. Therefore from 44° and 46°, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$AC,AH + $\square$AE,AH.

48. From 36° it follows (*per prop.* 16. *Elem.* 6.) that . . . . . } $\square$ED,M = $\square$AC,AH.

49. Likewiſe from 37°, that . . . } $\square$ED,Q = $\square$AE,AH.

50. And by adding the Equation in 49° to that in 48°, it makes . . . } $\square$ED,M + $\square$ED,Q = $\square$AC,AH + $\square$AE,AH

51. Therefore from 47° and 50°, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$ED,AC $\sqsubset$ $\square$ED,M + $\square$ED,Q.

52. Therefore from 51°, by caſting away the common altitude ED, . . } AC $\sqsubset$ M + Q.

53. And

53. And from 52°, by taking in the common altitude AC, ... } $\square$AC $\sqsubset$ $\square$AC,M + $\square$AC,Q.

54. But from 38° it follows (*per prop.* 16. *Elem.* 6.) that . . . . . } $\square$AE,M = $\square$AC,Q.

55. Therefore from 53° and 54°, (*per Ax.* 6. *Chap.* 2.) . . . . . } $\square$AC $\sqsubset$ $\square$AC,M + $\square$AE,M.

56. And from 55°, by ſubtracting $\square$AC,M from each part, . . . } $\square$AC — $\square$AC,M $\sqsubset$ $\square$AE,M.

57. And by adding $\frac{1}{4}\square$M to each part in 56°, ... } $\square$AC + $\frac{1}{4}\square$M — $\square$AC,M $\sqsubset$ $\square$AE,M + $\frac{1}{4}\square$M.

58. From 52° 'tis evident, that . . . } AC $\sqsubset$ $\frac{1}{2}$M.

59. And by *Theor.* 5. *Chap.* 4. . . . } $\square$:AC — $\frac{1}{2}$M: = $\square$AC + $\frac{1}{4}\square$M — $\square$AC,M.

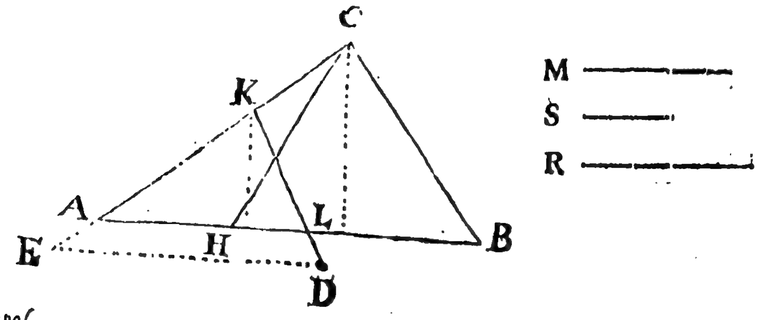
60. Therefore from 57° and 59°, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square$:AC — $\frac{1}{2}$M: $\sqsubset$ $\square$AE,M + $\frac{1}{4}\square$M.

61. And becauſe if one Plane exceeds another, the ſide of a Square equal to the former ſhall exceed the ſide of a Square equal to the latter, therefore from 60°, ... } AC — $\frac{1}{2}$M $\sqsubset$ $\sqrt{:\square$AE,M + $\frac{1}{4}\square$M:

62. Therefore from 61°, by adding $\frac{1}{2}$M to each part, ... } AC $\sqsubset$ $\frac{1}{2}$M + $\sqrt{:\square$AE,M + $\frac{1}{4}\square$M:

Which was to be Dem. The like Demonſtration may be applied to *Fig.* 2, and 3. after N is ſet in the place of B in *Fig.* 2. and C in the place of P in *Fig.* 3.

The Compoſition of the preceding Probl. 18.



M ———

S ——

R ———

Suppoſ.

63. △ ABC is given.

64. AH and HB are in a given Reaſon.

65. D is a point given without the △ ABC.

66. DE is ‖ AB, and given.

67. AE the Increaſe of CA continued until it cut DE is given.

Req. to find

68. AK ſuch a ſegment of AC, that DK being drawn, it may make

69. △ ALK . LKCB :: AH . HB.

Conſtruction.

70. Suppoſing (by what hath been ſaid in 1°, 2°, 3°, 4°, 5°, 33° and 34°,) it be diſcovered, that the line of partition to be drawn from D muſt cut the ſides AB and AC; draw DE parallel to AB, and cutting CA continued in E.

71. Then by *Probl.* 8. *Chap.* 5. let it be made, as ED to AC, ſo AH to a fourth proportional line M, therefore } ED . AC :: AH . M.

72. Find a mean proportional line, as S, between M and AE, therefore . . . . . } M . S :: S . AE.

73. Then eſteeming the line S to be the mean of three Proportionals, and the line M the difference of the extremes, find out the extremes, (*per Probl.* 12. *Chap.* 5.) the greater whereof ſuppoſe to be the line R, whence the leſſer ſhall be equal to R — M, therefore theſe are Proportionals, *viz.* . . . . . } R — M . S :: S . R.
That is, in 30°, the laſt ſtep of the Reſolution, . . } $a - m$. $\sqrt{mg}$ :: $\sqrt{mg}$. a.

74. From AC cut off AK = R, which may be done, for that AC is greater than R, I prove thus;

By *Conſtr.* in 73°, ...	R — M . S :: S . R.
Therefore by the Theorem in 24° of *Prob.* 12. *Chap.* 5.	R = ½M + √: □S + ¼□M :
And becauſe from 72°, (*per prop.* 17. *Elem.* 6.)	□AE, M = □S.
Therefore from the two laſt ſteps, (*per Ax.* 6. *Chap.* 2.)	R = ½M + √: □AE, M + ¼□M :
But it hath been ſhewn in 62°, that . . .	AC ⊏ ½M + √: □AE, M + ¼□M :
Therefore from the two laſt preceding ſteps, (*per Ax.* 3. *Chap.* 2.)	AC ⊏ R. Which was to be Dem.

75. Laſtly, draw the line DK, cutting AB in L; then ſhall the Triangle ALK be to the Trapezium LKCB, as AH to HB; which was required. The truth whereof will be made manifeſt by the following Demonſtration, formed out of the foregoing Reſolution by a repetition of its ſteps in a backward order.

76. . . . *Req. demonſtr.* . . . . . △ALK . LKCB :: AH . HB.

Demonſtration.

77. By *Conſtr.* in 73°, ...	R — M . S :: S . R.
That is, in 30°, (the laſt ſtep of the Reſolution,)	*a* — *m* . √*mg* :: √*mg* . *a*.
78. Therefore from 77°, (*per prop.* 17. *Elem.* 6.)	□R — □MR = □S.
79. And becauſe by *Conſtr.* in 74°, . . .	AK = R.
80. And from the *Conſtr.* in 72°, (*per prop.* 17. *Elem.* 6.)	□M, AE = □S.
81. Therefore from 78°, 79° and 80°, by exchanging equal quantities, ...	□AK — □M, AK = □M, AE.
That is, in 29°, ...	. *aa* — *ma* = *mg*.
82. Therefore from 81°, by adding □M, AK to each part, ...	□AK = □M, AK + □M, AE.
That is, in 28°, ...	. *aa* = *ma* + *mg*.
83. Therefore from 82°, (*per prop.* 14. *Elem.* 6.)	AK . M :: AK + AE . AK.
That is, in 27°, ...	*a* . *m* :: *a* + *g* . *a*.
84. But becauſe △EKD and △AKL are like, (for by *Conſtr.* in 70°, ED ∥ AL,) therefore (*per prop.* 4. *Elem.* 6.)	ED . AL :: AK + AE . AK.
That is, in 26°, ...	*b* . $\frac{ha}{a+g}$ :: *a* + *g* . *a*.
85. Therefore from 83° and 84°, (*per prop* 11. *Elem* 5.)	AK . M :: ED . AL.
That is, in 25°, ...	*a* . *m* :: *b* . $\frac{ha}{a+g}$ (= $\frac{bc}{a}$.)
86. But by *Conſtr.* in 71°, ...	M . AC :: AH . ED.
87. Therefore from 85° and 86°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. concerning Inordinate proportion, ...	AK . AC :: AH . AL.
That is, in 18°, ...	*a* . *b* :: *c* . $\frac{bc}{a}$.
88. And becauſe △ALK and △ACH have a common angle, to wit, ∠KAL, and (as appears by the Analogy in 87°,) the ſides about that angle are reciprocally proportional, therefore (*per prop* 15. *Elem.* 6.)	△ALK = △ACH.
89. Therefore by ſubtracting △ALK and △ACH ſeverally from the △ABC, the remaining ſpaces ſhall be equal one to another, *viz.*	Trapez. LKCB = △HCB.
90. But (*per prop.* 1. *Elem.* 6.)	△ACH . △HCB :: AH . HB.
91. Therefore from 88°, 89° and 90°, by exchanging equal ſpaces, ...	△ALK . LKCB :: AH . AB.

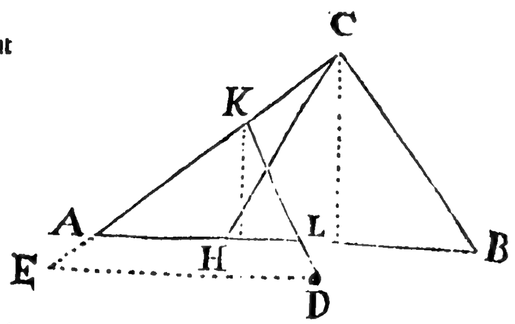
Which was to be demonſtrated. Therefore that is done which the Problem required.

An

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Probl. 18.

Suppoſ.

92. △ABC is given.
93. D is a point given without the △ABC.
94. DE ∥ AB.
95. CAE is a right line.
96. AB = 200.
97. AC = 160.
98. BC = 120.
99. DE = 140.
100. AE = 25.
101. AH = 70.
102. HB = 130.



} in the given Reaſon of 7 to 13.

Req. to find

103. The quantity of AK in number, ſuch, that DK being drawn, it may make △ALK . LKCB :: AH . HB :: 7 . 13.

Solution Arithmetical.

104. AK = 100, found out by the Canon in 32°, by the help of the lines given in numbers in 99°, 97°, 101° and 100°.

105. AL = 112, for { AK + AE . ED :: AK . AL. / 125 . 140 :: 100 . 112.

106. CL (⊥ AB) = 96, found out by the three ſides of △ABC, by the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

107. KH (⊥ AL) = 60, for { AC . CL :: AK . KH. / 160 . 96 :: 100 . 60.

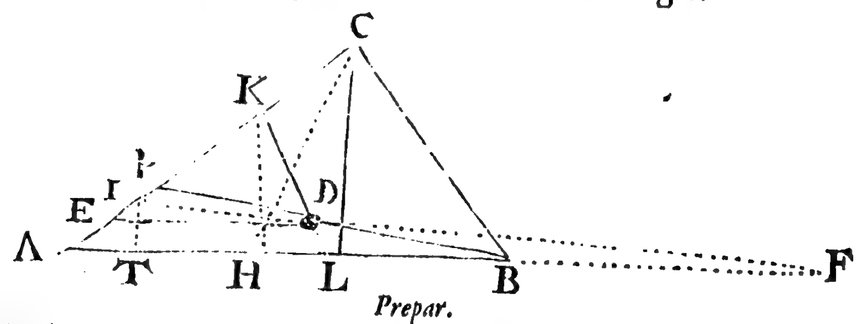
108. ½AB × CL = 9600, the Area of △ABC.
109. ½AL × KH = 3360, the Area of △ALK.
110. △ABC — △ALK = 6240, the Area of LKCB.

The Proof.

111. . . { △ALK . LKCB :: AH . HB :: 7 . 13. / 3360 . 6240 :: 70 . 130 :: 7 . 13.

Probl. XIX.

To divide a given Triangle ABC into two parts which ſhall be in any poſſible Reaſon given, ſuppoſe as AH to HB, by a right line KL that ſhall paſs by a given point D within the Triangle.



Prepar.

1. By the given point D draw a Parallel to one of the ſides of the given Triangle, as DE parallel to the Baſe AB, and cutting AC in E.
2. Divide the Baſe AB into two parts in the given Reaſon, as in the point H, and draw CH; therefore, (*per prop.* 1. *Elem.* 6.) △ACH . △HCB :: AH . HB.
3. Then ſuppoſing it be diſcovered (by the Rule hereafter given in 31° of this Problem,) that the line of partition required to paſs by the given point D muſt cut AB and AC, the Scope of the Reſolution is to find a point in AC, as K, from which a right line being drawn to paſs by D, as KDL, the Triangle AKL ſhall be equal to the Triangle AHC,

AHC, whence it evidently follows that LKCB = △HCB, and △AKL . LKCB :: AH . HB. Theſe things being premiſed, the Reſolution of the Problem propounded may be formed in manner following.

Suppoſ.

4. △ABC is given.
5. D is a point given within the △ABC.
6. AH and HB are in a given Reaſon.
7. b = AC is given.
8. c = AH is given.
9. g = AE is given.
10. h = ED (‖ AB) is given.

Req. to find

11. AK ſuch a ſegment of AC, that KDL being drawn, it may make
12. △AKL . LKCB :: AH . HB :: △ACH . △HCB.

Reſolution.

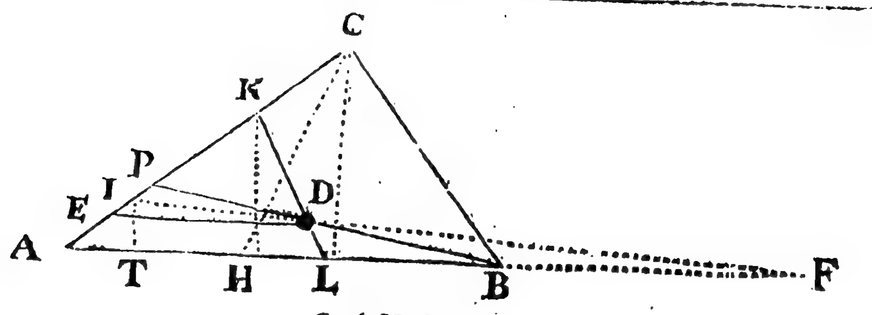
13. Suppoſe that done which is required, and put } a = AK.
14. Then by conſidering well what is required, and viewing the Diagram, it will appear that △AKL = △ACH, and that the ∠CAL is common to both thoſe Triangles, therefore *(per prop.* 15. *El.* 6.) } AK . AH :: AC . AL.
15. That is, in the letters of the Reſolution, . . } $a . c :: b . \frac{bc}{a}$.
16. And becauſe △AKL and △EKD are like, (for by *Conſtr.* in 1°, DE ‖ AL,) therefore (*per prop.* 4. *Elem.* 6.) . . . . . } EK . AK :: ED . AL.
17. That is, in the letters of the Reſolution, . . } $a-g . a :: h . \frac{ha}{a-g}$.
18. And becauſe the fourth Proportional in 14° is the ſame with the fourth in 16°, therefore the fourth Proportionals in 15° and 17°, ſhall be equal to one another, *viz.* . . . . . } $\frac{bc}{a} = \frac{ha}{a-g}$.
19. Now to avoid an Equation between Solids, let it be made as h to b, ſo c to a fourth Proportional, which may be called m, therefore . . . } $h . b :: c . m$.
20. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . } $hm = bc$.
21. Therefore from 18° and 20°, by exchanging equal Rectangles, 'tis evident that . . . . } $\frac{hm}{a} = \frac{ha}{a-g} = \frac{bc}{a}$.
22. Whence 'tis eaſie to inferr that theſe are Proportionals, *viz.* . . . . . } $a . m :: h . \frac{ha}{a-g} (= \frac{bc}{a}.)$
23. But it hath been ſhewn in 17°, that . . . } $a-g . a :: h . \frac{ha}{a-g}$.
24. Therefore from 22° and 23°, *(per prop.* 11. *El.* 5.) } $a . m :: a-g . a$.
25. And from 24°, by comparing the Rectangle of the extremes to the Rectangle of the means, . . } $aa = ma - mg$.
26. From which Equation, after due tranſpoſition, this ariſeth, *viz.* . . . . . } $ma - aa = mg$.
27. Which laſt Equation may be reſolved into theſe Proportionals, *viz.* . . . . . } $m-a . \sqrt{mg} :: \sqrt{mg} . a$.

28. Of which three Proportionals, the mean, to wit, $\sqrt{mg}$ is given, as alſo m the ſumm of the extremes, therefore the extremes ſhall be given ſeverally *(per Probl.* 13. *Chap.* 5.) the values whereof, by the Theorem in 21° of the ſaid *Probl.* 13. *Chap.* 5. will be found equal to theſe right lines, (or numbers,) *viz.*

$$\left\{\begin{array}{l} \frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg:} \\ \frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg:} \end{array}\right\}$$ the two Roots of the Equation in 26°.

Hence this

CANON.



CANON.

Let it be made as ED to AC, ſo AH to a fourth Proportional, which may be called M. Then to and from the half of M, add and ſubtract the ſquare Root of the exceſs of the Square of half M above the Rectangle of M into AE, ſo ſhall the ſumm and remainder made by that Addition and Subtraction be the two values of AK ſought, (repreſented by a in the foregoing Reſolution.)

29. But to the end the ſaid values of AK may be Real, that is, greater than nothing, the given lines muſt be ſubject to this

Determination. g not $\sqsubset \frac{1}{4}m$, or $\frac{bc}{4h}$; that is, in words,

The line AE muſt not exceed the right line ariſing by the Application of the Rectangle made of the lines AC and AH to the quadruple of the line ED.

The truth of this Determination, which is diſcovered both by the Analogy in 27°, and by the values of a in 28°, may be proved thus,

It hath been diſcovered in 27°, that theſe are Proportionals, *viz.* . . . . . } $m-a . \sqrt{mg} :: \sqrt{mg} . a$.
Of which Proportionals the ſumm of the extremes is evidently m, and the mean is $\sqrt{mg}$; therefore (as hath been ſhewn in 20° of *Probl.* 13. *Chap.* 5.) . . . } $\sqrt{mg}$ not $\sqsubset \frac{1}{2}m$.
Whence, by ſquaring each part it follows, that . . } mg not $\sqsubset \frac{1}{4}mm$.
And by Application of each part to m, ... } g not $\sqsubset \frac{1}{4}m$.
But by *Conſtr.* in 19°, ... } $\frac{bc}{h} = m$.
And conſequently, by taking $\frac{1}{4}$ of each part, . . } $\frac{bc}{4h} = \frac{1}{4}m$.
Therefore from the three laſt ſteps, ... } g not $\sqsubset \frac{1}{4}m$, or $\frac{bc}{4h}$.
Which was to be Demonſtr.

30. It is alſo eaſie to perceive by the two Roots or extreme Proportionals found out in 28°, that if $g = \frac{1}{4}m$, and conſequently $mg = \frac{1}{4}mm$, then thoſe Roots will be equal to one another, *viz.* each Root equal to $\frac{1}{2}m$, which, if it fall within the limits hereafter declared in the Rule in 31° of this Problem, ſhall be equal to the line AK ſought; But if $g \sqsupset \frac{1}{4}m$, then the ſaid Roots will be unequal, and the Equation in 26° may be expounded by each of thoſe Roots; and ſometimes either of them may be taken for the value of the line AK ſought, but ſometimes only one of the ſaid Roots, and ſometimes neither of them. To diſcover therefore whether there be a poſſibility of effecting the Problem or not, by the lines given in ſuch manner as before is expreſt, the value of the line AP muſt be enquired; to which end, firſt, draw a right line that may paſs by the angular point B and the given point D, (in the precedent Diagram,) then is EABD a Trapezium having (by *Conſtr.* in 1°,) two parallel ſides AB and ED, and the other two ſides AE and BD, which are not Parallels, being continued will meet in ſome point in AC, as in P. Now if AB, ED and AE be ſeverally given in numbers, the line EP ſhall be alſo given in number, for putting g = AE, and h = ED, (as before in the Reſolution,) alſo k = AB; the line EP (by the Theorem in 9° of *Probl.* 18. *Chap.* 7.) will be found equal to $\frac{gh}{k-h}$, to which adding g, that is, AE, it makes $\frac{gk}{k-h}$ for the value of AP.

It is

It is alſo evident, that if from any point in A C between E and P, a right line be drawn to paſs by the given point D, as I D F, it ſhall neceſſarily cut the Baſe A B continued without the Triangle A B C, as in F; but if a right line be drawn from any point in A C between P and C to paſs by the point D, as K D L, it ſhall neceſſarily cut the Baſe A B between A and B. From the premiſſes we may interr this following

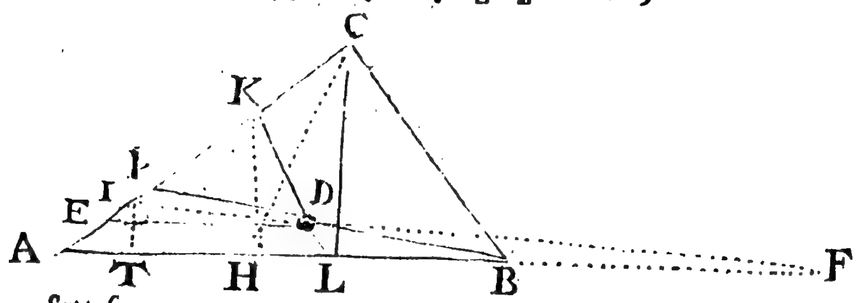
R u L E.

31. First, if $\frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg}$: and $\frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg}$: (the Roots before found out in 28° for the values of A K,) be unequal, and each of them leſs than A C, but neither of them leſs than $\frac{gk}{k-h}$, (the value of A P,) then two right lines equal to thoſe Roots or values of A K being ſet from A towards C will end in two points, from which if two right lines be drawn to paſs by the given point D, each of them ſhall divide the given Triangle A B C into two parts in the given Reaſon.

Secondly, if only one of thoſe Roots or values of A K happens to be leſs than A C, and not leſs than the ſaid value of A P, then the given Triangle A B C can be divided only by one right line paſſing by the given point D, ſo as to cut both A B and A C, or A C only, to divide the ſaid Triangle into two parts in the given Reaſon.

Thirdly and laſtly, if neither of the ſaid Roots or values of A K happens to be within the limits above expreſt, then 'tis impoſſible to draw a right line that ſhall paſs by the given point D, and cut both A B and A C, or A C only, and paſs by the angular point B, to divide the given Triangle into two parts in the given Reaſon: And if the like impoſſibility happen after a Parallel is drawn by the given point D to each of the other two ſides A C, B C, and tryal made as before, it ſhall be impoſſible to perform what the Problem requires; but when there is a poſſibility, then oftentimes which two ſides you pleaſe may be cut by the line of partition.

The Compoſition of the foregoing Probl. 19.



Suppoſ.

32. △ A B C is given.
33. A H and H B are in a given Reaſon.
34. D is a point given within the △ A B C.
35. D E (‖ A B) is given.
36. A E is given, and not greater than $\frac{\square AC,AH}{4ED}$, agreeable to the preceding *Determination* in 29°.

Req. to find

37. A K ſuch a ſegment of A C, that K D L being drawn, it may make
38. △ A L K . L K C B :: A H . H B :: △ A C H . △ H C B.

Conſtruction.

39. By the given point D draw . . . . . } D E ‖ A B.
40. Then ſuppoſing it be diſcovered by the preceding Rule in 31°, that the line of partition which is to paſs by the given point D may cut A B and A C, let it be made (by *Probl.* 8. *Chap.* 5.) as E D to A C, ſo A H to a fourth proportional line, which may be called M, therefore, ... } E D . A C :: A H . M.
41. Find a mean proportional line, as S, between M and A E, therefore . . . . . } M . S :: S . A E.
42. Divide the line M (before found) into two ſuch parts, that the line S may be a mean Proportional between the parts, which may be done (by *Probl.* 14. *Chap.* 5.) if S be not greater than $\frac{1}{2}$M; but that S is not greater than $\frac{1}{2}$M, I prove thus:

From the foregoing *Conſtr.* in 40°, it is eaſie to perceive that . . . . . } $M = \frac{\square AC,AH}{ED}$.
Whence, by taking $\frac{1}{4}$ of each part, ... } $\frac{1}{4}M = \frac{\square AC,AH}{4ED}$.
By the Determination in 36°, ... } A E not $\sqsubset \frac{\square AC,AH}{4ED}$.
Therefore from the two laſt ſteps, (*per Ax.* 3. *Chap.* 2.) } A E not $\sqsubset \frac{1}{4}$M.
Therefore by drawing M into each part, . . . } $\square M,AE$ not $\sqsubset \frac{1}{4}\square M$.
And becauſe from 41°, (*per prop.* 17. *Elem.* 6.) . } $\square S = \square M,AE$.
Therefore from the two laſt preceding ſteps, (*per Ax.* 4. *Chap.* 2.) . . . . . } $\square S$ not $\sqsubset \frac{1}{4}\square M$.
But if one Square exceeds another, the ſide of the former exceeds alſo the ſide of the latter, therefore from the laſt ſtep, ... } S not $\sqsubset \frac{1}{2}$M.

Which was to be Demonſtr. Therefore 'tis poſſible to cut the line M into two ſuch parts, that the line S may be a mean Proportional between them; ſuppoſe then the right line R be found the greater part, and T the leſſer, therefore theſe are Proportionals, *viz.*

$$\left\{\begin{array}{l} M - R \;.\; S :: S \;.\; R. \\ M - T \;.\; S :: S \;.\; T. \end{array}\right.$$

That is, in 27°, . . . $m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a$.

Which two lines R and T do anſwer to the two Roots or values of A K before expreſt in the Canon in 28°, *viz.*

$$\left\{\begin{array}{l} R = \frac{1}{2}m + \sqrt{:\frac{1}{4}mm - mg}: \\ T = \frac{1}{2}m - \sqrt{:\frac{1}{4}mm - mg}: \end{array}\right\} \text{ the values of A K.}$$

43. Now ſuppoſing that by the limits in 31° it be diſcovered that the greater Root or line R is leſs than A C, and not leſs than A P, let the line R be ſet from A towards C, as to K, *viz.* make A K = R, and draw the right line K D L; then ſhall △ A K L be equal to △ A C H, and conſequently △ A K L . L K C B :: A H . H B :: △ A C H . △ H C B, as was required.

And if the leſſer Root or line T happens to be leſs than A C, but not leſs than A P, let the line T be ſet alſo from A towards C, (as to I in the following Diagram belonging to the latter Example of this Problem,) and then a right line being drawn from the point in A C where T doth end, to paſs by the given point D, it ſhall likewiſe divide the given Triangle A B C into two parts in the given Reaſon.

But if either of the ſaid lines R and T, which are to be ſet (as before) from A towards C, happens to fall between E and P, then the line of partition to paſs by D will cut the Baſe A B continued without the Triangle A B C; as, if A I be ſuppoſed equal to T, and the line I D be drawn and continued till it concurr wth the Baſe A B continued, as in F, then although △ A I F be equal to △ A C H, yet it ſolves not the Problem, in regard part of △ A I F lyes without the given △ A B C.

It remains to prove that △ A L K . L K C B :: A H . A B, but this will be made manifeſt by the following Demonſtration, formed out of the preceding Reſolution by a repetition of its ſteps in a backward (not direct) order.

44. . . . *Req. demonſtr.* . . . . . △ A L K . L K C B :: A H . H B.

Demonſtration.

45. By *Conſtr.* in 42°, ... } M — R . S :: S . R.
That is, in 27°, (the laſt ſtep of the Reſolution,) } $m - a \;.\; \sqrt{mg} :: \sqrt{mg} \;.\; a$.
46. Therefore from 45°, (*per prop.* 17. *Elem.* 6.) } $\square MR - \square R = \square S$.
47. And becauſe by *Conſtr.* in 43°, . . . } A K = R.
48. And from the *Conſtr.* in 41°, (*per prop.* 17. *Elem.* 6.) . . . . . } $\square M,AE = \square S$.
49. Therefore from 46°, 47° and 48°, by exchanging equal quantities, ... } $\square M,AK - \square AK = \square M,AE$.
That is, in 26°, ... } $ma - aa = mg$.

50. There-

50. Therefore from 49°, by adding □ AK, and ſubtracting □ M,AE from each part, That is, in 25°, ... } □AK = □M,AK — □M,AE. / . aa = ma — mg.

51. Therefore by reſolving the Equation in 50° into Proportionals, That is, in 24°, ... } AK . M :: AK — AE . AK. / a . m :: a — g . a.

52. But becauſe △ EKD and △ AKL are like, (for by *Conſtr.* in 39°, ED ‖ AL,) therefore (*per prop.* 4. *Elem.* 6.) That is, in 23°, ... } ED . AL :: AK—AE . AK. / h . $\frac{ha}{a-g}$ :: a — g . a.

53. Therefore from 51° and 52°, (*per prop.* 11. *Elem.* 5.) That is, in 22°, ... } AK . M :: ED . AL. / a . m :: h . $\frac{ha}{a-g}$ (= $\frac{bc}{a}$.)

54. But by *Conſtr.* in 40°, ... } ED . AC :: AH . M.

55. Therefore from 53° and 54°, (*per prop.* 23. *Elem.* 5.) agreeable to *Defin.* 8. *Chap.* 3. That is, in 15°, ... } AK . AC :: AH . AL. / a . b :: c . $\frac{bc}{a}$.

56. And becauſe △ AKL and △ ACH have a common angle, to wit, ∠ KAL, and (as appears by the Analogy in 55°,) the ſides about that angle are reciprocally proportional, therefore (*per prop.* 15. *Elem.* 6.) . . . . } △ AKL = △ ACH.

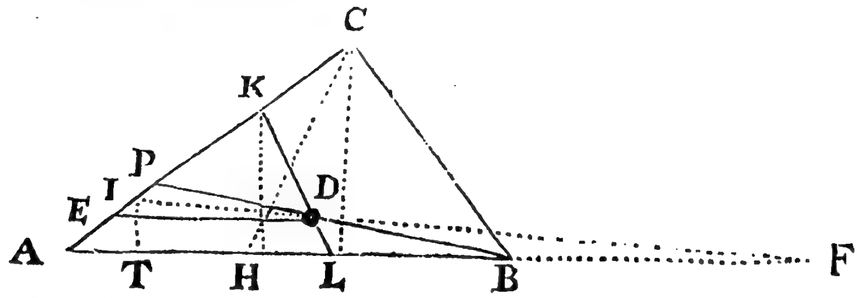
57. Therefore by ſubtracting △ AKL and △ ACH ſeverally from △ ABC, the remaining Spaces ſhall be equal one to another, *viz.* . . . . . } Trapez. LKCB = △ HCB.

58. But (*per prop.* 1. *Elem.* 6.) . . . . } △ ACH . △ HCB :: AH . HB.

59. Therefore from 56°, 57° and 58°, by exchanging equal Spaces, ... } △ AKL . LKCB :: AH . HB.

Which was to be demonſtrated. Therefore that is done which the Problem required.

An Example in Numbers, to illuſtrate the precedent Reſolution *of* Probl. 19. *in which Example the greater of the two Roots or values of* AK *before expreſt in* 28°, *is only capable of ſolving the* Problem.



Suppoſ.

60. AB = $231\frac{1}{4}$ the Baſe
61. AC = 185 } the leggs } of △ ABC are given.
62. BC = $138\frac{1}{4}$ }
63. D is a point given within the △ ABC.
64. DE ‖ AB, and DE = 112 is given.
65. EA = 25 is given.
66. AH = $94\frac{88}{148}$ }
67. HB = $136\frac{27}{148}$ } in a given Reaſon, *viz.* as 560 to 809.
68. CL, KH, IT are each ⊥ AB.

Req. to find in number,

69. AK, ſuch, that the right line KDL being drawn, it may make
70. △ AKL . LKCB :: AH . HB :: 560 . 809.

Solution

Solution Arithmetical.

71. AK = 125 = the greater of the two Roots in 28°, found out by the Canon there expreſt.

The Proof.

72. AL = 140; for { AK — AE . ED :: AK . AL. / 100 . 112 :: 125 . 140.

73. CL = 111, found out by the three ſides of △ ABC given in 60°, 61° and 62°, with the help of *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

74. KH = 75; for { AC . CL :: AK . KH. / 185 . 111 :: 125 . 75.

75. $\frac{1}{2}$AB × CL = $12834\frac{1}{8}$ the Area of △ ABC.
76. $\frac{1}{2}$AL × KH = 5250 the Area of △ AKL.
77. △ ABC — △ ALK = $7584\frac{1}{8}$ the Area of LKCB.
78. { △ AKL . LKCB :: AH . HB :: 560 . 809. / 5250 . $7584\frac{1}{8}$ :: $94\frac{88}{148}$. $136\frac{27}{148}$ :: 560 . 809.

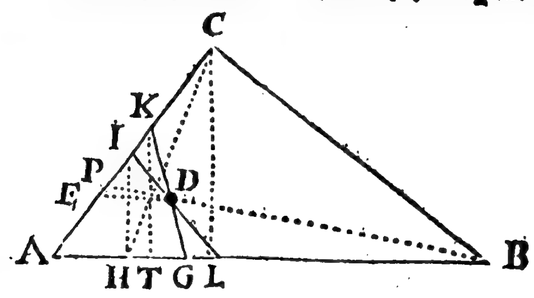
Which was to be done.

79. *Note.* In this Example, AI the leſſer of the two Roots or values of AK in 28° falls between E and P, and therefore it cannot ſolve the Problem; for if ID be drawn and continued, it will cut AB continued without the △ ABC, as in F. But 'tis worth obſerving, that the △ AIF is equal to the △ AKL, as the following Proof will make manifeſt.

80. AI = $31\frac{1}{4}$ = the leſſer Root, found out by the Canon in 28°.
81. EA = 25, given in 65°.
82. EI = $6\frac{1}{4}$ = AI — AE.
83. DE = 112, given in 64°.
84. AF = 560; for { EI . ED :: AI . AF. / $6\frac{1}{4}$. 112 :: $31\frac{1}{4}$. 560.
85. IT = $18\frac{1}{4}$; for { AK . KH :: AI . IT. / 125 . 75 :: $31\frac{1}{4}$. $18\frac{1}{4}$.
86. $\frac{1}{2}$AF × IT = 5250 = $\frac{1}{2}$AL × KH, therefore △ AIF = △ AKL.

Which was to be proved.

Another Example in Numbers, referring to the ſubſequent Figure, where each of the two Roots or values of AK *before expreſt in* 28° *is capable of ſolving the foregoing* Probl. 19.



Suppoſ.

87. AB = $231\frac{1}{4}$ the Baſe
88. AC = $138\frac{1}{4}$ } the leggs } of △ ABC are given.
89. BC = 185 }
90. D is a point given within the △ ABC.
91. DE ‖ AB, and DE = $36\frac{88}{111}$ is given.
92. EA = 38 is given.
93. AH = 40 }
94. HB = $191\frac{1}{4}$ } in a given Reaſon, *viz.* as 32 to 153.
95. CL, KT, IH are each ⊥ AB.

Req. to find in numbers,

96. AK and AI, ſuch, that the right lines KDG and IDL being drawn; theſe Analogies ſhall enſue, *viz.*

97. { △ AKG . GKCB :: AH . HB :: 32 . 153. Alſo, / △ AIL . LICB :: AH . HB :: 32 . 1585.

Solution Arithmetical.

98. AK $= 84\frac{6}{251}$,
99. AI $= 69\frac{1}{8}$, } the two Roots of the Equation in 26°, found out by the *Canon* in 28°.

The Proof by the greater Root AK.

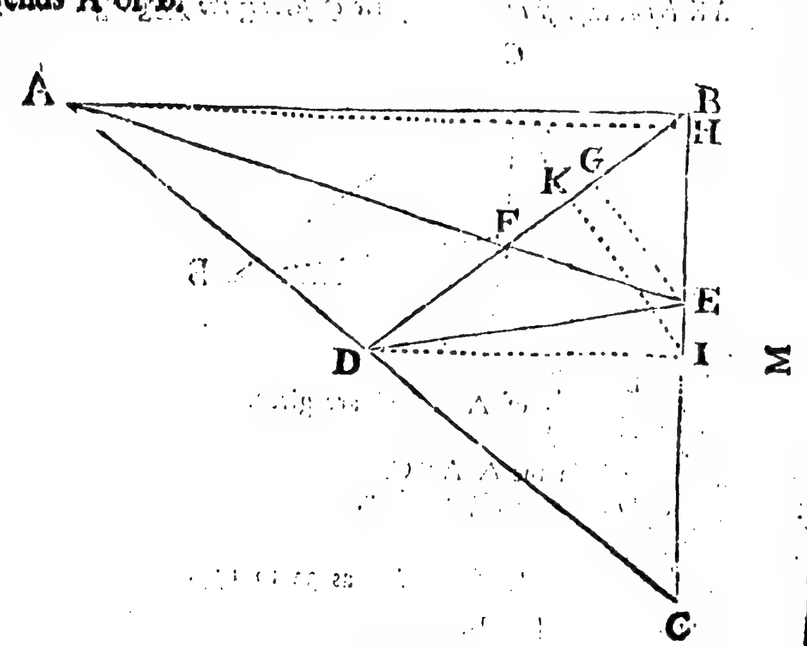
100. AK $= 84\frac{6}{251}$ before found out in 98°.
101. AE $= 38$ given in 92°.
102. EK $= 46\frac{6}{251} =$ AK $-$ AE.
103. ED $= 36\frac{20}{111}$ given in 91°.
104. AG $= 66\frac{1}{19}$; for { EK . ED :: AK . AG. / $46\frac{6}{251}$. $36\frac{20}{111}$ :: $84\frac{6}{251}$. $66\frac{1}{19}$.
105. CL $= 111$ found out by the three given sides of $\triangle$ ABC.
106. KT $= 67\frac{11}{251}$; for { AC . CL :: AK . KT. / $138\frac{2}{4}$. 111 :: $84\frac{6}{251}$. $67\frac{11}{251}$.
107. $\frac{1}{2}$AB × CL $= 12834\frac{1}{8} =$ the Area of $\triangle$ ABC.
108. $\frac{1}{2}$AG × KT $= 2220 =$ the Area of $\triangle$ AKG.
109. $\triangle$ ABC $-$ $\triangle$ AKG $= 10614\frac{1}{8} =$ the Area of GKCB.
110. { $\triangle$ AKG . GKCB :: AH . HB :: 32 . 153. / 2220 . $10614\frac{1}{8}$ :: 40 . $191\frac{1}{4}$ :: 32 . 153.

The Proof by the lesser Root AI, *found out in* 99°.

111. AL $= 80$; for { AI $-$ AE . ED :: AI . AL. / $31\frac{1}{8}$. $36\frac{20}{111}$ :: $69\frac{1}{8}$. 80.
112. IH $= 55\frac{1}{2}$; for { AC . CL :: AI . IH. / $138\frac{2}{4}$. 111 :: $69\frac{1}{8}$. $55\frac{1}{2}$.
113. $\frac{1}{2}$AB × CL $= 12834\frac{1}{8} =$ $\triangle$ ABC.
114. $\frac{1}{2}$AL × IH $= 2220 =$ $\triangle$ AIL.
115. $\triangle$ ABC $-$ $\triangle$ AIL $= 10614\frac{1}{8} =$ LICB.
116. { $\triangle$ AIL . LICB :: AH . HB :: 32 . 153. / 2220 . $10614\frac{1}{8}$ :: 40 . $191\frac{1}{4}$ :: 32 . 153.

Probl. XX.

To find the length of a right line AB, when we cannot come to either of its ends A or B.



Prepar.

First, supposing C to be a Station where A and B may be seen; and that there is liberty to measure from C towards AB without impediment, measure CD a distance at pleasure, (in Feet, or what equal parts you please,) yet so, as CDA may be a straight line;

line; measure likewise CE, a distance at pleasure, yet so, as CEB may be a straight line. Again, measure ED; also DF, that is, a distance directly towards B, until you come to the point F, where DB and EA cut one another; measure also FE.

Secondly, in solving this Problem Arithmetically according to the following Resolution, by the help of those five lines measured as above is directed, these are two principal Cases, *viz.* first, when the angle ACB is acute; secondly, when 'tis obtuse; and in regard the three sides of the Triangle CDE are supposed (as above) to be severally given in numbers, we may (by the Corollary in 45° of *Probl.* 10. *Chap.* 7.) discover the kind of every one of the angles of the said Triangle. First then, suppose it be found that the angles DCE and CED in the Diagram before expresst are each of them acute, then if a Perpendicular be let fall from the angle CDE, as DI, upon the Base CE, it will necessarily fall within the said Triangle CDE; likewise if the angle DCE be acute, and the angle CED obtuse, then a Perpendicular from E will fall within the said Triangle upon the opposite side CD; but if the angle DCE be obtuse, as in the following Diagram belonging to the latter Example of this Problem 'tis supposed to be,) then a Perpendicular from D or E will fall without the Triangle DCE.

Thirdly, let a Perpendicular be supposed to fall from E upon DB, as EG; this Perpendicular shall be less than the Perpendicular DI, for EG is manifestly less than the Perpendicular IK, which is less than ID the Hypothenusal of the right-angled Triangle IKD.

Fourthly, supposing a Perpendicular to fall from A upon CB produced *infinitely*, it will either fall upon the point B, or else within or without the Triangle CBA; but in this Example I shall suppose that Perpendicular to be AH, falling within the Triangle CBA. Now by the help of those five lines CD, CE, ED, DF, FE measured in Feet, or what equal parts you please, the length of the inaccessible distance AB may be found out in the same kind of parts, in manner following.

Suppos.

1. b = CE = 152
2. c = CD = 210
3. d = DF = 90
4. f = FE = 100
5. g = DE = 170

} Lines given.

6. p = DI = 168
7. k = IE = 26
8. l = IC = 126
9. n = EG = 80

} These are consequently given, for they may be found out by the help of the given sides of $\triangle$ CED and $\triangle$ DFE, (*per Theor.* in 29° and 30° of *Probl.* 9. *Chap.* 7.)

I. *Req.* to find EB and FB.

Resolution.

10. Put . . . . . } a = EB.
11. Then it follows from 7° and 10°, that . . } $a + k$ = IB.
12. The Square of the last Equation gives . . . } $aa + 2ka + kk$ = □IB.
13. To which Square adding the Square of p, that is, } . . . . . pp = □DI.
14. The summ makes . . . . . } $aa + 2ka + kk + pp$ = □DB.
15. And because in $\triangle$ EID right-angled at I, . } $gg = kk + pp$ = □DE.
16. Therefore from 14° and 15°, (*per Ax.* 6. *Ch.* 2.) } $aa + 2ka + gg$ = □DB.
17. And by extracting the square Root of the last Equation, this ariseth, ... } $\sqrt{:aa + 2ka + gg:}$ = DB.
18. By *Supposition* the angle CED is acute, and consequently (*per Coroll prop.* 13. *Elem.* 1.) the angle DEB is obtuse, therefore (*per prop.* 41. *Elem.* 1.) } □DI,EB=□EG,DB(=2$\triangle$DEB)
19. That is, in the letters of the Resolution, . . } $pa = n \times \sqrt{:aa + 2ka + gg:}$
20. Which last preceding Equation may be resolved into these Proportionals, *viz.* . . . . . } $p . n :: \sqrt{:aa + 2ka + gg:} . a.$
21. The Squares of which Proportionals are also Proportionals, *viz.* . . . . . } $pp . nn :: aa + 2ka + gg . aa.$
22. Now to avoid an Equation between Solids; to p and n find a third Proportional, call it q, therefore } $p . n :: n . q.$

23. There-

23. Therefore from the laſt Analogy this will ariſe, *(per Coroll. of 20. prop. 6. Elem.)* . . } $pp \cdot nn :: p \cdot q.$

24. Therefore from the Analogies in 21° and 23° this is manifeſt, *(per 11. prop. 5. Elem.)* } $p \cdot q :: aa + 2ka + gg \cdot aa.$

25. And from 24°, by Diviſion of Reaſon, . } $p - q \cdot q :: 2ka + gg \cdot aa.$

26. Then unto $2k$ and g find a third Proportional, which may be called s, therefore . . } $2k \cdot g :: g \cdot s.$

27. Therefore by reſolving that Analogy into an Equation, it gives . . . . . } $2ks = gg.$

28. Suppoſe . . . . . } $r = p - q.$

29. Then from 25°, 27° and 28°, by exchange of equal quantities, ... } $r \cdot q :: 2ka + 2ks \cdot aa.$

30. Let it be made . . . . . } $r \cdot q :: 2k \cdot$ (to a fourth) $t.$

31. Then from 29° and 30°, it's manifeſt *(per 11. prop. 5. Elem.)* that . . . . . } $2k \cdot t :: 2ka + 2ks \cdot aa.$

32. And by drawing $a + s$ into $2k$ and t ſeverally, this Analogy is manifeſt, ... } $2k \cdot t :: 2ka + 2ks \cdot ta + ts.$

33. Therefore from 31° and 32°, *(per 11. prop. 5. Elem.)* . . . . . } $2ka + 2ks \cdot aa :: 2ka + 2ks \cdot ta + ts.$

34. In which laſt Analogy, the firſt Term is equal to the third, therefore the ſecond ſhall be equal to the fourth, *viz.* . . . . . } $aa = ta + ts.$

35. Therefore by ſubtracting ta from each part of the laſt Equation, ... } $aa - ta = ts.$

36 Therefore (by the Canon in 43° of *Probl.* 12. *Chap.* 5.) . . . . . } $a = \frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = EB.$

From 22°, 26°, 28°, 30° and 36°, 'tis eaſie to deduce this following

CANON 1.

37. Firſt, to p and n, (that is, DI and EG,) find a third Proportional, which may be called q; therefore, ... } $p \cdot n :: n \cdot q \ (= \frac{242}{11}.)$

Secondly, unto $2k$ and g, (that is, 2IE and DE,) find a third Proportional, which may be called s; therefore, ... } $2k \cdot g :: g \cdot s \ (= 22\frac{8}{13})$

Thirdly, make . . . . . } $\ldots r = p - q \ (= 22\frac{8}{11}.)$

Fourthly, let it be made as r to q, ſo $2k$ to a fourth Proportional, which may be called t; therefore, } $r \cdot q :: 2k \cdot t \ (= \frac{1452}{341}.)$

Fifthly and laſtly, by the Canon in 43° of *Probl.* 12. *Ch.* 5. the quantity of EB (repreſented by a in the Reſolution) will be made known, *viz.* } $\frac{1}{2}t + \sqrt{:\frac{1}{4}tt + ts:} = 100 = EB = a.$

The Demonſtration of the ſaid Canon is manifeſt by the preceding Reſolution, which is argued Geometrically as well as Arithmetically, and no quantity in any ſtep thereof exceeds the dimenſions of a Plane. Moreover, by converting the Analogy in 21° into an Equation according to the vulgar way, the following Canon will ariſe, which is the ſame in ſubſtance with the former, but not ſo apt for Demonſtration.

CANON 2.

38. Multiply k, and the Square of g, ſeverally by the Square of n, and divide each Product by the exceſs of the Square of p, above the Square of n; then add the latter Quotient to the Square of the firſt, and extract the ſquare Root of the ſumm; laſtly, the ſaid ſquare Root added to the firſt Quotient will give the ſame value of a, (that is, 100 = EB) as before.

39. Now in order to find FB, firſt, . . . } IE + EB = IB = 126.

40. Then . . . . . } $\sqrt{:\square IB + \square ID:} = DB = 210.$

41. And . . . . . } DB − DF = FB = 120.

II. To find DA.

42. The lines CD, DB, BC, DF, CE being given or found out, (as before,) the length of DA may be diſcovered by the Canon in 17° of *Probl.* 19. *Chap.* 7. *viz.*

First,

First, . . . DF . DB :: CE . (to a fourth) M . $(= 354\frac{8}{13}.)$

Then, . . . M − CB . EB :: CD . DA $(= 204\frac{8}{11}.)$

III. To find AH and CH.

43. By *Suppoſition* AH is perpendicular to CH, and each of them may be found out thus,

CD . CA :: { DI . AH $(= 331\frac{7}{11}.)$ / CI . CH $(= 248\frac{8}{11}.)$

IV. To find AB the diſtance required.

44. Becauſe in this Example CH happens to be leſs than CB, (that is, CE + EB,) ſubtract CH from CB, and the remainder is HB, *viz.*

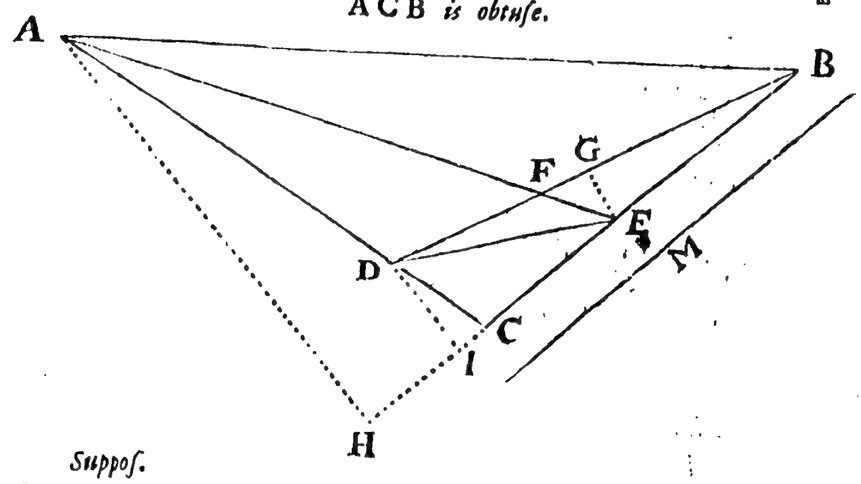
$$CB - CH = HB = 3\tfrac{3}{11}.$$

Then in △ AHB right-angled at H,

$$\sqrt{:\square AH + \square HB:} = AB = \frac{\sqrt{13309200}}{11}, \text{ that is, } 331, 652, \&c.$$

45. *Note.* When CH happens to exceed CB, then the Perpendicular AH will fall without the △ CAB, upon CB produced; in which Caſe GB is to be ſubtracted from CH, and then the Square of the remainder being added to the Square of the Perpendicular AH, the ſquare Root of the ſumm ſhall be the deſired diſtance AB. And laſtly, when CH happens to be found equal to CB, then the Perpendicular AH is the ſame with the diſtance AB.

The Arithmetical Solution of Caſe 2. Probl. 20. viz. *when the angle* ACB *is obtuſe.*



Suppoſ.

46. { $b = CE = 63\frac{1}{3}.$ / $c = CD = 41\frac{2}{3}.$ / $d = DF = 60\frac{129}{169}.$ / $f = FE = 29\frac{121}{169}.$ / $g = DE = 85.$ } Lines given.

{ $p = DI = 40.$ / $k = IE = 75.$ / $l = CI = 11\frac{2}{3}.$ / $n = EG = 20\frac{4}{13}.$ } Theſe are conſequently given, for they may be found out by the help of the given ſides of △CED and △DFE, *(per Theor. 2. in 31° and 32° of Probl. 10. Chap. 7.)*

Req. to find AB.

Operation Arithmetical.

First, by *Canon* 2. in 38° of the preceding Reſolution you will find by the help of the reſpective numbers given in 46°, ... } EB = $82\frac{1}{2}.$

Secondly, ... } IE + EB = IB = $157\frac{1}{2}.$

Thirdly, in △ DIB right-angled at I, . . . } $\sqrt{:\square IB + \square ID:} = DB = 162\frac{1}{4}.$

Fourthly, (by the Canon in 17° of *Probl.* 19. *Chap.* 7.) let it be made,

{ DF . DB :: CE . (to a fourth,) M = $169\frac{114221}{308071}$; then / M − CB . EB :: CD . (to a fourth,) DA = $146\frac{141}{11603}.$

Fifthly, ... } CD + DA = CA = $187\frac{144442}{348091}.$

Sixthly,

23. Therefore from the laſt Analogy this will ariſe, *(per Coroll.* of 20. *prop.* 6. *Elem.)* . . } $pp \cdot nn :: p \cdot q.$

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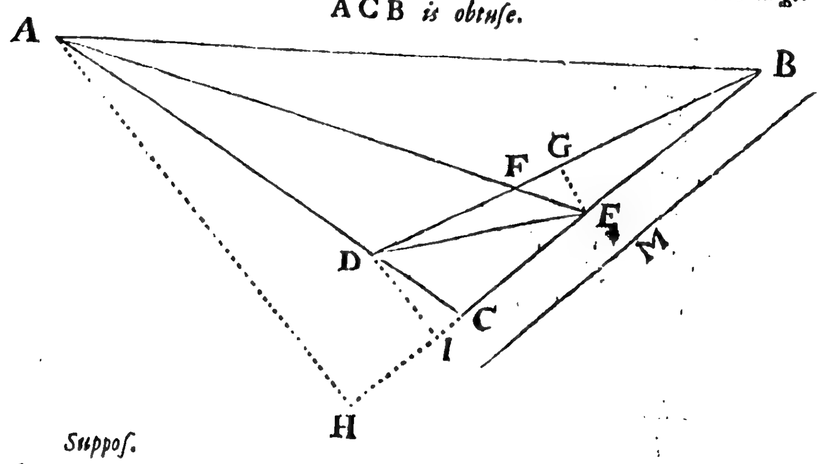
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{ M − CB . EB :: CD . (to a fourth,) DA = $146\frac{1412}{11603}$.

Fifthly, ... } CD + DA = CA = $187\frac{444412}{348091}$.

Sixthly,

Sixthly, becauſe △ CDI and △ CAH are like, theſe are Proportionals, *viz.*

$$\left\{\begin{array}{l} CD \,.\, CI :: CA \,.\, CH \;(= 52\tfrac{26604}{174045}.) \\ CD \,.\, DI :: CA \,.\, AH \;(= 180\tfrac{10248}{38015}.) \end{array}\right.$$

Seventhly, ... ⟩ $CH + CB = HB = 198\tfrac{41061}{116030}$.

Laſtly, in the △ AHB right-angled at H, ⟩ $\sqrt{:\square HB + \square HA:} = AB = 268\tfrac{3}{1000}$, *&c.*

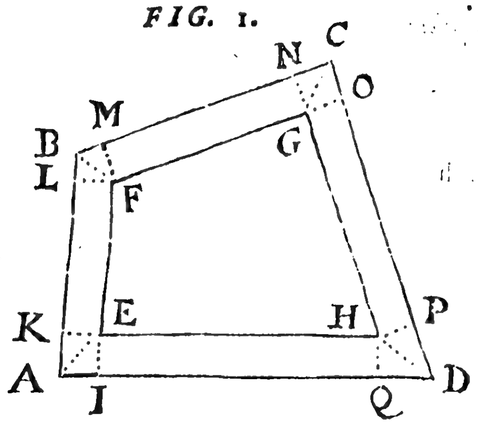
Note. When the angle ACB happens to be a right angle, then after EB and DA are found out in like manner as before in 38° and 42° of this Problem, there will be given CA and CB the ſides about the right angle ACB of △ ACB; therefore (*per prop.* 47. *Elem.* 1.) the ſquare Root of the ſumm of the Squares of thoſe ſides ſhall be the quantity of the Hypothenuſal AB, to wit, the diſtance ſought.

LEMMA 1. *leading to* Probl. 21.

Suppoſ.

1. ABCD is a Trapezium.
2. EFGH is a Trapezium.
3. EF ‖ AB.
4. FG ‖ BC.
5. GH ‖ CD.
6. EH ‖ AD.
7. AE, BF, CG, DH are right lines.
8. EI and HQ ⊥ AD.
9. EK and FL ⊥ AB.
10. FM and GN ⊥ BC.
11. GO and HP ⊥ CD.
12. EI = EK = FL = FM = GN = GO = HP = HQ.

FIG. 1.



Req. demonſtr.

13. AI = AK . BL = BM . CN = CO . PD = DQ.
14. $\left\{\begin{array}{l} \leqslant EAI = \leqslant EAK \,.\, \leqslant FBL = \leqslant FBM. \\ \leqslant GCN = \leqslant GCO \,.\, \leqslant HDP = \leqslant HDQ. \end{array}\right.$

Demonſtration.

15. By *Suppoſ.* in 8° EI ⊥ AD, therefore (*per defin.* 10. *Elem.* 1.) . . . . . ⟩ . . . $\leqslant AIE = \perp$.
16. Therefore (*per prop.* 47. *Elem.* 1.) . . ⟩ $\square AI + \square IE = \square AE$.
17. Likewiſe, ... ⟩ $\square AK + \square KE = \square AE$.
18. Therefore from 16° and 17°, (*per Ax.* 1. *Chap.* 2.) . . . . . ⟩ $\square AI + \square IE = \square AK + \square KE$.
19. By *Suppoſ.* in 12°, IE = KE, and conſequently ⟩ . . . . . $\square IE = \square KE$.
20. Therefore from 18° and 19°, (*per Ax.* 9. *Chap.* 2.) . . . . . $\square AI = \square AK$.
21. Therefore from 20°, (*per Schol. prop.* 48. *Elem.* 1.) . . . . . $AI = AK$.

Which was to be Demonſtr.

22. Again, becauſe AE is common to the Triangles AIE and AKE, and the other ſides of thoſe Triangles are correſpondently equal to one another, (as appears in 12° and 21°,) therefore (*per prop.* 8. *Elem.* 1.) . . . $\leqslant EAI = \leqslant EAK$.

Which was alſo to be Demonſtr.

And by the like argumentation the truth of the reſt of the Equations in 13° and 14° may be demonſtrated.

LEMMA 2.

Let the ſame things be ſuppoſed here as before in *Lemma* 1. then, (reſpect being had to *Fig.* 1.)

Req.

1. . . . *Req. demonſtr.* As Radius is to the ſumm of the Tangents of $\left\{\begin{array}{l} \leqslant AEI, \\ \leqslant BFL, \\ \leqslant CGN, \\ \leqslant DHP; \end{array}\right.$ So EI (or EK) to AI + LB + NC + PD.

Demonſtration.

2. By *Suppoſition* in 12° of *Lemma* 1. . . . EI = FL = GN = HP.
3. Therefore theſe four following Analogies will be manifeſt by a vulgar Axiom in the Doctrine of plain Triangles, *viz.*

Radius . EI :: Tangent $\leqslant$ AEI . AI.
Radius . FL, or EI :: Tangent $\leqslant$ BFL . LB.
Radius . GN, or EI :: Tangent $\leqslant$ CGN . NC.
Radius . HP, or EI :: Tangent $\leqslant$ DHP . PD.

4. And from thoſe four Analogies this that follows is deducible, (*per Schol. prop.* 12. *Elem.* 5.) *viz.*

As . . . . . 4 Rad.
To . . . . . 4 EI;
So is the ſumm of the Tangents of theſe angles, to wit, $\left\{\begin{array}{l} \leqslant AEI, \\ \leqslant BFL, \\ \leqslant CGN, \\ \leqslant DHP; \end{array}\right.$
To . . . . . AI + LB + NC + PD.

5. But (by *prop.* 15. *Elem.* 5.) . . . 4 Rad. 4 EI :: Rad. EI.
6. Therefore from the two laſt preceding Analogies in 4° and 5° this ariſeth, (*per prop.* 11. *Elem.* 5.) *viz.*

As . . . . . Rad.
To . . . . . EI;
So is the ſumm of the Tangents of . . . . . $\left\{\begin{array}{l} \leqslant AEI, \\ \leqslant BFI, \\ \leqslant CGN, \\ \leqslant DHP; \end{array}\right.$
To . . . . . AI + LB + NC + PD.

7. Therefore alternately, As Radius is to the ſumm of the Tangents of $\left\{\begin{array}{l} \leqslant AEI, \\ \leqslant BFL, \\ \leqslant CGN, \\ \leqslant DHP; \end{array}\right.$ So EI to AI + LB + NC + PD.

Which was to be Demonſtr.

LEMMA 3.

Let the ſame things be ſuppoſed here as before in *Lemma* 1. then (reſpect being had to *Fig.* 1.)

Req. demonſtr.

1. $\left\{\begin{array}{l} \square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD} \end{array}\right\}$ = Space AEFBCGHDA.

That is, in words,

The Rectangle made of the parallel diſtance EI, (= EK,) and the exceſs by which the ſumm of the four ſides AD, AB, BC, CD of the Trapezium ABCD, exceeds the ſumm of the four ſegments AI, LB, NC, PD, is equal to the Interval or Space AEFBCGHDA.

Demonſtration.

2. By *Suppoſ.* in 8° and 12° of *Lemma* 1. ⟩ EI and HQ ⊥ IQ. Alſo, EI = HQ.
3. Therefore (*per prop.* 27, 33, 34. *Elem.* 1.) ⟩ EH = IQ.
4. It is evident by *Fig.* and *Lemma* 1. reſpect being had to the laſt Equation, that . . . IQ = EH = AD − AI − QD (PD.)
5. And by adding AD to each part of the laſt Equation, ... AD + EH = 2AD − AI − QD (PD.)
6. And by arguing as in 4° and 5°, this Equation will be manifeſt, *viz.* . . . . AB + EF = 2AB − AK (AI) − LB.

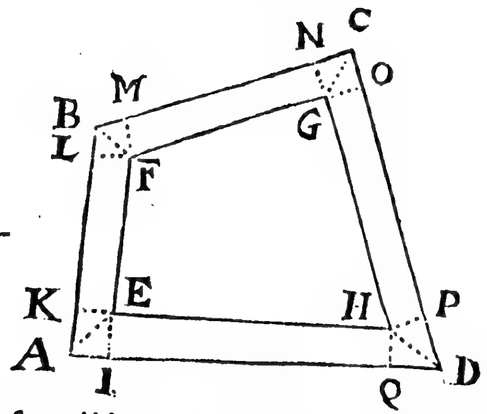
7. Likewise, ... > $BC + FG = 2BC - BM(LB) - NC$.
8. Likewise, ... > $CD + GH = 2CD - CO(NC) - PD(QD.)$
9. And by comparing the half summ of the first parts of the four last preceding Equations to the half summ of the latter parts, this Equation ariseth, *viz.* . $\left.\begin{matrix}\frac{1}{2}AD + \frac{1}{2}EH + \\ \frac{1}{2}AB + \frac{1}{2}EF + \\ \frac{1}{2}BC + \frac{1}{2}FG + \\ \frac{1}{2}CD + \frac{1}{2}GH\end{matrix}\right\} = \left\{\begin{matrix}AD - AI + \\ AB - LB + \\ BC - NC + \\ CD - PD.\end{matrix}\right.$
10. By *Suppos.* the Trapezium AEHD hath two parallel sides AD and EH, and EI $\perp$ AD, therefore (*per Theor.* 2. in 13° of *Probl.* 18. *Chap.* 7.) . . $\square EI \times \overline{\frac{1}{2}AD + \frac{1}{2}EH} = $ Trapez. AEHD.
11. In like manner, (respect being had to the *Suppos.* in 12° of *Lemma* 1.) . $\square EI \times \overline{\frac{1}{2}AB + \frac{1}{2}EF} = $ Trapez. AEFB.
12. Likewise, ... > $\square EI \times \overline{\frac{1}{2}BC + \frac{1}{2}FG} = $ Trapez. BFGC.
13. Likewise, ... > $\square EI \times \overline{\frac{1}{2}CD + \frac{1}{2}GH} = $ Trapez. DHGC.
14. By viewing *Fig.* 1. it will be evident, that the summ of the four Trapezia expresst in the four last preceding Equations is equal to the Interval or Space AEFBCGHDA, therefore from 9°, 10°, 11°, 12° and 13°, by exchanging equal quantities this Equation ariseth, *viz.*

$$\left.\begin{matrix}\square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD}\end{matrix}\right\} = \text{Space AEFBCGHDA.}$$

Which was to be Demonstr.

LEMMA 4.

FIG. 1.



G

Let the same things be suppos'd here as in *Lemma* 1. and let the line G be found out by this Analogy, *viz.*

1. As the summ of the Tangents of . . . . . $\left\{\begin{matrix}\lessdot AEI, \\ \lessdot BFL, \\ \lessdot CGN, \\ \lessdot DHP,\end{matrix}\right.$
   To . . . . . Radius;
   So . . . . . AB + BC + CD + DA,
   To a fourth Proportional . . . . . G. Then
2. . . . *Req. demonstr.* . . . . . $\frac{1}{2}G \sqsubset EI$, (the parallel distance.)

Demonstration.

3. By inverting the Terms of the Analogy demonstrated in 7° of the foregoing *Lemma* 2. these are Proportionals, *viz.*

4. As the summ of the Tangents of . . . . . $\left\{\begin{matrix}\lessdot AEI, \\ \lessdot BFL, \\ \lessdot CGN, \\ \lessdot DHP,\end{matrix}\right.$
   To . . . . . Radius;
   So is . . . . . AI + LB + NC + PD,
   To the parallel distance . . . . . EI.

5. Therefore

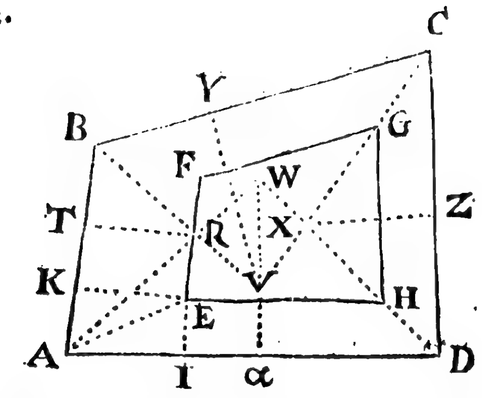
5. Therefore from 3° and 4°, (*per prop.* 11. *Elem.* 5.)
   AB + BC + CD + DA . G :: AI + LB + NC + PD . EI.
6. But by *prop.* 15. *Elem.* 5.
   AB + BC + CD + DA . G :: $\frac{1}{2}$AB + $\frac{1}{2}$BC + $\frac{1}{2}$CD + $\frac{1}{2}$DA . $\frac{1}{2}$G.
7. Therefore from 5° and 6°, (*per prop.* 11. *Elem.* 5.)
   $\frac{1}{2}$AB + $\frac{1}{2}$BC + $\frac{1}{2}$CD + $\frac{1}{2}$DA . $\frac{1}{2}$G :: AI + LB + NC + PD . EI.
8. In which last Analogy the first Term is greater than the third; for 'tis evident by *Fig.* 1. and by what hath been demonstrated in *Lemma* 1. that the double of the first Term is greater than the double of the third, in regard the double of the third is but part of the double of the first: Therefore (*per prop.* 14. *Elem.* 5.) the second Term is greater than the fourth, *viz.*

   $\frac{1}{2}G \sqsubset EI$ (the parallel distance.)

Which was to be Demonstr.

LEMMA 5.

FIG. 2.



If the angles A, B, C, D of the Trapezium ABCD be severally divided into two equal parts by the right lines AW, DW, BV, CV; and if from the points W, R, V and X where every two of the said lines that lye next to one another do intersect, four right lines Wα, RT, VY and XZ be drawn perpendicularly upon AD, AB, BC and CD; and if the Perpendicular RT be shorter, or, if not shorter, yet not longer than any one of the other three Perpendiculars VY, XZ and Wα: And lastly, if a Trapezium be made within the before-mentioned Trapezium ABCD, so, as that the sides of the one are parallel to the sides of the other, and every where separated by an equal parallel distance: Then I say that the said parallel distance shall be less than the said Perpendicular RT.

For if it be said that the parallel distance is equal to the Perpendicular RT, then by the point R let ERF be drawn parallel to AB, and finish the Trapezium EFGH so as FG may be parallel to BC, likewise GH$\parallel$CD and EH$\parallel$AD; draw also AE: Now if the sides of the interiour Trapezium EFGH be every where distant from the sides of the exteriour Trapezium by an equal parallel distance, then by *Lemma* 1. the angle EAI is equal to the angle EAK; but by *Supposition* in this *Lemma* 5. $\lessdot$RAI = $\lessdot$RAK, and how short soever the parallel line ERF be drawn, the angle EAI will be but part of, and consequently less than $\lessdot$RAI, ($=\lessdot$RAK,) as is evident by *Fig.* 2. therefore the angle EAI cannot be equal to the angle EAK, and consequently it contradicts *Lemma* 1. before demonstrated. The like contradiction will ensue if a Parallel be drawn to AB at a distance greater than RT. Wherefore I conclude, that if a Trapezium be made within a Trapezium in such manner as is above supposed, the parallel distance shall be less than RT, which is suppos'd to be shorter, or, if not shorter, yet not longer than any one of the three Perpendiculars VY, XZ and Wα before mentioned.

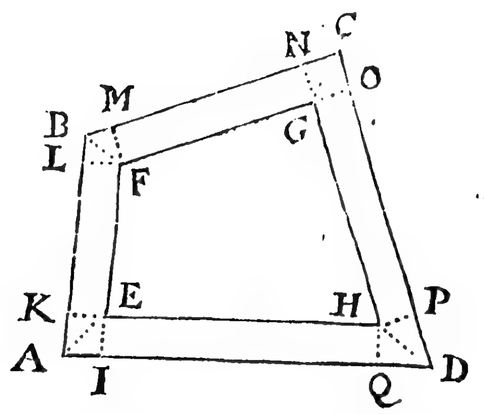
Probl. XXI.

A Trapezium being given by Position, as also the quantities of all its sides and angles severally, to make a Trapezium within the former, in such manner that the sides of the one may be every where separated

from the ſides of the other by an equal parallel diſtance; and that the Space lying between the ſides of both the Trapezia may be equal to any poſſible Space (or Figure) given.

Note. This Problem hath two Caſes, *viz.* Firſt, when each of the Diagonals of the given Trapezium lyes within the ſame; Secondly, when one of the Diagonals lyes without. I ſhall handle the firſt Caſe only, for this well underſtood, will be a ſufficient light to ſhew the induſtrious Analyſt how to ſolve the latter Caſe, which is briefly done by the Learned *Fran. à Schooten*, in his *Tractat. de concinnandis Demonſtrationibus ex Calculo Algebraico.*

FIG. 1.



Suppoſ.

1. ABCD is a Trapezium given, whoſe Diagonals AC and BD do lye within the ſame.
2. c = a right line given, equal to the ſumm of AB, BC, CD, DA.
3. r = the Radius, or Semidiameter of a Circle is given.
4. s = a right line given equal to the ſumm of the Tangents of the angles AEI, BFL, CGN and DHP, agrecable to the Radius r; which angles are the complements of the halves of the given angles of the Trapezium ABCD.
5. b = the ſide of a given Square equal to the Interval or Space lying between the ſides of the given Trapezium ABCD, and the ſides of the Trapezium required to be made within the former.

Req. to make

6. EFGH a Trapezium, ſuch, that EF ∥ AB; FG ∥ BC; GH ∥ CD, and EH ∥ AD. Alſo,
7. Space AEFBCGHDAE = □b, (or bb.) Alſo,
8. The Perpendiculars EI, FL, GN and HP to be equal between themſelves, that there may be an equal parallel diſtance between the ſides of the Trapezium given, and of that required.

Reſolution of CASE 1.

9. Suppoſe that done which is required, and put a for the parallel diſtance, *viz.* . . . . . $a = EI = EK = FM = GO$.
10. Then by the foregoing *Lemma* 2. this Analogy is manifeſt, *viz.*

As Radius, to the ſumm of the Tangents of . . . . $\left\{\begin{matrix} \lessgtr AEI, \\ \lessgtr BFL, \\ \lessgtr CGN, \\ \lessgtr DHP; \end{matrix}\right.$

So is the parallel diſtance EI, to . . AI + LB + NC + PD.

11. Therefore in the letters of the Reſolution,

$$r \cdot s :: a \cdot \frac{sa}{r} \quad (= AI + LB + NC + PD.)$$

12. By the preceding *Lemma* 3. this Equation is manifeſt, *viz.*

$$\left.\begin{matrix} \square EI \times \overline{AD - AI} + \square EI \times \overline{AB - LB} + \\ \square EI \times \overline{BC - NC} + \square EI \times \overline{CD - PD} \end{matrix}\right\} = \text{Space AEFBCGHDA.}$$

13. Therefore in the letters of the Reſolution,

$$a \times c - \frac{sa}{r} = bb \quad (= \text{Space AEFBCGHDA.}$$

14. Which

14. Which Equation in 13° may be reſolved into theſe Proportionals, *viz.* . . . . . $b \cdot c - \frac{sa}{r} :: a \cdot b.$
15. And by drawing r as a common Factor into each of the two firſt Terms of the laſt preceding Analogy, this ariſeth, *viz.* . . . . . $br \cdot cr - sa :: a \cdot b.$
16. Now to avoid an Equation between Solids, which would ariſe by comparing the Rectangle of the extremes to the Rectangle of the means of the laſt Analogy, let it be made, as s to r, ſo b to a fourth Proportional, which may be called d, therefore . . . $s \cdot r :: b \cdot d.$
17. Whence, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation is produced, *viz.* . . . . . $sd = br.$
18. Again, let it be made, as s to r, ſo c to a fourth Proportional, call it g, therefore . . . . . $s \cdot r :: c \cdot g.$
19. Therefore from the laſt Analogy, by comparing the Rectangle of the extremes to the Rectangle of the means, ... $sg = cr.$
20. Therefore from 15°, 17° and 19°, by exchanging equal Rectangles, this Analogy ariſeth, *viz.* . . $sd \cdot sg - sa :: a \cdot b.$
21. Therefore from 20°, by caſting the common Factor s out of the two firſt Terms, ... $d \cdot g - a :: a \cdot b.$
22. From which laſt Analogy, by comparing the Rectangle of the means to the Rectangle of the extremes, this Equation is produced, *viz.* . . . . . $ga - aa = bd.$
23. Which Equation may be reſolved into theſe Proportionals, *viz.* . . . . . $g - a \cdot \sqrt{bd} :: \sqrt{bd} \cdot a.$
24. But of thoſe three continual Proportionals, the mean, to wit, $\sqrt{bd}$ is given, as alſo g the ſumm of the extremes $g - a$ and a, therefore (*per Probl.* 13. *Chap.* 5.) the extremes ſhall be given ſeverally, which by the Theorem in 21° of the ſaid *Probl.* 13. are equal to theſe right lines, (or numbers,) *viz.*

$$\left.\begin{matrix} \frac{1}{2}g + \sqrt{:\frac{1}{4}gg - bd:} \\ \frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} \end{matrix}\right\} = \text{the extreme Proportionals in 23°.}$$

Which extreme Proportionals, (or Roots of the Equation in 22°,) are equal one to the other when $\frac{1}{2}g = \sqrt{bd}$, in which Caſe each of the ſaid extremes is evidently equal to $\frac{1}{2}g$; but if $\frac{1}{2}g \sqsubset \sqrt{bd}$, then the ſaid extremes are unequal, as they happen to be in the Reſolution of this Problem, but the leſſer of them only (for the reaſon hereafter given in 26°) ſhall be the parallel diſtance ſought. Hence this

CANON.

25. $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} = EI = EK$ the parallel diſtance. That is, in words,

Let it be made, As (s) the ſumm of the Tangents of the complements of the halves of the angles of the given Trapezium ABCD, is to the Radius (r); So (b) the given ſide of a Square equal to the preſcribed Interval or Space between the ſides of both the Trapezia, to a fourth Proportional (d); and ſo (c) the ſumm of the four ſides of the given Trapezium to a fourth Proportional (g). Then ſubtract the ſquare Root of the exceſs whereby the Square of half g exceeds the Rectangle made of b and d, from half g; the remainder ſhall be EI (= EK) the parallel diſtance ſought.

Which Canon may be propounded in the form of a Theorem, the Demonſtration whereof may be eaſily framed by a repetition of the ſteps of the foregoing Reſolution, by proceeding in a direct order from the beginning to the end thereof; for the Argumentation therein uſed is clearly Geometrical, as well as Arithmetical: But waving the Demonſtration of the Canon, I ſhall in the next place ſhew what Determinations are neceſſary for limiting the given lines, that there may be a poſſibility of effecting the Problem propounded.

Determinat. 1.

26. . . . $\frac{1}{2}g \sqsubset \sqrt{bd}$; that is, $\frac{cr}{2s} \sqsubset \sqrt{:b \times \frac{br}{s}:}$

Although

Although there be a poſſibility to find out the extreme Proportionals in 24°, (or the two Roots of the Equation in 22°,) if $\frac{1}{2}g$ be not leſs than $\sqrt{bd}$, yet the parallel diſtance is not poſſible unleſs $\frac{1}{2}g$ be greater than $\sqrt{bd}$; for if $\frac{1}{2}g = \sqrt{bd}$, then each of the ſaid extreme Proportionals is equal to $\frac{1}{2}g$; but by the preceding *Lemma* 4. $\frac{1}{2}g$ is greater than EI or EK (the parallel diſtance,) therefore neither of thoſe equal extremes can be the parallel diſtance: But if $\frac{1}{2}g \sqsubset \sqrt{bd}$, as *Determinat.* 1. requires, then the two extreme Proportionals before mentioned are unequal; for if $\frac{1}{2}g$, that is, half the ſumm of the extremes of three Proportionals be greater than the mean, to wit, $\sqrt{bd}$, then the extremes are unequal, and conſequently in ſuch Caſe $\frac{1}{2}g$ is greater than the leſſer extreme; therefore agreeable to *Lemma* 4. the leſſer extreme ſhall be the parallel diſtance required; but the greater extreme is to be neglected, becauſe 'tis greater than $\frac{1}{2}g$, and conſequently doth contradict the ſaid *Lemma* 4.

Determinat. 2.

27. . . . . $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:} \sqsupset$ RT.

The perpendicular line RT in *Fig.* 2. prefix'd before the precedent *Lemma* 5. is ſuppoſed to be ſhorter, or if not ſhorter, yet not longer than any one of the other three Perpendiculars VY, XZ and W*a*. But the quantities of the ſaid four Perpendiculars may be found out in numbers by the Doctrine of Plain Triangles; for in each of the Triangles ARB, BVC, CXD and AWD, the Baſe is given, as alſo the angles at the Baſe, to find the Perpendicular. Which Baſes are the given ſides of the Trapezium ABCD, and the angles at the Baſes are the halves of the given angles of the ſaid Trapezium, therefore the ſaid Perpendiculars are given alſo, whence it may be known which of them is the ſhorteſt. Now becauſe by the foregoing Canon in 25°, and by what hath been ſaid in 26°, the parallel diſtance is found equal to $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:}$: and becauſe by the preceding *Lemma* 5. the ſaid parallel diſtance muſt be leſs than RT, which is ſuppos'd to be the ſhorteſt of the ſaid four Perpendiculars, or if not the ſhorteſt, yet not longer than any one of the other three, therefore $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:}$ muſt be leſs than RT; otherwiſe that cannot be done which the Problem requires.

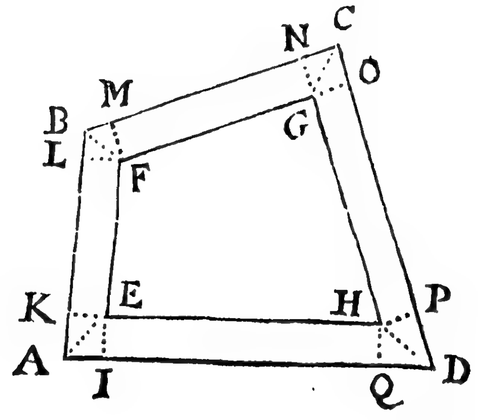
But if the given lines be qualified according to the import of the precedent Determinations, then the parallel diſtance EI or EK ſhall be given by the foregoing Canon; and then, after the angles A, B, C, D of the given Trapezium ABCD in *Fig.* 1. are ſeverally divided into two equal parts by the right lines AE, BF, CG and DH; and after right lines are drawn parallel to the ſides AB, BC, CD, DA, at the diſtance of the ſaid EI or EK; the points where thoſe Parallels do concurr in the lines biſecting the ſaid angles will form the deſired Trapezium EFGH: And it will not be difficult to demonſtrate, by a repetition of the ſteps of the foregoing Reſolution in a backward order, (in like manner as in divers preceding Problems of this Chapter,) that the Area of the Interval or Space lying between the ſides of both the Trapezia is equal to the Square of the given ſide *b*. But leaving the Compoſition of this Problem to the Learners practice, I ſhall prove the truth of its Solution by an Example in Numbers.

An Example in Numbers, to illuſtrate the preceding Reſolution of Probl. 21.

Suppoſ.

28. ABCD is a Trapezium.
29. EFGH is a Trapezium.
30. { EH ‖ AD . EF ‖ AB. / FG ‖ BC . GH ‖ CD.
31. AE, BF, CG, DH are right lines.
32. EI and HQ ⊥ AD.
33. EK and FL ⊥ AB.
34. FM and GN ⊥ BC.
35. GO and HP ⊥ CD.

36. AD = 40
37. AB = 24
38. BC = 29
39. CD = 36
} are given.



40. ∠EAI

			Gr.	Min.	
40.	∠EAI	= ∠EAK	= 44	: 6.	} Are given.
41.	∠FBL	= ∠FBM	= 55	: 56.	
42.	∠GCN	= ∠GCO	= 44	: 43.	
43.	∠HDP	= ∠HDQ	= 35	: 15.	
44.	∠AEI	= ∠AEK	= 45	: 54.	} Theſe are conſequently given, becauſe the angles at I, K, L, M, N, O, P, Q are right angles.
45.	∠BFL	= ∠BFM	= 34	: 4.	
46.	∠CGN	= ∠CGO	= 45	: 17.	
47.	∠DHP	= ∠DHQ	= 54	: 44.	

48. c = 129 = AD + AB + BC + CD is given; (from 36°, 37°, 38°, 39°.)
49. bb = $45^{c}\frac{17}{1000}$ is given for the Area of the Space AEFBCGHDA.
50. r = 100000 the Radius of a Circle is given.

Req. to find

51. EI = FL = GN = HP, (the parallel diſtance.)

Solution Arithmetical.

			Gr.	Min.	
52.	103192	= Tangent of	45	: 54	= ∠AEI = ∠AEK.
53.	67620	= Tangent of	34	: 04	= ∠BFL = ∠BFM.
54.	100994	= Tangent of	45	: 17	= ∠CGN = ∠CGO.
55.	141409	= Tangent of	54	: 44	= ∠DHP = ∠DHQ.

56. 413215 = s = the ſumm of thoſe four Tangents.

Then according to the foregoing Canon in 25°, let it be made,

57. As { s : r :: b : d. / 413215 . 100000 :: $\sqrt{:45^{0}\frac{17}{1000}}$. $5\frac{111}{1000}$, *&c.*

58. As { s : r :: c : g. / 413215 . 100000 :: 129 . $31\frac{216}{1000}$, *&c.*

. Laſtly, by the precedent Canon in 25°,

59. $\frac{1}{2}g - \sqrt{:\frac{1}{4}gg - bd:}$ = 4 (very near) = EI = EK.

The Proof.

To r, s and EI find a fourth Proportional, (agreeable to the foregoing *Lemma* 2.) *viz.*

60. . . { r : s :: EI : AI + LB + NC + PD. / 100000 . 413215 :: 4 . $16\frac{1216}{10000}$.

Then (by *Lemma* 3.)

EI × { AD − AI + AB − LB + / BC − NC + CD − PD } = Space AEFBCGHDA.

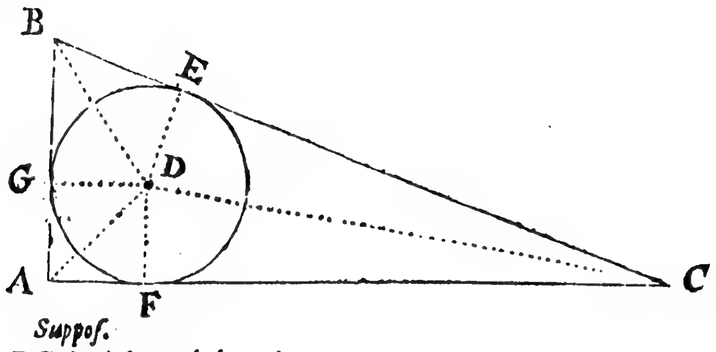
That is, 4 into $129 - 16\frac{1216}{10000} = 449\frac{111}{1000}$, *&c.*

Which Area doth not want $\frac{1}{20}$ of $450\frac{111}{1000}$ the preſcribed Area of the Interval AEFBCGHDA, the defect ariſing from this, that ſuch numbers as were found out near the truth, were aſſumed to be exactly true, to avoid tediouſneſs of Calculation. Therefore the truth of the Solution of the Problem propounded is evident.

Probl. XXII.

In a right-angled plain Triangle, the Area being given, as alſo the Perimeter, (that is, the ſumm of all the three ſides,) to find out the Triangle. But the quadruple Area muſt be leſs than the Square of the Perimeter.

Suppoſ.



AC = 156
AB = 65
BC = 169
DF = 26
DG = 26
DE = 26

Suppoſ.

1. △ ABC is right-angled at A.
2. □½AC, AB = the Area of △ ABC is given.
3. AC + AB + BC = the Perimeter is given.

Req. to find out AC, AB, BC ſeverally.

Prepar.

4. Suppoſing ABC to be the Triangle ſought, biſect (by *Probl.* 9. *Elem.* 1.) the angles BAC and ABC by the lines AD and BD meeting in D; and make DF ⊥ AC. DG ⊥ AB. DE ⊥ BC; and draw DC: Then is DF (= DG = DE) the Semidiameter of the inſcribed Circle DFGE, which toucheth the Triangle in the points F, G and E, (*per prop.* 4. *Elem.* 4.)

Reſolut. I.

5. From the premiſſes 'tis eaſie to perceive, that in any plain Triangle, the Semiperimeter multiplied by the Semidiameter of the inſcribed Circle produceth the Area of the Triangle; and conſequently the Area divided by the Semiperimeter gives the Semidiameter of the inſcribed Circle: As in △ ABC before expos'd, the Rectangle (or Product) of ½AC into DF is equal to the △ ADC, (*per prop.* 41. *Elem.* 1.) Likewiſe, □½AB, DG (DF) = △ ABD: Alſo, □½BC, DE (DF) = △ BCD; and thoſe three Triangles are evidently equal to △ ABC.
6. Moreover, in every right-angled plain Triangle, if the Diameter of the inſcribed Circle be ſubtracted from the Perimeter, the remainder is equal to the double of the Hypothenuſal: As in △ ABC before expos'd, if AF + AG, (= DF + DG = the Diameter of the inſcribed Circle DFGE,) be ſubtracted from the Perimeter AC + AB + BC, there evidently remains FC + GB + BC = 2BE + 2EC = 2BC; for FC = EC, and GB = BE. Therefore,

From 5° and 6° we may deduce

CANON 1.

7. Divide the given Area by the Semiperimeter, and ſubtract the double of the Quotient from the whole Perimeter; the half of the remainder ſhall be the Hypothenuſal; which ſubtracted from the Perimeter leaves the ſumm of the ſides about the right angle. Then the Hypothenuſal being given, as alſo the ſumm of the ſides about the right angle, the ſides ſhall be given ſeverally, both Geometrically and Arithmetically, by *Probl.* 4. *Chap.* 8.

Another way to find out the Hypothenuſal.

Suppoſ.

8. △ ABC is right-angled at A.
9. cc = □½AC, AB = △ ABC is given.
10. b = AC + AB + BC is given.

Req. to find BC the Hypothenuſal.

Reſolut. II.

11. Put a for the Hypothenuſal, *viz.* . . . . . } a = BC.
12. Therefore from 10° and 11°, the ſumm of the ſides about the right angle is . . . . . } $b - a$ (= AC + AB.)
13. And from 11° the Square of the Hypothenuſal is . } aa.
14. And from 12°, the Square of the ſumm of the ſides about the right angle is . . . . . } $aa - 2ba + bb$.

15. Therefore

15. Therefore from 9°, 13° and 14°, (by *Theor.* 1. in 12° of *Probl.* 15. *Chap.* 8.) this Equation ariſeth, *viz.* } $aa - 2ba + bb = aa + 4cc$.
16. From which Equation reduced, the Hypothenuſal will be made known, *viz.* . . . . . } $a = \frac{1}{2}b - \frac{2cc}{b}$.

Hence

CANON 2.

17. From the Semiperimeter ſubtract the Quotient of the double Area divided by the Perimeter, and the remainder ſhall be the Hypothenuſal. Then the ſides about the right angle ſhall be given as before in *Canon* 1.
18. But to the end there may be a poſſibility of finding out a right-angled Triangle to ſolve the Problem propounded, it is evident by *Canon* 2. that the given quantities b and cc muſt be ſuch, that . . . . . } $\frac{1}{2}b \sqsubset \frac{2cc}{b}$.
19. Whence, by doubling each part, ... } $b \sqsubset \frac{4cc}{b}$.
20. Therefore from 19°, by multiplying each part into b, } $bb \sqsubset 4cc$.
21. And conſequently, ... } $4cc \sqsupset bb$.

Therefore the reaſon of the Determination added to *Probl.* 22. is manifeſt; and the Canons may be exemplified by any right-angled Triangle in Rational numbers.

CHAP. X.

The fourth Claſſis *of Examples of the Reſolution and Compoſition of Plane Problems.*

IN which Examples, the Reſolution ends in an Analogy conſiſting of three Squares in continual Proportion, whereof the Mean is given; as alſo a Square equal either to the Difference, or elſe to the Summ of the Extremes; and therefore the Extremes ſhall be given ſeverally, by *Probl.* 15, or 16. of *Chap.* 5.

Probl. I.

The difference of the Squares and the Rectangle of two right lines being given ſeverally, to find out thoſe lines.

Suppoſ.

1. d = the given ſide of a Square equal to the difference of the Squares of two right lines.
2. m = the given ſide of a Square equal to the Rectangle of the ſame lines.

Req. to find out the lines.

Reſolution.

3. For the leſſer of the two right lines ſought put . } a.
4. Therefore the Square of the leſſer line is . . . } aa.
5. And from 1° and 4°, the Square of the greater line is . . . . . } $aa + dd$.
6. Therefore from 5° the greater line is . . . . } $\sqrt{:aa + dd:}$.
7. And from 3° and 6° the Rectangle made of the two right lines ſought is . . . . . } $a \times \sqrt{:aa + dd:}$.
8. Which Rectangle (according to the import of the Problem) muſt be equal to the Square of the given right line m; therefore, ... } $a \times \sqrt{:aa + dd:} = mm$.
9. And that Equation may be reſolved into theſe Proportionals, *viz.* . . . . . } $\sqrt{:aa + dd:} . m :: m . a$.

Which Analogy is qualified in every reſpect like that in 60° of *Probl.* 15. *Chap.* 5. therefore the lines ſought, which are repreſented by a and $\sqrt{:aa + dd:}$ ſhall be given by the Geometrical Conſtruction of the ſaid *Probl.* 15. and their quantities in Numbers ſhall be given alſo by the Canon in 77° of the ſame Problem.

Probl. II.

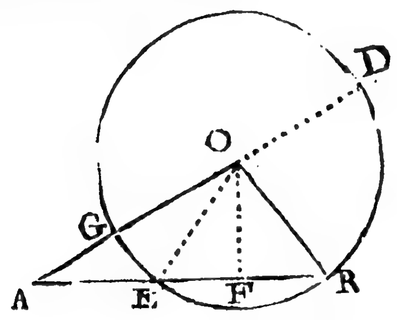
In a plain Triangle, any two ſides and the Area being ſeverally given, to find out the Triangle. But eight times the given Area muſt not exceed the exceſs by which the Square of the ſumm of the two given ſides is greater than the Square of their difference.

Note. The two given ſides of the Triangle ſought are either equal one to the other, or unequal; if equal, then the Triangle is either equicrural, or equilateral; each of which is compos'd of two equal right-angled Triangles, having a common Perpendicular falling from the angle contain'd under two equal ſides upon the middle of the third ſide, (or Baſe,) in which Caſe, the Problem propounded may be ſtated thus; *viz.* The Hypothenuſal of a right angled Triangle is given, as alſo the Area, (that is, half the given Area,) to find the ſides about the right angle; which ſides may be found out both Geometrically and Arithmetically, by *Probl.* 15. *Chap* 8. and then either of thoſe ſides about the right angle being doubled, gives the Baſe of a Triangle, either equicrural or equilateral, to ſolve the Problem. But if the two ſides given be unequal, then the Triangle ſought is either right-angled, or elſe obtuſe-angled, or hath unequal acute angles at the Baſe; if right-angled, then the third ſide or Baſe ſhall be given by *prop.* 47. *Elem.* 1. by which *Prop.* it may be diſcovered whether the Triangle ſought, when the two given ſides are unequal, and expreſt by numbers, be right-angled at the Baſe or not; for if the Square of the leſſer ſide given be ſubtracted from the Square of the greater, and the ſquare Root of the remainder be multiplied by half the ſaid leſſer ſide, and if that Product be equal to the given Area, then the Triangle ſought is right-angled at the Baſe. Now when the Triangle ſought is neither right-angled at the Baſe, nor hath equal acute angles at the Baſe, the Reſolution of the Problem propos'd may be formed in manner following.

Suppoſ.

1. ARO is a △ whoſe angles A and R are acute and unequal, alſo $\angle R \sqsubset \angle A$, and conſequently $AO \sqsubset RO$, (*per prop.* 19. *Elem.* 1.)
2. $b = AO$ is given.
3. $k = RO$ is given.
4. $c = b + k = AD$ is given.
5. $d = b - k = AG$ is given.
6. $m =$ a given right line, whoſe Square is equal to the Triangle ARO.

Req. to find out the Triangle.



AR = 21
AO = 17
RO = 10
OF = 8
AE = 9
EF = 6
FR = 6
AF = 15
AG = 7
□M = 84 = △ARO.

Reſolution.

7. Put a for the Baſe (or third ſide) ſought, *viz.* . . . . . $a = AR$.
8. Then by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. this Equation is manifeſt, *viz.*
$\sqrt{:\frac{1}{4}aa - \frac{1}{4}dd}:$ into $\sqrt{:\frac{1}{4}cc - \frac{1}{4}aa}: = mm.$
9. Which Equation may be reſolved into theſe Proportionals, *viz.*
$\sqrt{:\frac{1}{4}aa - \frac{1}{4}dd}: \; . \; m \;::\; m \; . \; \sqrt{:\frac{1}{4}cc - \frac{1}{4}aa}:$
10. And their Squares are alſo Proportionals, *viz.*
$\frac{1}{4}aa - \frac{1}{4}dd \; . \; mm \;::\; mm \; . \; \frac{1}{4}cc - \frac{1}{4}aa.$
11. Which Proportionals being ſeverally quadrupled give theſe, *viz.*
$aa - dd \; . \; 4mm \;::\; 4mm \; . \; cc - aa.$

12. And

12. And by extracting the ſquare Root out of every one of the four laſt Proportionals, theſe alſo ſhall be Proportionals, *viz.*
$\sqrt{:aa - dd}: \; . \; 2m \;::\; 2m \; . \; \sqrt{:cc - aa}:$
13. Now foraſmuch as the two laſt Analogies in 11° and 12° are in every reſpect like to thoſe in 57° of *Probl.* 16. *Chap.* 5. therefore by the Geometrical Conſtruction of that Problem, reſpect being had to what hath been deliver'd in the ſaid *Sect.* 57. two right lines may be found out, either of which may be taken for the value of a, that is, the Baſe of the Triangle ſought; which right lines are equal to one another when $8mm = cc - dd$, but unequal when $8mm$ is leſs than $cc - dd$; in which latter Caſe, two Triangles may be found out having unequal Baſes, but each Triangle ſhall have the ſame Area and leggs, as will hereafter appear in the Compoſition of this Problem. But to the end there may be a poſſibility of effecting the ſame by the help of the lines given, they muſt be ſubject to the following Determination, which is diſcovered by the Analogy in 11°.

Determination.

14. The octuple of the given Area muſt not be greater than the exceſs by which the Square of the ſumm of the two given ſides exceeds the Square of their difference.

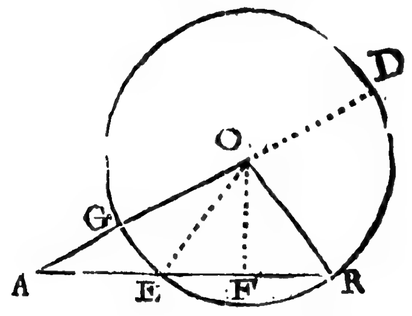
The truth of this Determination will be evident by the following

L E M M A.

15. The octuple of the Area of a plain Triangle having unequal leggs, is never greater than the exceſs by which the Square of the ſumm of the leggs exceeds the Square of their difference.

Suppoſ.

16. AR the Baſe . . . . } of △ ARO.
17. AO and OR the leggs }
18. $AO \sqsubset OR$.
19. M is a right line, ſuch, that □M = △ARO.
20. AD = AO + OR.
21. AG = AO − OR.
22. . . . *Req. demonſtr.* . . . . 8□M, or 8△ARO, not $\sqsubset$ □AD−□AG.



M ——
S ——————
T ——————

Prepar.

23. By *Probl.* 4. *Chap.* 5. find a right line S, ſuch, that its Square may be equal to □AR − □AG, therefore,
$S = \sqrt{:\square AR - \square AG}:$
24. Likewiſe by the ſame *Probl.* find a right line T, ſuch, that its Square may be equal to □AD − □AR, therefore,
$T = \sqrt{:\square AD - \square AR}:$

Demonſtration.

25. Foraſmuch as by *Theor.* 5. in 69° of *Probl.* 8. *Chap* 8. (reſpect being had to the Diagram here in view,)
□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG}:$ × $\sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR}:$ = △ARO.
26. And by *Suppoſition* in 19°, ... □M = △ARO.
27. Therefore from 25° and 26°, (*per Axiom.* 1. *Chap.* 2.)
□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG}:$ × $\sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR}:$ = □M.

28. And

28. And by reſolving the laſt Equation into Proportionals, this Analogy is manifeſt, *viz.*
$\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \;.\; M \;::\; M \;.\; \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:}$
29. The Squares of which Proportionals are alſo Proportionals, (*per prop.* 22. *Elem.* 6.) *viz.*
$\frac{1}{4}\square AR - \frac{1}{4}\square AG \;.\; \square M \;::\; \square M \;.\; \frac{1}{4}\square AD - \frac{1}{4}\square AR.$
30. And by quadrupling all in 29°,
$\square AR - \square AG \;.\; 4\square M \;::\; 4\square M \;.\; \square AD - \square AR.$
31. But the ſides of proportional Squares are alſo Proportionals, therefore from 30°,
$\sqrt{:\square AR - \square AG:} \;.\; 2M \;::\; 2M \;.\; \sqrt{:\square AD - \square AR:}$
32. And becauſe by *Conſtr.* in 23°, . . . $S = \sqrt{:\square AR - \square AG:}$
33. Alſo by *Conſtr.* in 24°, ... $T = \sqrt{:\square AD - \square AR:}$
34. Therefore from 31°, 32° and 33°, by exchanging equal right lines, this Analogy is manifeſt, *viz.*
$S \;.\; 2M \;::\; 2M \;.\; T.$
35. And becauſe the laſt Analogy conſiſts of three Proportionals, therefore by what hath been ſaid in 20° of *Probl.* 13. *Chap.* 5.
$4M \text{ not} \sqsubset S \dotplus T.$
36. And by ſquaring each part in 35°,
$16\square M \text{ not} \sqsubset \square S \dotplus \square T \dotplus 2\square S,T.$
37. And becauſe by *Conſtr.* in 23°, . . . $\square AR - \square AG = \square S.$
38. And by *Conſtr.* in 24°, ... $\square AD - \square AR = \square T.$
39. Therefore by adding the two laſt Equations together,
$\square AD - \square AG = \square S \dotplus \square T.$
40. Therefore from 36° and 38°, by exchanging equal Squares,
$16\square M \text{ not} \sqsubset \square AD - \square AG \dotplus 2\square S,T.$
41. But from 34°, (*per* 17. *prop. Elem.* 1.) . . . $4\square M = \square S,T.$
42. And conſequently, ... $8\square M = 2\square S,T.$
43. Therefore from 40° and 42°,
$16\square M \text{ not} \sqsubset \square AD - \square AG \dotplus 8\square M.$
44. Wherefore from 43°, by ſubtracting $8\square M$ from each part,
$8\square M \text{ not} \sqsubset \square AD - \square AG.$

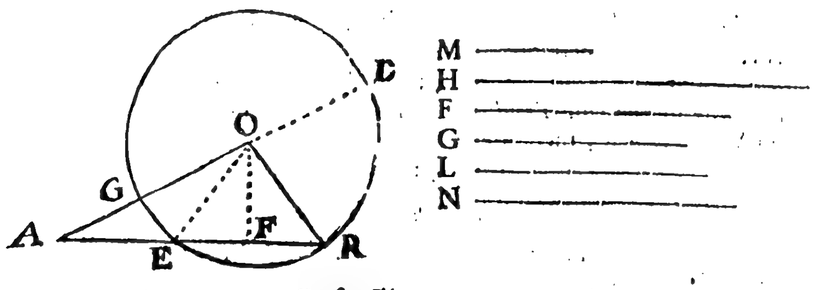
Which was to be Demonſtr. Therefore the truth both of the *Lemma* and *Determination* is manifeſt.

The Compoſition of Probl. 2.

Suppoſ.

45. AO and OR the leggs of the Triangle ARO are given ſeverally.
46. $AO \sqsubset OR.$
47. $AD = AO \dotplus OR$ is given.
48. $AG = AO - OR$ is given.
49. M = the given ſide of a Square equal to $\triangle ARO.$
50. $8\square M \text{ not} \sqsubset \square AD - \square AG.$ (*Determination.*)

Req. to find out the Triangle.



Conſtruction.

51. By *Probl.* 4. *Chap.* 5. find a right line H, ſuch, that its Square may be equal to $\square AD - \square AG$; therefore,
$\square H = \square AD - \square AG.$

52. Then

52. Then making H to be the Hypothenuſal of a right-angled Triangle, and 2M a mean Proportional between the Baſe and Perpendicular, find out (*per Probl.* 16. *Chap.* 5.) the Baſe and Perpendicular, but ſuch a Triangle cannot be found out unleſs it be proved that $\frac{4\square M}{H}$ is not greater than $\frac{1}{2}H$; for, (agreeable to the Determination annexed to the ſaid *Probl.* 16.) the right line ariſing by the Application of the Square of the given mean Proportional to the given Hypothenuſal, muſt not be greater than half the Hypothenuſal; I ſhall therefore firſt prove that $\frac{4\square M}{H}$ is not greater than $\frac{1}{2}H.$

Req. demonſtr. . . . . . $\frac{4\square M}{H}$ not $\sqsubset \frac{1}{2}H.$

Demonſtration.

By the Determination in 50°, ... $8\square M \text{ not} \sqsubset \square AD - \square AG.$
And by *Conſtr.* in 51°, ... $\ldots \square H = \square AD - \square AG.$
Therefore, (*per Ax.* 3. *Chap.* 2.) . . . . $8\square M \text{ not} \sqsubset \square H.$
And conſequently, ... $4\square M \text{ not} \sqsubset \frac{1}{2}\square H.$
Wherefore by Application of each part to H, . $\frac{4\square M}{H} \text{ not} \sqsubset \frac{1}{2}H.$

Which was to be Demonſtr.

53. Having proved that 'tis poſſible to find out a right-angled Triangle which ſhall have the right line H for a Hypothenuſal, and the double of the right line M for a mean Proportional between the Baſe and Perpendicular, ſuppoſe that (by *Probl.* 16. *Chap.* 5.) the right lines F and G are found equal to the ſaid Baſe and Perpendicular; therefore by ſuch Conſtruction,
$F \;.\; 2M \;::\; 2M \;.\; G.$
Alſo, ... $\square F \dotplus \square G = \square H.$
54. By *Probl.* 2. *Chap.* 5. find out a right line L, ſuch, that its Square may be equal to $\square AG \dotplus \square G$; therefore,
$\square L = \square AG \dotplus \square G.$
55. Likewiſe, find out a right line N, ſuch, that its Square may be equal to $\square AG \dotplus \square F$; therefore,
$\square N = \square AG \dotplus \square F.$
56. Now either of the ſaid right lines N and L, (being the two values of *a* in the foregoing Reſolution,) ſhall be the Baſe of a Triangle, to ſatisfie the Problem propounded; therefore let a Triangle, as ARO be made, ſo, that its Baſe AR may be equal to the right line N, and the leggs equal to the given right lines AO, OR; which may be done, (by *prop.* 22. *Elem.* 1.) if thoſe three right lines AR, AO and OR be ſuch that every two taken together as one right line, is longer than the third; but that the ſumm of every two of the ſaid right lines is longer than the third, I demouſtrate thus:
57. . . . I. *Req. demonſtr.* . . . . . $AR \dotplus AO \sqsubset OR.$

Demonſtration.

58. Foraſmuch as by *Suppoſition* in 46°, ... $AO \sqsubset OR.$
59. Therefore much more . . . . . $AR \dotplus AO \sqsubset OR.$
60. . . . II. *Req. demonſtr.* . . . . . $AR \dotplus OR \sqsubset AO.$

Demonſtration.

61. By *Conſtr.* in 55°, ... $\square AG \dotplus \square F = \square N.$
62. Alſo by *Conſtr.* in 56°, AR = N, and conſequently, ... $\square AR = \square N.$
63. Therefore from 61° and 62°, (*per Ax.* 1. *Chap.* 2.) . . . . . $\square AR = \square AG \dotplus \square F.$
64. Therefore from 63°, ... $\square AR \sqsubset \square AG.$
65. And conſequently, ... $AR \sqsubset AG.$
66. But by *Suppoſ* in 48°, ... $AG = AO - OR.$
67. Therefore from 65° and 66°, (*per Ax* 3. *Ch.* 2.) . . . . $AR \sqsubset AO - OR.$
68. Wherefore by adding OR to each part in 67°, . . . $AR \dotplus OR \sqsubset AO.$

Which was to Demonſtr.

69. III. *Req.*

69. . . . III. *Req. demonſtr.* . . . . . AO + OR ⌐ AR.

Demonſtration.

70. By *Conſtruction* in 51°, ... } □AD − □AG = □H.
71. Alſo by *Conſtr.* in 53°, ... } □F + □G = □H.
72. Therefore from 70° and 71°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □AD − □AG = □F+□G.
73. And from 72°, by adding □AG to each part, ... } □AD = □F+□G+□AG.
74. And from 73°, by equal ſubtraction of □G, } □AD−□G = □F+□AG.
75. But by *Conſtr.* in 55°, ... } . . . . □N = □F+□AG.
76. Therefore from 74° and 75°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □AD−□G = □N.
77. And becauſe by *Conſtr.* in 56°, AR = N, and conſequently; . . . . . } . . . □AR = □N.
78. Therefore from 76° and 77°, (*per Ax.* 1. *Chap.* 2.) . . . . . } □AD−□G = □AR.
79. And by adding the □G to each part in 78°, } □AD = □AR+□G.
80. From 79° 'tis evident, (*per Ax.* 25. *Chap.* 2.) that . . . . . } □AD ⌐ □AR.
81. And conſequently, ... } AD ⌐ AR.
82. And by *Suppoſ.* in 47°, ... } AD = AO + OR.
83. Wherefore from 81° and 82°, (*per Ax.* 4. *Chap.* 2.) . . . . . } AO + OR ⌐ AR.

Which was to be Demonſtr.

84. Having proved that of the ſaid three right lines AR, AO and OR every two taken together as one right line is longer than the third, it is poſſible to make a Triangle of them, (by *prop.* 22. *Elem.* 1.) ſuppoſe it then to be △ARO; I ſay the Triangle ſo made is that which was required. Now we muſt ſhew that it will ſatisfie the Problem propounded.

85. Firſt then, by *Conſtruction* in 84°, the leggs AO and OR are equal to the two right lines given in 45° to be the leggs of the Triangle ſought; and that the △ARO is equal to the Square of the given right line M, the following Demonſtration, formed by a retrograde repetition of the ſteps of the Reſolution aforegoing, will make manifeſt.

86. . . . *Req. demonſtr.* . . . . . △ARO = □M.

Demonſtration.

87. By *Conſtruction* in 53°, ... } F . 2M :: 2M . G.
88. Therefore, (*per* 22. *prop.* 6. *Elem.*) . . . } □F . 4□M :: 4□M . □G.
89. By *Conſtr.* in 55°, ... } □N = □AG + □F.
90. Alſo by *Conſtr.* in 56°, N = AR, and conſequently, ... } □N = □AR.
91. Therefore from 89° and 90°, (*per Ax.* 1.) } □AR = □AG + □F.
92. And from 91°, by equal ſubtraction of □AG, } □AR − □AG = □F.
93. It hath been proved in 79°, that . . . . } □AD = □AR + □G.
94. And conſequently, by ſubtracting □AR from each part, ... } □AD − □AR = □G.
95. Therefore from 88°, 92° and 94°, by exchanging equal Planes, this Analogy will be manifeſt, *viz.* □AR−□AG . 4□M :: 4□M . □AD−□AR.

That is, in 11°, $aa - dd \;.\; 4mm :: 4mm \;.\; cc - aa.$

96. And by taking a quarter of all in 95°,

$$\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG \;.\; \square M :: \square M \;.\; \tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR.$$

That is, in 10°, $\frac{1}{4}aa - \frac{1}{4}dd \;.\; mm :: mm \;.\; \frac{1}{4}cc - \frac{1}{4}aa.$

97. And becauſe (*per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are alſo Proportionals, therefore from 96°,

$$\sqrt{:\tfrac{1}{4}\square AR - \tfrac{1}{4}\square AG:} \;.\; M :: M \;.\; \sqrt{:\tfrac{1}{4}\square AD - \tfrac{1}{4}\square AR:}$$

That is, in 9°, $\sqrt{:\frac{1}{4}aa - \frac{1}{4}dd:} \;.\; m :: m \;.\; \sqrt{:\frac{1}{4}cc - \frac{1}{4}aa:}$

98. And

98. And from 97°, (*per* 17. *prop.* 6. *Elem.*)

□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:} = \square M.$

That is, in 8°,

$$\sqrt{:\tfrac{1}{4}aa - \tfrac{1}{4}dd:} \text{ into } \sqrt{:\tfrac{1}{4}cc - \tfrac{1}{4}aa:} = mm.$$

99. But by *Theor.* 5. in 69° of *Probl.* 8. *Chap.* 8. reſpect being had to the laſt preceding Diagram,

□ of $\sqrt{:\frac{1}{4}\square AR - \frac{1}{4}\square AG:} \times \sqrt{:\frac{1}{4}\square AD - \frac{1}{4}\square AR:} = \triangle ARO.$

100. Therefore from 98° and 99°, (*per Axiom.* 1. *Chap.* 2.) . . . . . } △ARO = □M.

Which was to be Demonſtr.

Therefore one Triangle is found out to ſolve the Problem; and by the like Conſtruction and Argumentation another Triangle may be made upon the right line L as a Baſe, which latter Triangle ſhall have its Area and leggs correſpondently equal to the Area and leggs of the firſt Triangle.

101. But in order to raiſe an Arithmetical Canon for the finding out of the third ſide (or Baſe) of the Triangle ſought, firſt let us ſuppoſe the given quantities to be expreſt by numbers, and then let the Analogy in 11° be here repeated, *viz.*

$$aa - dd \;.\; 4mm :: 4mm \;.\; cc - aa.$$

102. Which Analogy is reducible to this following Equation, *viz.*

$$ccaa + ddaa - aaaa = 16mmmm + ccdd.$$

103. Now if that Equation be reſolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. this following Canon will thence ariſe, whereby the Baſe of the Triangle ſought may be found out Arithmetically.

C A N O N.

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd + \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dddd - \tfrac{1}{2}ccdd - 16mmmm:}:} = a.$$

Alſo,

$$\sqrt{:\tfrac{1}{2}cc + \tfrac{1}{2}dd - \sqrt{:\tfrac{1}{4}cccc + \tfrac{1}{4}dddd - \tfrac{1}{2}ccdd - 16mmmm:}:} = a.$$

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Probl. 2.

Suppoſ.

104. $b = 17$ }
105. $k = 10$ } the leggs of a plain Triangle are given; whence,
106. $c = 27$ the ſumm of the leggs is given, and
107. $d = 7$ the difference of the leggs is given.
108. $mm = 84$ the Area of the ſame Triangle is given:

Req. to find out the Baſe or third ſide.

Solution Arithmetical.

109. By the help of the numbers given in 106°, 107°, 108°, and of the Canon in 103°, the numbers 21 and $\sqrt{337}$ will be found out, either of which may be taken for the value of *a* in the foregoing Reſolution, that is, the Baſe of a Triangle which ſhall have 17 and 10 for its leggs, and 84 for its Area: For as well from theſe three ſides 21, 17 and 10, as from theſe, $\sqrt{337}$, 17 and 10, the Area will be found 84, (by *Canon* 1. in 77° of *Probl.* 8. *Chap.* 8.) And therefore the Problem propounded is ſolved both Arithmetically and Geometrically.

Probl. III.

The Baſe, Perpendicular and Rectangle of the leggs of a plain Triangle being given ſeverally, to find out the Triangle. But the given quantities muſt be ſubject to the Determinations hereafter expreſt.

Suppoſ.

1. b = the Baſe of a Triangle is given.
2. p = the Perpendicular is given.
3. r = the ſide of a Square equal to the Rectangle of the leggs is given.

Req. to find the Triangle.

4. *Vieta*,

4. *Vieta*, in *Probl.* 1. of his Appendix to his *Appollonius Gallus*, ſhews the Conſtruction of this Problem with great facility, by the help of this Theorem, *viz.* The Rectangle of any two ſides containing an angle of a plain Triangle is equal to the Rectangle of the Perpendicular falling from that angle upon the oppoſite ſide, and the Diameter of the Circle circumſcribing the ſame Triangle. Afterwards *Marinus Ghetaldus* in his Treatiſe of Mathematical Reſolution and Compoſition ſupplies *Vieta*'s Conſtruction with Determinations: But it will be difficult for Learners to deduce the Arithmetical Solution from that Geometrical Effection; I ſhall therefore form Reſolutions and Compoſitions to ſolve this Problem both Arithmetically and Geometrically in all Caſes, which are theſe five, *viz.*

Caſe 1.	When	$rr = \frac{1}{4}bb \dot{+} pp.$
Caſe 2.	When	$rr \sqsubset \frac{1}{4}bb \dot{+} pp.$
	And	$rr \sqsubset \frac{1}{2}bb.$
Caſe 3.	When	$rr \sqsubset \frac{1}{4}bb \dot{+} pp.$
	And	$rr \sqsupset \frac{1}{2}bb$
Caſe 4.	When	$rr \sqsubset \frac{1}{4}bb + pp.$
	And	$rr = \frac{1}{2}bb.$
Caſe 5.	When	$rr \sqsupset \frac{1}{4}bb \dot{+} pp.$

Now to ſolve the propoſed Problem in every one of thoſe Caſes, I ſhall give a peculiar Reſolution, with a Canon and an Example in numbers; and if the given Quantities be expreſt by numbers, it will be eaſie to diſcern under which of thoſe Caſes they fall.

The Reſolution of CASE 1.

5. One Triangle may be always eaſily found out to ſolve the Problem in *Caſe* 1. *viz.* when $rr = \frac{1}{4}bb \dot{+} pp$; for if upon the middle of the given Baſe, the given Perpendicular be erected, and from the ends of the Baſe two right lines be drawn to meet at the top of the Perpendicular, then the equicrural Triangle ſo formed will ſatisfie the Problem. For in every Iſoſceles, or equicrural Triangle, the Square of half the Baſe, (or a quarter of the Square of the whole Baſe,) together with the Square of the Perpendicular is (*per prop.* 47. *Elem.* 1.) equal to the Square of either of the equal leggs, that is, the Rectangle of the leggs.

Moreover, when in *Caſe* 1. it happens that $b \sqsubset 2p$, then beſides that equicrural Triangle, another having unequal leggs may be found out to ſolve the Problem, by this following Reſolution, *viz.*

6. For the unknown difference of the leggs put . . . . . } a.
7. Then by *Theor.* 7. *Ch.* 4. the Square of the ſumm of the leggs is } $aa \dot{+} 4rr$.
8. Therefore from 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.)

$$aa \dot{+} 4rr - bb \;.\; aa \dot{+} 4rr - bb - 4pp :: bb \;.\; aa.$$

9. And becauſe by *Suppoſ.* in *Caſe* 1. . . . . } $rr = \frac{1}{4}bb \dot{+} pp.$
10. And conſequently, ... } $4rr = bb \dot{+} 4pp.$
11. Therefore from 8° and 10°, by exchanging equal quantities,

$$aa \dot{+} bb \dot{+} 4pp - bb \;.\; aa \dot{+} bb \dot{+} 4pp - bb - 4pp :: bb \;.\; aa \text{ . Proportionals.}$$

12. Whence, by caſting away ſuch equal quantities as deſtroy one another by reaſon of contrary ſigns $\dot{+}$ and $-$, this Analogy ariſeth,

$$aa \dot{+} 4pp \;.\; aa :: bb \;.\; aa.$$

13. Therefore from 12°, (*per prop.* 14. *Elem.* 5.) this Equation enſues, *viz.* . . . . . } $aa \dot{+} 4pp = bb.$
14. And by ſubtracting $4pp$ from each part of that Equation, . } $aa = bb - 4pp.$
15. Therefore by extracting the ſquare Root out of each part of the laſt Equation, ... } $a = \sqrt{:bb - 4pp:}$
16. And out of 7° and 14°, the Square of the ſumm of the leggs will be found equal to . . . . . } $bb - 4pp \dot{+} 4rr.$
17. And from 10° and 16°, by exchanging $4rr$ for its equivalent quantity $bb \dot{+} 4pp$, the Square of the ſumm of the leggs is alſo equal to . . . . . } $bb - 4pp \dot{+} bb \dot{+} 4pp.$

18. Which

18. Which laſt quantity being firſt contracted, and then the ſquare Root extracted, the ſumm of the leggs will be made known, to wit, . . . } $\sqrt{2bb}$.

The preceding 15th and 18th ſteps do afford this

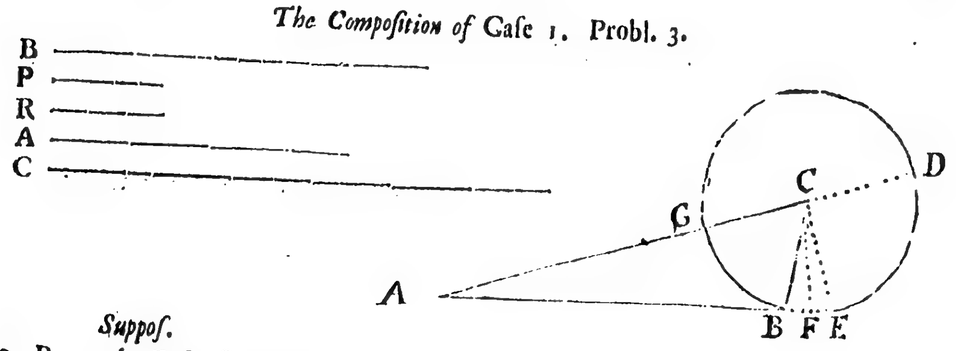
CANON.

19. . . { $\sqrt{:bb - 4pp:}$ = the Difference; $\sqrt{2bb}$ = the Summ } of the Leggs ſought.

Then the ſumm and difference of the leggs ſought being given, the leggs ſhall be given ſeverally, (*per Theor.* 9. *Chap.* 4.) and conſequently the Triangle.

By which Canon it appears, that the Triangle ſought in *Caſe* 1. cannot have unequal leggs unleſs the given Baſe exceeds the double of the given Perpendicular: For by the Canon, the difference of the leggs is equal to $\sqrt{:bb - 4pp:}$: which cannot be a real Quantity, (to wit, greater than nothing,) unleſs $b \sqsubset 2p$.

The Compoſition of Caſe 1. Probl. 3.



Suppoſ.

20. B = the Baſe of a Triangle is given.
21. P = the Perpendicular is given.
22. R = the ſide of a Square equal to the Rectangle of the leggs is given.
23. $\square R = \frac{1}{4}\square B \dot{+} \square P$. Alſo, $B \sqsubset 2P$.

Req. to find the Triangle.

Conſtruction.

24. By *Probl.* 4. *Chap.* 5. find a right line A, ſuch, that its Square may be equal to $\square B - 4\square P$: Which Effection is poſſible, for by *Suppoſ.* in 23°, $B \sqsubset 2P$; therefore,

$$A = \sqrt{:\square B - 4\square P:} \text{ (that is, } a = \sqrt{:bb - 4pp:}\text{)}$$

25. Again, by *Probl.* 2. *Chap.* 5. find a right line C, ſuch, that its Square may be equal unto $2\square B$, therefore, $C = \sqrt{2}\square B \; (= \sqrt{2}bb.)$

Thus far the Conſtruction hath been made according to the direction of the Canon in 19°.

26. Now let a Triangle be made of theſe three right lines, to wit, B, $\frac{1}{2}C \dot{+} \frac{1}{2}A$, and $\frac{1}{2}C - \frac{1}{2}A$; which Effection is poſſible, (*per prop.* 22. *Elem.* 1.) if $C \sqsubset A$, and the ſumm of every two of thoſe lines be greater than the third: But that thoſe lines are ſo qualified, I prove thus;

Firſt, from the Conſtruction in 24° and 25° it is eaſie to inferr that $C \sqsubset A$, and conſequently $\frac{1}{2}C - \frac{1}{2}A$ is equal to ſome real right line.

Secondly, it is manifeſt that the ſumm of the right lines B and $\frac{1}{2}C \dot{+} \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C - \frac{1}{2}A$.

Thirdly, that the ſumm of the two lines $\frac{1}{2}C \dot{+} \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line B, *viz.* that $C \sqsubset B$, is evident by *Conſtr.* in 25°.

Fourthly, that the ſumm of the two lines B and $\frac{1}{2}C - \frac{1}{2}A$ is greater than the third line $\frac{1}{2}C \dot{+} \frac{1}{2}A$, *viz.* that $B \sqsubset A$, I prove thus;

By *Conſtr.* in 24°, ...	$\square B - 4\square P = \square A.$
Therefore by adding $4\square P$ to each part, . . .	$\square B = \square A \dot{+} 4\square P.$
Therefore, (*per Ax.* 25. *Chap.* 2.)	$\square B \sqsubset \square A.$
Therefore, ...	$B \sqsubset A.$

Which was to be demonſtrated in the fourth and laſt place.

Having proved that $\frac{1}{2}C - \frac{1}{2}A$ is a real right line, and that the ſumm of every two of theſe three right lines, to wit, B, $\frac{1}{2}C \dot{+} \frac{1}{2}A$ and $\frac{1}{2}C - \frac{1}{2}A$, is greater than the third,

'tis

'tis poſſible to make a Triangle of thoſe three lines; *(per prop. 22. Elem 1.)* Suppoſe then it be made, and that the Triangle found out is A B C, having its Baſe AB = B, (the Baſe preſcribed in the Problem,) and the greater legg AC = ½C + ½A, and the leſſer legg BC = ½C − ½A: I ſay the Triangle A B C will ſatisfie the Problem propounded. But to render the Demonſtration thereof the more clear and eaſie, I ſhall premiſe a few things in ſeven ſteps next following.

27. If the given Quantities be expreſt by numbers, the kind of the Triangle ſought in *Caſe* 1. when the leggs are unequal, as they were ſuppoſed in the foregoing Reſolution, may be diſcovered by the help of the Canon in 19° of this Problem, and of *Theor.* 3. in 35°. of *Probl.* 8. *Chap* 8. Suppoſing then it be diſcovered that the Perpendicular falls without the Triangle A B C upon A B increaſed, from C as a Center at the diſtance of C B (= ½C − ½A) deſcribe the Circle C B G D cutting C A in G; then produce A C and A B to the Circumference in D and E; draw alſo the Semidiameter C E, and from the Center C make CF ⊥ BE, whence FE = FB, *(per prop. 3. Elem. 3.)*

28. Then becauſe *(per Defin. 15. Elem 1.)* . . . CB = CD.
29. It follows, (by adding C A to each part,) that . . . CA + CB = CD + CA = AD.
30. And becauſe by *Conſtr* in 26°, ... ½C + ½A = CA.
31. Alſo by *Conſtr.* in 26°, ... ½C − ½A = CB = CD = CG.
32. Therefore the ſumm of the Equations in 30° and 31° gives . . . . . C = CA + CB = AD.
33. And by ſubtracting the Equation in 31° from that in 30°, ... A = CA − CB = AG.
34. Now I ſhall ſhew that the Triangle A B C made as before will ſatisfie the Problem. Firſt then by *Conſtr.* in 26° the Baſe A B is equal to the preſcribed Baſe B. Secondly, that the Rectangle of the leggs A C and B C is equal to the Square of the given line R, (to wit, the preſcribed Rectangle,) I ſhall here demonſtrate.

35. . . . *Req. demonſtr.* . . . . . □AC,BC = □R.

Demonſtration.

36. By *Conſtr.* in 24° and 26°, ... □A = □AB − 4□P.
37. And from 33°, ... □A = □AG.
38. Therefore from 36° and 37°, *(per Axiom. 1. Chap. 2.)* . . . . . □AG = □AB − 4□P.
39. And by adding 4□P to each part in 38°, . . 4□P + □AG = □AB.
40. And by adding □ AB to each part of the laſt Equation, ... □AB + 4□P + □AG = 2□AB.
41. But from 32°, 25° and 26°, ... □AD = 2□AB.
42. Therefore from 40° and 41°, *(per Axiom. 1. Chap. 2.)* . . . . . □AB + 4□P + □AG = □AD.
43. And by ſubtracting □ AG from each part in 42°, ... □AB + 4□P = □AD − □AG.
44. From 32° and 33°, AD is the ſumm, and AG the difference of CA and CB, therefore, *(per Theor. 7. Chap. 4.* . . . . . 4□AC,BC = □AD − □AG.
45. Therefore from 43° and 44°, *(per Axiom. 1. Chap. 2.)* . . . . . □AB + 4□P = 4□AC,BC.
46. But from 23° and 26°, ... □AB + 4□P = 4□R.
47. Therefore from 45° and 46°, *(per Ax. 1.)* 4□AC,BC = 4□R.
48. Therefore, *(per Ax. 21. Chap. 2.)* . . . . □AC,BC = □R.

Which was to be Demonſtr.

Thirdly and laſtly, that the Perpendicular C F is equal to the given Perpendicular P, I ſhall here demonſtrate by a retrograde repetition of the ſteps of the preceding Reſolution.

49. . . . *Req. demonſtr.* . . . . . CF = P.

Demonſtration.

50. By *Conſtr.* in 24°, ... $A = \sqrt{:\square B - 4\square P:}$
51. And it hath been ſhewn in 33°, that . . . A = AG.

52. There.

52. Therefore from 50° and 51°, *(per Axiom. 1. Chap. 2.)* . . . . . $AG = \sqrt{:\square B - 4\square P:}$
53. And becauſe by *Conſtr.* in 26° AB = B, therefore from 52°, ... $AG = \sqrt{:\square AB - 4\square P:}$
That is, in 15°, ... $\cdot\cdot a = \sqrt{:bb - 4pp:}$
54. And from 53°, by comparing the Squares of each part one to another, ... $\square AG = \square AB - 4\square P.$
That is, in 14°, ... $\cdot\ aa = bb - 4pp.$
55. And by adding 4 □P to each part of the Equation in 54°, ... $\square AG + 4\square P = \square AB.$
That is, in 13, ... $\cdot\cdot aa + 4pp = bb.$

56. And from 55°, *(per prop. 7. Elem. 5.)* this Analogy ariſeth, *viz.*

$$\square AG + 4\square P \ . \ \square AG :: \square AB \ . \ \square AG.$$

That is, in 12°,

$$aa + 4pp \ . \ aa :: bb \ . \ aa.$$

57. It is manifeſt that the Analogy in 56° is equal to this that follows, (for □ AB − □ AB = 0. Alſo, 4□P − 4□P = 0,) *viz.*

$$\left.\begin{array}{r} \square AG + \square AB + 4\square P - \square AB \ . \\ \square AG + \square AB + 4\square P - \square AB - 4\square P :: \\ \square AB \ . \\ \square AG \ . \end{array}\right\} \text{Proportionals.}$$

That is, in 11°,

$$\left.\begin{array}{r} aa + bb + 4pp - bb \ . \\ aa + bb + 4pp - bb - 4pp :: \\ bb \ . \\ aa \ . \end{array}\right\} \text{Proportionals.}$$

58. From 23° and 26°, ... 4□R = □AB + 4□P.
That is, in 10°, ... $4rr = bb + 4pp.$

59. Therefore from 57° and 58°, by exchanging equal quantities, this Analogy ariſeth,

$$\left.\begin{array}{r} \square AG + 4\square R - \square AB \ . \\ \square AG + 4\square R - \square AB - 4\square P :: \\ \square AB \ . \\ \square AG \ . \end{array}\right\} \text{That is, in 8°,} \left\{\begin{array}{r} aa + 4rr - bb \ . \\ aa + 4rr - bb - 4pp :: \\ bb \ . \\ aa \ . \end{array}\right.$$

60. But it hath been ſhewn in 47°, that . . . 4□AC, BC = 4□R.

61. Therefore by ſetting 4□AC, BC in the place of 4□R in the Analogy in 59°, this ariſeth, *viz.*

$$\left.\begin{array}{r} \square AG + 4\square AC,BC - \square AB \ . \\ \square AG + 4\square AC,BC - \square AB - 4\square P :: \\ \square AB \ . \\ \square AG \ . \end{array}\right\} \text{are Proportionals.}$$

62. And becauſe from 32° and 33°, A D is the ſumm, and A G the difference of A C and B C, therefore by *Theor.* 7. *Chap.* 4.

$$\square AD = \square AG + 4\square AC,BC.$$

63. Therefore from 61° and 62°, by exchanging equal quantities, this Analogy ariſeth, *viz.*

$$\square AD - \square AB \ . \ \square AD - \square AB - 4\square P :: \square AB \ . \ \square AG$$

64. And from 63°, by Converſion of Reaſon,

$$\square AD - \square AB \ . \ 4\square P :: \square AB \ . \ \square AB - \square AG.$$

65. And from 65°, by altern and inverſe Reaſon,

$$\square AB \ . \ \square AB - \square AG :: \square AD - \square AB \ . \ 4\square P.$$

66. But by *Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.

$$\square AB \ . \ \square AB - \square AG :: \square AD - \square AB \ . \ 4\square CF.$$

67. Therefore from 65° and 66°, (*per prop.* 11. and 14. *Elem.* 5.)
$4 \square CF = 4 \square P$. And conſequently, $CF = P$.

Which was to be Demonſtrated in the laſt place; and therefore *Caſe* 1. *Probl.* 4. is ſolved.

An Example in Numbers, to illuſtrate the foregoing Reſolution of Caſe 1. Probl. 3.

Suppoſ.

68. $b = 20$ the Baſe of a Triangle is given.
69. $p = 6$ the Perpendicular is given.
70. $rr = 136$ the Rectangle of the leggs is given.
71. $rr = \frac{1}{4}bb + pp$; alſo,
72. $b \sqsubset 2p$, ... } agreeable to *Caſe* 1.

Req. to find the Triangle.

Solution Arithmetical.

73. $\sqrt{136} =$ the difference of the leggs, is found out of 68° and 69°, by the Canon in 19° of this Problem.
74. $\sqrt{800} =$ the ſumm of the leggs, is found out of 68°, by the ſame Canon.
75. $\left\{\begin{matrix}\sqrt{200} + 8 = \\ \sqrt{200} - 8 =\end{matrix}\right\}$ the leggs are found out of 73° and 74°, (*per Theor.* 9. *Chap* 4.)

The Proof.

76. $\square$ of $\left\{\begin{matrix}\sqrt{200}, + 8, \\ \sqrt{200} - 8\end{matrix}\right\} = 136$ the given Rectangle. Alſo,

$$\left\{\begin{matrix} 20 = \text{the Baſe} \ldots \\ \sqrt{200} + 8, \\ \sqrt{200} - 8 \end{matrix}\right\} \text{the leggs of a Triangle.}$$

77. If . .

78. Then (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) the Perpendicular will be found 6, which is the ſame with the given Perpendicular in 69°.

Moreover, $\sqrt{136}$ and $\sqrt{136}$ ſhall be the leggs of an equicrural Triangle having the ſame Baſe, Perpendicular and Rectangle of the leggs as are before given in 68°, 69°, and 70°.

The Reſolution of CASE 2. Probl. 3.

Suppoſ.

1. $b =$ the Baſe of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ a right line given, whoſe Square is equal to the Rectangle of the leggs of the ſaid Triangle.
4. $rr \sqsubset \frac{1}{4}bb + pp$, } by *Suppoſ.* in Caſe 2.
5. $rr \sqsubset \frac{1}{2}bb$, ... }

Req. to find the Triangle.

6. The Triangle ſought in this ſecond Caſe, as alſo in the third, fourth and fifth Caſes of the Problem propounded, cannot be equicrural, for in every equicrural Triangle $rr = \frac{1}{4}bb + pp$, which agrees only with the *Suppoſ.* in *Caſe* 1. and conſequently the leggs of the Triangle ſought in all the other Caſes are unequal; therefore,
For the difference of the leggs ſought put . . . } a.
7. Then from 4°, 5° and 6°, (*per Theor.* 7. *Chap.* 4.) the Square of the ſumm of the leggs ſhall be . . . } $aa + 4rr$.
8. Therefore from the premiſſes (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*

$$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$$

9. Therefore alternately,

$$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$$

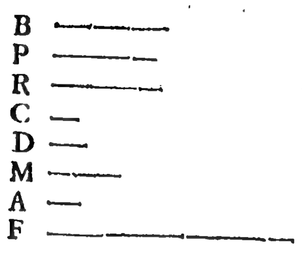
10. From 4° 'tis evident that . . . . . } . . $4rr - bb - 4pp \sqsubset 0$.
11. And from 10°, by adding aa to each part, . . . } $aa + 4rr - bb - 4pp \sqsubset aa$.
12. There-

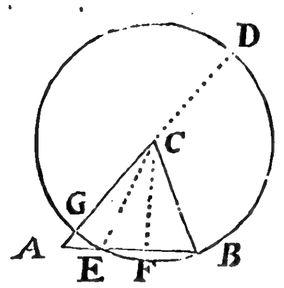
12. Therefore from the Analogy in 9°, by Diviſion of Reaſon, (which is evidently poſſible from the Equation in 11°,) theſe are Proportionals, *viz.*

$$aa + 4rr - 2bb \;.\; bb :: 4rr - bb - 4pp \;.\; aa.$$

13. From 5° 'tis eaſie to perceive that $4rr - 2bb$ is greater than nothing, ſuppoſe therefore . . } $cc = 4rr - 2bb$.
14. It appears alſo in 10°, that $4rr$ exceeds $bb + 4pp$, we may therefore ſuppoſe . . . . . } $dd = 4rr - bb - 4pp$.
15. Then from 12°, 13° and 14°, by exchanging equal quantities, this Analogy ariſeth, *viz.* . . } $aa + cc \;.\; bb :: dd \;.\; aa$.
16. And from 15°, (*per prop.* 22. *Elem.* 6.) . . } $\sqrt{:aa + cc:} \;.\; b :: d \;.\; a$.
17. Then between b and d find a mean Proportional, ſuppoſe it to be m, therefore, ... } $b \;.\; m :: m \;.\; d$.
18. Therefore from 16° and 17°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap.* 3. concerning Inordinate proportion,) this Analogy ariſeth, *viz.* . . . . . } $\sqrt{:aa + cc:} \;.\; m :: m \;.\; a$.
19. Which laſt Analogy doth evidently conſiſt of three Proportionals, whereof the greater extreme, to wit, $\sqrt{:aa + cc:}$ is equal to the Hypothenuſal of a right-angled Triangle, having c for its Baſe, and a for its Perpendicular: Now in that right-angled Triangle the Baſe c is given, alſo m, a mean Proportional between the Hypothenuſal $\sqrt{:aa + cc:}$ and the Perpendicular a, is given; therefore the Perpendicular a ſhall be given alſo, (*per Probl.* 15. *Chap.* 5.) which Perpendicular is equal to the difference of the leggs of the Triangle ſought by this *Probl.* 3. in *Caſe* 2. Then the Rectangle and the difference of the leggs being given, the leggs ſhall be given ſeverally, by *Probl.* 1. *Chap.* 9. Therefore the Triangle ſought is given alſo.

The Compoſition of CASE 2. Probl. 3.





Suppoſ.

20. $B =$ the Baſe of a Triangle is given.
21. $P =$ the Perpendicular is given.
22. $R =$ the ſide of a Square equal to the Rectangle of the leggs, is given.
23. $\square R \sqsubset \frac{1}{4} \square B + \square P$, } by *Suppoſ.* in *Caſe* 2.
24. $\square R \sqsubset \frac{1}{2} \square B$, ... }

Req. to make the Triangle.

Conſtruction.

25. By *Probl.* 4. *Chap* 5. find a right line C, ſuch, that its Square may be equal to $4 \square R - 2 \square B$, (which Effection is evidently poſſible from the *Suppoſ.* in 24°;) therefore,

$$\square C = 4 \square R - 2 \square B.$$

26. Find likewiſe a right line D, ſuch, that its Square may be equal to $4 \square R - \square B - 4 \square P$, (which may be done, for 'tis eaſie to inferr from the *Suppoſ.* in 23°, that $4 \square R \sqsubset \square B + 4 \square P$,) therefore,

$$\square D = 4 \square R - \square B - 4 \square P.$$

27. By

27. By *Probl* 9. *Chap.* 5 find a mean proportional line M between the given Baſe B and the line D, (found out in 26°;) therefore,

B . M :: M . D.

28. Then eſteeming the line C to be the Baſe of a right-angled Triangle, and the line M to be a mean Proportional between the Hypothenuſal and Perpendicular, find out (by *Probl.* 15. *Chap.* 5.) the Perpendicular it ſelf, ſuppoſe it be the right line A: Then if A be the Perpendicular, and C the Baſe, the Hypothenuſal ſhall be equal to $\sqrt{:\square A + \square C}$: and from that Effection this Analogy ariſeth, *viz.*

$\sqrt{:\square A + \square C}$: . M :: M . A.

Which line A ſhall be equal to the difference of the leggs of the Triangle ſought.

29. By *Probl.* 2. *Chap* 5. find a right line F, ſuch, that its Square may be equal to $4\square R + \square A$; therefore,

$\square F = 4\square R + \square A$.

Which line F ſhall be equal to the ſumm of the leggs of the Triangle ſought.

30. Then let a Triangle be made of theſe three right lines, *viz.* B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$; which may be done (by *Prop.* 22. *Elem.* 1.) if F be greater than A, and the ſumm of every two of thoſe three lines be greater than the third: Firſt, that $F \sqsubset A$, is evident from the *Conſtr.* in 29°. Secondly, it is alſo evident that the ſumm of B and $\frac{1}{2}F + \frac{1}{2}A$ exceeds $\frac{1}{2}F - \frac{1}{2}A$. Thirdly, that the ſumm of $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ exceeds the Baſe B, *viz.* that $F \sqsubset B$, I demonſtrate thus;

By *Conſtr.* in 29°, ...	$4\square R + \square A = \square F$.
And from the *Conſtr.* in 25°, ...	$2\square R - \frac{1}{2}\square C = \square B$.
But 'tis evident (*per Axiom.* 25. *Chap.* 2.) that	$4\square R + \square A \sqsubset 2\square R - \frac{1}{2}\square C$.
Therefore from the three laſt preceding ſteps, (*per Ax.* 3. *Chap.* 2.) . . .	$\square F \sqsubset \square B$.
And conſequently, ...	$F \sqsubset B$.

Which was to be Demonſtr.

Fourthly and laſtly, that the ſumm of B and $\frac{1}{2}F - \frac{1}{2}A$ is greater than $\frac{1}{2}F + \frac{1}{2}A$; *viz.* that $B \sqsubset A$, the following Demonſtration, formed by a repetition of the ſteps of the foregoing Reſolution in a retrograde order, will make manifeſt.

31. . . . *Req. demonſtr.* . . . . . $B \sqsubset A$.

Demonſtration.

32. Becauſe by *Conſtr.* in 28°, . . .	$\sqrt{:\square A + \square C}$: . M :: M . A.
That is, in 18°, (the laſt ſtep of the Reſolution,)	$\sqrt{:aa + cc}$: . m :: m . a.
33. And by *Conſtr.* in 27°, ...	M . B :: D . M.
34. Therefore from 32° and 33°, by exchanging the mean Proportionals, (according to *Defin.* 8. *Chap.* 3.) this Analogy ariſeth, *viz.*	$\sqrt{:\square A + \square C}$: . B :: D . A.
That is, in 16°, ...	$\sqrt{:aa + cc}$: . b :: d . a.
35. But the Squares of proportional lines are alſo Proportionals, therefore from 34°, (*per prop.* 22. *Elem.* 6.) . .	$\square A + \square C$. $\square B$ :: $\square D$. $\square A$.
That is, in 15°, ...	$aa + cc$. bb :: dd . aa.
36. And becauſe by *Conſtr.* in 25°, . .	$4\square R - 2\square B = \square C$.
37. Alſo by *Conſtr.* in 26°, ...	$4\square R - \square B - 4\square P = \square D$.

38. Therefore from 35°, 36° and 37°, by exchanging equal quantities, this Analogy ariſeth, *viz.*

$\square A + 4\square R - 2\square B$. $\square B$ :: $4\square R - \square B - 4\square P$. $\square A$.

That is, in 12°,

$aa + 4rr - 2bb$. bb :: $4rr - bb - 4pp$. aa.

39. There-

39. Therefore from 38°, by Compoſition of Reaſon,

$\square A + 4\square R - \square B$. $\square B$ :: $\square A + 4\square R - \square B - 4\square P$. $\square A$.

That is, in 9°,

$aa + 4rr - bb$. bb :: $aa + 4rr - bb - 4pp$. aa.

40. But the firſt Proportional in 39° is evidently greater than the third, therefore (*per prop.* 14. *Elem.* 5.) the ſecond ſhall be greater than the fourth, *viz.*

$\square B \sqsubset \square A$; therefore $B \sqsubset A$.

Which was to be Demonſtr.

41. Now becauſe it hath been demonſtrated that $F \sqsubset A$, and conſequently $\frac{1}{2}F - \frac{1}{2}A$ is equal to a real right line, and that the ſumm of every two of theſe three right lines, to wit, B, $\frac{1}{2}F + \frac{1}{2}A$ and $\frac{1}{2}F - \frac{1}{2}A$ is longer that the third, 'tis poſſible to make a Triangle of them (*per prop.* 22. *Elem.* 1.) Suppoſe then it be done, and that the Triangle ſo made is A B C, having its Baſe A B equal to the right line B, (the Baſe preſcribed in the Problem;) alſo $AC = \frac{1}{2}F + \frac{1}{2}A$, and $BC = \frac{1}{2}F - \frac{1}{2}A$; I ſay the Triangle A B C will ſatisfie *Caſe* 2. *Probl.* 3. before propounded. But to render the Demonſtration thereof the more eaſie, I ſhall premiſe a few things in ſeven ſteps next following.

42. Firſt, if the quantities given in the Problem be expreſt by numbers, the kind of the Triangle ſought in *Caſe* 2. ſhall be known: For the Baſe was firſt given, and by what hath been ſaid in 19°, the leggs are given alſo; therefore by the Corollary in 45° of *Probl.* 10. *Chap.* 7. it may be diſcovered whether the Triangle ſought be obtuſe-angled, acute-angled, or right-angled at the Baſe. Suppoſing then it be found that the Perpendicular falls upon the Baſe A B within the Triangle; from the Center C, at the diſtance of C B, ($= \frac{1}{2}F - \frac{1}{2}A$,) the leſſer legg of the Triangle A B C made as before, deſcribe the Circle C B G D cutting C A in G; then produce A C to the Circumference in D, and draw the Semidiameter C E; and from the Center C let fall $CF \perp EB$, therefore $FE = FB$, (*per prop.* 3. *Elem.* 3.) Then,

43. Becauſe (*per defin.* 15. *Elem.* 1.) . .	CD = CB = CG.
44. Therefore by adding A C to each part,	AD = AC + CB.
45. By *Conſtr.* in 41°, ...	$AC = \frac{1}{2}F + \frac{1}{2}A$.
46. Alſo by *Conſtr.* in 41°, ...	$CB = \frac{1}{2}F - \frac{1}{2}A$.
47. Therefore the ſumm of the Equations in 45° and 46° gives	$AD = F = AC + CB$.
48. And the Equation in 46° being ſubtracted from that in 45°, leaves	$AG = A = AC - CB$.

Now I ſhall ſhew that the Triangle A B C, formed as before is expreſt in 41°, will ſatisfie *Caſe* 2. *Probl.* 3. Firſt then, by *Conſtr.* in 41° the Baſe A B is equal to the preſcribed Baſe B. Secondly, that the Rectangle made of the leggs A C and B C is equal to the given Rectangle, that is, the Square of the right line R, I ſhall here-under demonſtrate.

49. . . . *Req. demonſtr.* . . . . . $\square AC, BC = \square R$.

Demonſtration.

50. By *Conſtr.* in 29°, ...	. $\square F = 4\square R + \square A$.
51. And becauſe from 47°, ...	$\square AD = \square F$.
52. And from 48°, ...	$\square AG = \square A$.
53. Therefore from 50°, 51° and 52°, by exchanging equal quantities, ...	$\square AD = 4\square R + \square AG$.
54. And from 53°, by ſubtracting $\square AG$ from each part, ...	$\square AD - \square AG = 4\square R$.
55. It hath been ſhewn in 47° and 48°, that A D is the ſumm, and A G the difference of A C and B C, therefore (*per Theor.* 7. *Chap.* 4)	$\square AD = 4\square AC, BC + \square AG$.
56. And from 55°, by ſubtracting $\square AG$ from each part, ...	$\square AD - \square AG = 4\square AC, BC$.

57. There-

57. Therefore from 54° and 56°, (*per Axiom.* 1. *Chap.* 2.) . . . . . } 4□AC,BC = 4□R.
58. Therefore from 57°, (*per Ax.* 21. *Chap.* 2.) . . } □AC,BC = □R.
Which was to be Demonſtr.

Thirdly and laſtly, that the Perpendicular CF is equal to the given Perpendicular P, the following Demonſtration, form'd out of the ſteps of the preceding Reſolution by a repetition of its ſteps in a backward order, will make manifeſt.

59. . . . *Req. demonſtr.* . . . . . CF = P.

Demonſtration.

It hath been proved in 39°, that

60. . □A + 4□R − □B . □B :: □A + 4□R − □B − 4□P . □A.
That is, in 9°,
aa + 4*rr* − *bb* . *bb* :: *aa* + 4*rr* − *bb* − 4*pp* . *aa*.
61. And becauſe from 41° and 48°, . . . □AB = □B; and □AG = □A.
62. Therefore from 60° and 61°, by exchanging equal quantities,
□AG + 4□R − □AB . □AB :: □AG + 4□R − □AB − 4□P . □AG.
63. Therefore from 62°, by alternate Reaſon,
□AG + 4□R − □AB . □AG + 4□R − □AB − 4□P :: □AB . □AG.
That is, in 8°,
aa + 4*rr* − *bb* . *aa* + 4*rr* − *bb* − 4*pp* :: *bb* . *aa*.
64. It hath been proved in 53°, that . . . . . □AD = □AG + 4□R.
65. Therefore from 63° and 64°, by exchanging equal quantities,
□AD − □AB . □AD − □AB − 4□P :: □AB . □AG.
66. Therefore from 65°, by Converſion of Reaſon,
□AD − □AB . 4□P :: □AB . □AB − □AG.
67. And from 66°, by Altern and Inverſe Reaſon,
□AB . □AB − □AG :: □AD − □AB . 4□P.
68. But by *Theor.* 4. in 68° of *Probl.* 8.
□AB . □AB − □AG :: □AD − □AB . 4□CF.
69. Therefore from 67° and 68°, (*per prop.* 11. *Elem.* 5.)
□AD − □AB . 4□CF :: □AD − □AB . 4□P.
70. Therefore from 69°, (*per prop.* 14. *Elem.* 5.) . . . . } 4□CF = 4□P.
71. Therefore from 70°, (*per Ax.* 21. *Chap* 2.) . . . . } □CF = □P.
72. But the ſides of equal Squares are alſo equal, therefore from 71°, } . CF = P.
Which was to be Demonſtr.

73. An Arithmetical Canon to find out the difference of the legs of the Triangle ſought in *Caſe* 2. may be deduced from the Analogy in 12°, for by comparing the Rectangle of the extremes to the Rectangle of the means of that Analogy, this ariſeth, *viz.*

aaaa + 4*rraa* − 2*bbaa* = 4*rrbb* − *bbbb* − 4*ppbb*.

Which Equation being reſolved according to the Canon in 77° of *Probl.* 15. *Chap.* 5. will give this following

CANON.

74. . . . . $\sqrt{(2) : \sqrt{4rrrr - 4bbpp} + bb - 2rr : } = a.$

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Caſe 2. Probl. 3.

Suppoſ.

75. *b* = 14 the Baſe of a Triangle is given.
76. *p* = 12 the Perpendicular is given.
77. *rr* = 195 the Rectangle of the leggs is given.
78. *rr* ⊏ $\frac{1}{4}bb$ + *pp*, } (*per Suppoſ.* in *Caſe* 2.)
79. *rr* ⊏ $\frac{1}{2}bb$, }
Req. to find the Triangle.

Solution Arithmetical.

80. 2 = the difference of the leggs, is found out of 74°, 75°, 76° and 77°.
81. 28 = the ſumm of the leggs, is found out of 7°, 77° and 80°.
82. 15 and 13 = the leggs are found out of 80° and 81°, by *Theor.* 9, *Chap.* 4.

The

The Proof.

83. 15 × 13 = 195 the given Rectangle; and if
84. { 14 = the Baſe, 13 } 15 } = the leggs } of a Triangle, then
85. 12 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 76°.

The Reſolution of CASE 3. Probl. 3.

Suppoſ.

1. *b* = the Baſe of a Triangle is given.
2. *p* = the Perpendicular is given.
3. *r* = the ſide of a Square equal to the Rectangle of the leggs is given.
4. *rr* ⊏ $\frac{1}{4}bb$ + *pp*, }
5. *rr* ⊐ $\frac{1}{2}bb$, } (*per Suppoſ.* in *Caſe* 3.)
Req. to find the Triangle.

The leggs of the Triangle ſought in this *Caſe* 3. are unequal, for the reaſon before given in 6° of the Reſolution of *Caſe* 2. therefore,

6. For the difference of the leggs put . . . . . } *a*.
7. Therefore from 3° and 6°, the Square of the ſumm of the leggs (*per Ther.* 7. *Chap.* 4.) ſhall be . . . . . } *aa* + 4*rr*.
8. And from 1°, 2°, 4°, 6° and 7°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*
aa + 4*rr* − *bb* . *aa* + 4*rr* − *bb* − 4*pp* :: *bb* . *aa*.
9. Therefore alternately,
aa + 4*rr* − *bb* . *bb* :: *aa* + 4*rr* − *bb* − 4*pp* . *aa*.
10. From 4° 'tis evident that . . . . . } 4*rr* − *bb* − 4*pp* ⊏ 0.
11. Whence it follows, by adding *aa* to each part, . . } *aa* + 4*rr* − *bb* − 4*pp* ⊏ *aa*.
12. Therefore from the Analogy in 9°, by Diviſion of Reaſon, (which is evidently poſſible from the Equation in 11°,) theſe are Proportionals, *viz.*
aa + 4*rr* − 2*bb* . *bb* :: 4*rr* − *bb* − 4*pp* . *aa*.
13. It is evident by the firſt Term of the laſt Analogy, that } *aa* + 4*rr* ⊏ 2*bb*.
14. And 'tis eaſie to infert from the *Suppoſ.* in 5°, that . . } 2*bb* ⊏ 4*rr*.
15. Therefore from 13°, by ſubtracting 4*rr* from each part, } *aa* ⊏ 2*bb* − 4*rr*.
16. It is evident, that if 2*bb* − 4*rr* be ſubtracted from *aa*, the remainder *aa* + 4*rr* − 2*bb* is the ſame with the firſt Term of the Analogy in 12°; therefore if *cc* be put equal to 2*bb* − 4*rr*, and *aa* − *cc* be taken inſtead of the ſaid firſt Term *aa* + 4*rr* = 2*bb* that Analogy will be converted into this, *viz.*
aa − *cc* . *bb* :: 4*rr* − *bb* − 4*pp* . *aa*.
17. And by putting *dd* = 4*rr* − *bb* − 4*pp*, the laſt preceding Analogy will be converted into this that follows, *viz.*
aa − *cc* . *bb* :: *dd* . *aa*.
18. But the ſides of proportional Squares are Proportionals alſo, (*per prop.* 22. *Elem.* 6.) therefore,
$\sqrt{:aa - cc:}$. *b* :: *d* . *a*.
19. Between *b* and *d* find a mean Proportional, which may be called *m*; therefore,
b . *m* :: *m* . *d*.
20. Therefore from 18° and 19°, by exchanging the mean Proportionals, according to *Defin.* 8. *Chap.* 3. concerning Inordinate Proportion, this Analogy ariſeth, *viz.*
$\sqrt{:aa - cc:}$. *m* :: *m* . *a*.
21. Which three continual Proportionals laſt expreſt being well examined, it will appear that the greater extreme *a* may be eſteem'd the Hypothenuſal of a right-angled Triangle whoſe Baſe is *c*, and the Perpendicular is $\sqrt{:aa - cc:}$ Now in that right-angled Triangle the Baſe *c* is given, as alſo *m*, a mean Proportional between the Hypothenuſal *a* and the Perpendicular $\sqrt{:aa - cc:}$ therefore the Hypothenuſal, which is equal to the difference of the leggs of the Triangle ſought in *Caſe* 3. ſhall be given alſo, *(per Probl.* 15. *Chap.* 5.) Then the Rectangle and difference of the leggs being given ſeverally, the ſumm of the leggs ſhall be given alſo, (by *Probl.* 1. *Chap.* 9.) And laſtly, the ſumm and difference of the leggs being given, the leggs ſhall be given ſeverally, (by *Theor.* 9. *Chap.* 4.)

This

This third Caſe needs not any other Determination than what is implied in the Suppoſitions; and the Compoſition may be formed (by the help of *Probl.* 15. *Chap.* 5.) in like manner as before in *Caſe* 2.

22. But for the Learner's fuller ſatisfaction, an Arithmetical Canon to find out the value of *a*, to wit, the difference of the leggs of the Triangle ſought in *Caſe* 3. is deducible from the premiſſes: For the Rectangle of the extremes of the Analogy in 12° being compared to the Rectangle of the means, and it being obſerved (as is eaſie to inferr from the *Suppoſ.* in 5°,) that $2bb \sqsubset 4rr$, this following Biquadratick Equation ariſeth, *viz.*

$$aaaa - 2bb - 4rr \times aa = 4rrbb - bbbb - 4ppbb.$$

Which Equation, if it be reſolved according to the Canon in 5 8° of *Probl.* 15. *Chap.* 5. gives this

C A N O N.

23. . . . $a = \sqrt{:bb - 2rr + \sqrt{4rrrr - 4ppbb}:}$ = the difference of the leggs.

An Example in Numbers, to illuſtrate the precedent Reſolution *of* Caſe 3. Probl. 3.

Suppoſ.

24. $b =$ 10185 the Baſe of a Triangle is given.
25. $p =$ 4752 the Perpendicular is given.
26. $rr =$ 50307696 the Rectangle of the leggs is given.
27. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppoſ.* in *Caſe* 3.
28. $rr \sqsupset \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

29. 5529 = the difference of the leggs is found out of 23°, 24°, 25° and 26°.
30. 15225 = the ſumm of the leggs is found out of 7°, 26° and 29°.
31. $\left\{\begin{matrix}10377\\4848\end{matrix}\right\}$ = the leggs are found out of 29° and 30°, (by *Theor.* 9. *Chap.* 4.)

The Proof.

32. $10377 \times 4848 = 50307696$ the given Rectangle: And, if
33. $\left\{\begin{matrix}10185 = \text{the Baſe}\\10377\\4848 = \text{the leggs}\end{matrix}\right\}$ of a Triangle; then, by thoſe three ſides,
34. 4752 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the Perpendicular given in 25°.

The Reſolution of CASE 4. Probl. 3.

Suppoſ.

1. $b =$ the Baſe of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the ſide of a given Square equal to the Rectangle of the leggs.
4. $rr \sqsubset \frac{1}{4}bb + pp$, } (*per Suppoſ.* in *Caſe* 4.)
5. $rr = \frac{1}{2}bb$, }

Req. to find the Triangle.

6. For the difference of the leggs ſought put . . . . . a.
7. Then proceed as before in the Reſolution of *Caſe* 3. from the 6th ſtep to the 12th, and let the Analogy in that 12th ſtep be here repeated, *viz.*

$$aa + 4rr - 2bb \;.\; bb :: 4rr - bb - 4pp \;.\; aa.$$

8. Now becauſe by *Suppoſ.* in 5°, ... $4rr = 2bb$.
7. And conſequently from 8°, by ſubtracting bb from each part, $4rr - bb = bb$.
10. Therefore 'tis evident from 8° and 9°, that by exchanging equal quantities the Proportionals in 7° are reducible to theſe, *viz.*

$$aa \;.\; bb :: bb - 4pp \;.\; aa.$$

11. But the ſides of proportional Squares are alſo Proportionals, therefore from 10°,

$$a \;.\; b :: \sqrt{:bb - 4pp:} \;.\; a.$$

12. There-

12. Therefore from 11°, by comparing the Rectangle of the extremes to the Rectangle of the means, this Equation ariſeth, *viz.*

$$aa = b\sqrt{:bb - 4pp:}$$

13. Therefore from 12°, by extracting the ſquare Root out of each part, the difference of the leggs ſought is made known, *viz.*

$$a = \sqrt{:b \times \sqrt{bb - 4pp}:}$$

14. And becauſe by *Theor.* 7. *Chap.* 5. the Square of the ſumm of any two right lines (or numbers) is equal to four times the Rectangle together with the Square of their difference; therefore from 3°, 6°, 12° and 13°, the ſumm of the leggs ſought is alſo made known, *viz.*

$$\sqrt{:aa + 4rr:} = \sqrt{:b\sqrt{bb - 4pp} + 4rr:}$$

The two laſt preceding Equations give this following

C A N O N.

15. . . . . $\left\{\begin{matrix}\sqrt{:b\sqrt{bb - 4pp}:} = \text{the difference of the leggs.}\\\sqrt{:b\sqrt{bb - 4pp} + 4rr:} = \text{the ſumm of the leggs.}\end{matrix}\right.$

Then the ſumm and difference of the leggs being given, the leggs ſhall be given ſeverally by *Theor.* 9. *Chap.* 4.

By which Canon and Reſolution aforegoing, the Geometrical Effection and Demonſtration of *Caſe* 4. is very eaſie to be made, and therefore I ſhall leave the ſame to the Learner's practice.

An Example in Numbers, to illuſtrate the foregoing Reſolution *of* Caſe 4. Probl. 3.

Suppoſ.

16. $b =$ 5 the Baſe of a Triangle is given.
17. $p =$ 2 the Perpendicular is given.
18. $rr =$ $12\frac{1}{2}$ the Rectangle of the leggs is given.
19. $rr \sqsubset \frac{1}{4}bb + pp$, } agreeable to the *Suppoſ.* in *Caſe* 4.
20. $rr = \frac{1}{2}bb$, }

Req. to find the Triangle.

Solution Arithmetical.

21. $\sqrt{15}$ = the difference of the leggs is found out of 15°, 16° and 17°.
22. $\sqrt{65}$ = the ſumm of the leggs is found out of 15°, 16°, 17° and 18°.
23. $\left\{\begin{matrix}\sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}}\\\sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}}\end{matrix}\right\}$ = the leggs, found out of 21° and 22°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

24. . . . . $\sqrt{\frac{15}{4}} + \sqrt{\frac{15}{4}}$ into $\sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}} = 12\frac{1}{2}$ the given Rectangle.
25. And, if $\left\{\begin{matrix}5 = \text{the Baſe of a Triangle.}\\\sqrt{\frac{65}{4}} + \sqrt{\frac{15}{4}}\\\sqrt{\frac{65}{4}} - \sqrt{\frac{15}{4}}\end{matrix}\right\}$ the Leggs;
26. Then 2 = the Perpendicular will be found out, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 17°.

The Reſolution of CASE 5. Probl. 3.

Suppoſ.

1. $b =$ the Baſe of a Triangle is given.
2. $p =$ the Perpendicular is given.
3. $r =$ the ſide of a Square equal to the Rectangle of the leggs is given.
4. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the preceding *Suppoſ.* in *Caſe* 5.

Req. to find the Triangle.

5. For the difference of the leggs ſought put . . . . } a.
6. Then from 3° and 5°, (*per Theor.* 7. *Chap.* 4.) the Square of the ſumm of the leggs is . . . . } $aa + 4rr$.
7. And from 1°, 2°, 5° and 6°, (*per Theor.* 2. in 34° of *Probl.* 8. *Chap.* 8.) this Analogy ariſeth, *viz.*

$$aa + 4rr - bb \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa.$$

8. Therefore alternly,

$$aa + 4rr - bb \;.\; bb :: aa + 4rr - bb - 4pp \;.\; aa.$$

9. Therefore inverſly,

$$aa \;.\; aa + 4rr - bb - 4pp :: bb \;.\; aa + 4rr - bb.$$

10. By quadrupling all in 4°, and ſubtracting $\frac{1}{4}bb + pp$ from each part, it will be evident that . . . . } $4rr - bb - 4pp \sqsupset 0$.
11. And by adding aa to each part in 10°, ... } $aa + 4rr - bb - 4pp \sqsupset aa$.
12. Therefore from 9°, by Converſion of Reaſon, (which the 11th ſtep ſhews is poſſible,) theſe are Proportionals, *viz.*

$$aa \;.\; bb + 4pp - 4rr :: bb \;.\; 2bb - 4rr - aa.$$

13. It is evident from the *Suppoſ.* in 4°, that . . . } $bb + 4pp \sqsubset 4rr$.
14. Therefore for the exceſs whereby $bb + 4pp$ exceeds $4rr$ we may put dd, whence, ... } $dd = bb + 4pp - 4rr$.
15. By viewing the 11th ſtep it will appear that the third Proportional in 8° is leſs than the fourth, therefore the firſt is leſs than the ſecond, *viz.* . . . . } $aa + 4rr - bb \sqsupset bb$.
16. Whence, by adding bb to each part, ... } $aa + 4rr \sqsupset 2bb$.
17. And from 16°, by comparing the latter part to the firſt, 'tis eaſie to inferr that . . . . . } $2bb \sqsubset 4rr$.
18. Therefore for the exceſs whereby $2bb$ exceeds $4rr$, we may put cc, whence, ... } $cc = 2bb - 4rr$.
19. Then from the Analogy in 12°, by exchanging equal quantities according to the Poſitions in 14° and 18°, this Analogy ariſeth, *viz.* . . . . } $aa \;.\; dd :: bb \;.\; cc - aa$.
20. And becauſe (*per prop.* 22. *Elem.* 6.) the ſides of proportional Squares are alſo Proportionals, therefore from 19°, ... } $a \;.\; d :: b \;.\; \sqrt{:cc - aa}$:
21. Between b and d find a mean Proportional, which may be called m, therefore . . . . . } $b \;.\; m :: m \;.\; d$.
22. Therefore from 20° and 21°, by exchanging the mean Proportionals according to *Defin.* 8. *Chap.* 3. this Analogy ariſeth, *viz* . . . . . } $a \;.\; m :: m \;.\; \sqrt{:cc - aa}$:
23. Which laſt Analogy doth evidently conſiſt of three continual Proportionals, whereof the extremes a and $\sqrt{:cc - aa}$: may be eſteem'd the Baſe and Perpendicular of a right-angled Triangle whoſe Hypothenuſal c is given, as alſo m a mean Proportional between the Baſe and Perpendicular; therefore (*per Probl.* 16. *Chap.* 5.) the Baſe and Perpendicular ſhall be given ſeverally, either of which may be taken for the difference of the leggs of a Triangle to ſatisfie the Problem propounded in *Caſe* 5. For here, two different Triangles may be always found out that ſhall have theſe three things common, to wit, the Baſe, the Perpendicular, and Rectangle of the leggs, except when it happens that $\frac{bp}{r} = r$, for then the two values of a will be equal to one another, in which Caſe there can only one Triangle be found out to agree with the preceding *Suppoſ.* in 1°, 2°, 3° and 4°.

But that there may be a poſſibility of finding out a Triangle to ſolve *Probl.* 3. in *Caſe* 5. the right line repreſented by $\frac{bp}{r}$ muſt not be longer than r: Now to make it evident that this Determination is neceſſary in *Caſe* 5,

24. . . . *Req. demonſtr.* . . . . . $\frac{bp}{r}$ not $\sqsubset r$.

Demonſtr.

Demonſtration.

25. Firſt, if c be given for the Hypothenuſal and m for a mean Proportional between the Baſe and Perpendicular of a right-angled Triangle, (as before is ſuppoſed in 23°,) then to find out that Triangle by *Probl.* 16. *Chap.* 5. this Determination is neceſſary, *viz.* . . . . } $\frac{mm}{c}$ not $\sqsubset \frac{1}{2}c$.
26. Therefore from 25°, by multiplying each part into c, ... } mm not $\sqsubset \frac{1}{2}cc$.
27. But from 21°, (*per prop.* 17. *Elem.* 6.) } $bd = mm$.
28. Therefore from 26° and 27°, (*per Ax.* 4. *Chap.* 2.) . . . . . } bd not $\sqsubset \frac{1}{2}cc$.
29. And becauſe (as is evident in 14°,) . . } $\sqrt{:bb + 4pp - 4rr} = d$.
30. Therefore from 28° and 29°, by exchanging equal Factors, ... } $b\sqrt{:bb + 4pp - 4rr}$: not $\sqsubset \frac{1}{2}cc$.
31. And becauſe from 18°, ... } $bb - 2rr = \frac{1}{2}cc$.
32. Therefore from 30° and 31°, (*per Ax.* 3. *Chap.* 2.) . . . . . } $b\sqrt{:bb + 4pp - 4rr}$: not $\sqsubset bb - 2rr$.
33. And from 32°, (by dividing each part by b,) } $\sqrt{:bb + 4pp - 4rr}$: not $\sqsubset b - \frac{2rr}{b}$.
34. And from 33°, by ſquaring each part,

$$bb + 4pp - 4rr \text{ not } \sqsubset bb - 4rr + \frac{4rrrr}{bb}.$$

35. And from 34°, by adding $4rr$ unto, and ſubtracting bb from each part, ... } $4pp$ not $\sqsubset \frac{4rrrr}{bb}$.
36. And from 35°, by extracting the ſquare Root out of each part, ... } $2p$ not $\sqsubset \frac{2rr}{b}$.
37. And from 36°, by multiplying each part by b, } $2bp$ not $\sqsubset 2rr$.
38. And by halving each part in 37°, . . . } bp not $\sqsubset rr$.
39. Therefore from 38°, by dividing each part by r, ... } $\frac{bp}{r}$ not $\sqsubset r$. Which was to be Demſt.

40. Suppoſing then that $\frac{bp}{r}$ is not greater than r, the Compoſition of *Caſe* 5. *Probl.* 3. may be eaſily formed out of the laſt preceding Reſolution, by the help of *Probl.* 16. *Chap.* 5. in like manner as before in *Caſe* 2. and an Arithmetical Canon to find the difference of the leggs of the Triangle ſought in the ſaid 5th Caſe may be deduced from the Analogy in the foregoing 12th ſtep: For the Rectangle of the extremes of that Analogy being compared to the Rectangle of the means, this following Biquadratick Equation ariſeth, *viz.*

$$\overline{2bb - 4rr} \times aa, - aaaa = bbbb + 4ppbb - 4rrbb.$$

Which Equation being reſolved according to the Canon in 55° of *Probl.* 16. *Chap.* 5. gives this

CANON.

41. $a = \sqrt{:bb - 2rr + \sqrt{4rrrr - 4bbpp}}$: Alſo, $a = \sqrt{:bb - 2rr - \sqrt{4rrrr - 4bbpp}}$:
42. Now either of thoſe Roots or values of a may be taken for the difference of the leggs of a Triangle to ſolve *Caſe* 5. *Probl.* 3. before propounded; and if it happens that $\frac{bp}{r} \sqsupset r$, (and conſequently, $bp \sqsupset rr$,) then thoſe values of a are unequal, and there may be two different Triangles found out to ſatisfie the ſaid *Caſe* 5. But when $\frac{bp}{r} = r$, and conſequently, $bp = rr$,) then the ſaid values of a are equal to one another, each being equal to $\sqrt{:bb - 2rr}$: which ſhall be the difference of the leggs of a Triangle having a right angle oppoſite to the Baſe, (as appears by Canon 1. in 21° of *Probl.* 14. *Chap.* 8.) Which Triangle, when $\frac{bp}{r} = r$, is the only Triangle that can be found to ſolve *Caſe* 5. *Probl.* 3.

Examples

Examples in Numbers, to illuſtrate the preceding Reſolution of Caſe 5. Probl. 3.

Examp. 1. *Where two Triangles are found out to ſolve* Caſe 5.

Suppoſ.

43. $b = 51$ the Baſe of a Triangle is given.
44. $p = 12$ the Perpendicular is given.
45. $rr = 740$ the Rectangle of the leggs is given.
46. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppoſ.* in *Caſe* 5.
Req. to find the Triangle.

Solution Arithmetical.

47. 17 = the difference of the leggs is found out of 43°, 44° and 45°, by the leſſer Root in 41°.
48. 57 = the ſumm of the leggs is found out of 6°, 45° and 47°.
49. 37 and 20 = the leggs are found out of 47° and 48°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

50. $37 \times 20 = 740$ the given Rectangle.
51. And if { 51 = the Baſe, 37, 20 } = the leggs } of a Triangle; then,
52. 12 = the Perpendicular will be found out of 51°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is the ſame with the given Perpendicular in 44°.

Again, from the ſame things given as before in 43°, 44°, 45°, another Triangle may be found out by the greater Root in 41°, to ſolve *Caſe* 5. *Probl.* 3. the ſides of which latter Triangle are here-under expreſt, *viz.*

53. $\sqrt{1953}$ = the difference of the leggs.
54. $\sqrt{4913}$ = the ſumm of the leggs.
55. $\sqrt{\frac{4225}{4}} + \sqrt{\frac{1225}{4}}$
56. $\sqrt{\frac{4225}{4}} - \sqrt{\frac{1225}{4}}$ } the leggs.

The Proof is eaſie to be made, in like manner as in *Example* 1.

Examp. 2. *Where only one Triangle can be found out to ſolve* Caſe 5. Probl. 3.

57. $b = 169$ the Baſe of a Triangle is given.
58. $p = 60$ the Perpendicular is given.
59. $rr = 10140$ the Rectangle of the leggs is given.
60. $rr \sqsupset \frac{1}{4}bb + pp$, agreeable to the *Suppoſ.* in *Caſe* 5.
Req. to find the Triangle.

Solution Arithmetical.

61. 91 = the difference of the leggs is found out of 57°, 58° and 59°, by either of the Roots in 41°.
62. 221 = the ſumm of the leggs is found out of 6°, 59° and 61.
63. 156 and 65 = the leggs are found out of 61° and 62°, (*per Theor.* 9. *Chap.* 4.)

The Proof.

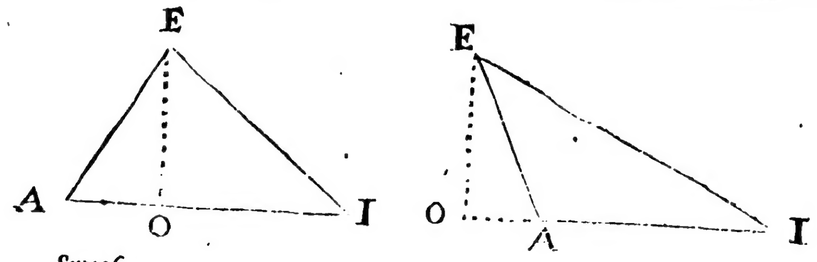
64. $156 \times 65 = 10140$ the Rectangle given in 59°.
65. And if { 169 = the Baſe, 156, 65 } = the leggs } of a Triangle, then
66. 60 = the Perpendicular will be found out of 65°, (*per Theor.* 4. in 68° of *Probl.* 8. *Chap.* 8.) which is equal to the given Perpendicular in 58°.

Note. This laſt Triangle hath a right angle oppoſite to the Baſe, agreeable to what was before hinted in 42°.

A LEMMA, *leading to the following* Probl. 4.

In any plain Triangle, As the Sine of an angle is to the Radius, (or total Sine;) So is the double Area of the Triangle, (that is, the Rectangle made of the Perpendicular and Baſe,) to the Rectangle of the leggs containing the ſaid angle.

Suppoſ.



Suppoſ.

1. EI the Baſe
2. { AE, AI } the leggs } of the oblique-angled △ AIE.
3. $\sphericalangle$A, (that is, $\sphericalangle$EAI) is contain'd under the leggs AE, AI.
4. EO $\perp$ AI.
5. R = the Radius, (or total Sine.)
6. S$\sphericalangle$A = the Sine of the angle A.
7. . . . *Req. demonſtr.* . . . . . S$\sphericalangle$A . R :: $\square$EO,AI . $\square$AE,AI.

Demonſtration.

8. By a vulgar *Axiom* in the Doctrine of plain Triangles this Analogy is manifeſt, *viz.* } S$\sphericalangle$A . R :: EO . AE.
9. Therefore by drawing AI into each of the two latter Terms of that Analogy, this ariſeth, *viz.* } S$\sphericalangle$A . R :: $\square$EO,AI . $\square$AE,AI.
Which was to be Demonſtr.

Probl. IV.

The Baſe of a plain Triangle being given, as alſo the Perpendicular, and angle oppoſite to the Baſe, to find the Triangle.

Conſtruction.

Let a Circle be deſcribed by a Radius (or Semidiameter) taken at pleaſure, and according to the *Note* at the beginning of *Probl.* 19. *Chap.* 8. find out a right line that ſhall be the Sine of an angle equal to the given angle. Then (by *Probl.* 9. *Chap.* 5.) find out a Square equal to a Rectangle made of the given Baſe and Perpendicular. That done, let it be made, (by *Probl.* 11. *Chap.* 5.) As the Sine, (found out as above,) to the Radius firſt aſſumed; So the ſaid Square to another Square, which Square (or fourth Proportional) found out, ſhall be equal to the Rectangle of the leggs containing the given angle, (as is evident by the foregoing *Lemma*) Now there is given the Baſe and Perpendicular, as alſo a Square equal to the given Rectangle of the leggs, to find out the Triangle; and therefore if thoſe given quantities be expreſt by numbers, this *Probl.* 4. may be ſolved both Geometrically and Arithmetically in all Caſes, by the help of the preceding *Probl.* 3.

Note. Some ſubtil Geometrical Problems, wherein the meaſures of angles as well as of right lines are given in numbers, may be ſolved Arithmetically by the Doctrine of Plain Triangles, without the help of *Algebra*; of ſuch kind is the following Problem, with which I ſhall conclude this Treatiſe.

Probl. V.

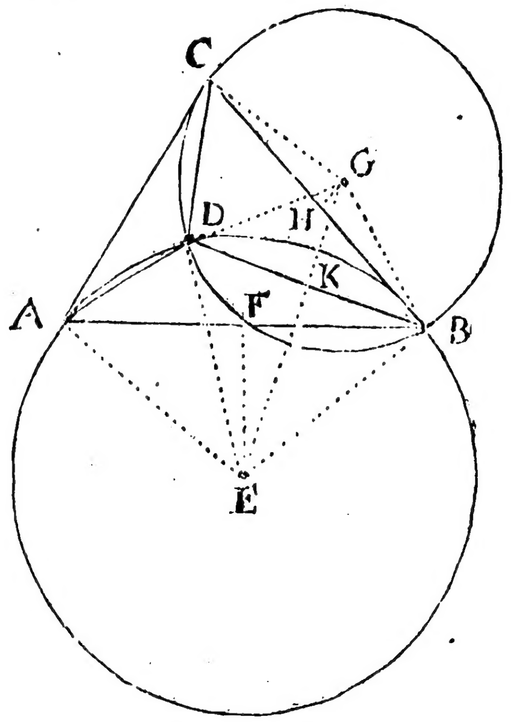
The diſtances AB, AC, BC between three Towers A, B and C not ſtanding in a ſtraight line, being given ſeverally in Feet. Alſo a fourth Tower being ſuppos'd to ſtand within the Triangle ABC, as at D; and the meaſures of the angles ADB, BDC and CDA being given ſeverally in Degrees; to find the diſtance between the fourth Tower D and each of the other three, *viz.* the meaſures of the three right lines DB, DA and DC in Feet.

Prepar.

Foraſmuch as by *Suppoſ.* the point D lyes within the Triangle ABC, and conſequently the three points A, D, B do not lye in a ſtraight line; let the Circumference of a Circle be ſuppoſed to paſs by thoſe points, as ADB, whoſe Center is E; ſuppoſe alſo the Semi-

Semidiameters E A, E D, EB to be drawn, and E F ⊥ A B. In like manner, ſuppoſing G to be the Center of a Circle whoſe Circumference paſſeth by the points B, D, C, draw the Semidiameters G B, G D, G C, and make G H ⊥ C B.

	Feet.
A B =	20000
A C =	15000
B C =	18000
E B =	12521.286
G B =	9317.445
D B =	13888.610
D A =	8283.832
D C =	8406.944



	Gr. Min.
∠ A D B =	127: 0
B D C =	105: 0
C D A =	128: 0
F E B =	53: 0
E B A =	37: 0
H G B =	75: 0
G B C =	15: 0
C B A =	46: 8
G B E =	98: 8
G E B =	33:41
D B A =	19:19
D B C =	26:49

Solution Arithmetical, by the Doctrine of plain Triangles.

Firſt, ſubtract the given ∠ A D B from two right angles, (*viz.* from 180. Degrees,) the remainder ſhall be the ſumm of the unknown angles D A B and D B A, (*per prop.* 32. *Elem.* 1.)

Secondly, foraſmuch as (by *prop.* 20. *Elem.* 3.) ∠ D E B = 2 ∠ D A B, and ∠ D E A = 2 ∠ D B A; it follows that ∠ A E B = 2 ∠ D A B + 2 ∠ D B A; therefore in △ F E B right-angled at F, the ∠ F E B (that is, ½ ∠ A E B) = ∠ D A B + ∠ D B A is given; and by *Suppoſ.* F B = ½ A B is given, therefore the Semidiameter E B = E D = E A ſhall be given alſo.

Thirdly, by arguing as above in the firſt and ſecond ſteps ∠ H G B = ∠ H G C is given; alſo G D = G C = G B the Semidiameter of the Circle G B D C is given.

Fourthly, becauſe E F ⊥ A B and ∠ F E B is given as before, therefore ∠ E B A the Complement of the ∠ F E B to a right angle is given; likewiſe the ∠ G B C the Complement of ∠ H G B to a right angle is given; and the ∠ C B A is given, for it may be found out by the three given ſides A B, A C, B C; therefore ∠ G B E the ſumm of thoſe three angles, E B A, C B A, G B C is given.

Fifthly, in △ G B E, the ſides G B and E B, (to wit, the Semidiameters of the two Circles G B D C and E A D B,) are given ſeverally, as alſo the angle G B E comprehended by thoſe ſides, therefore the angle G E B is given alſo.

Sixthly, becauſe the two Triangles E G B and E G D have two ſides G B, E B equal to the two ſides G D, E D, *viz.* G B = G D, and E B = E D, alſo the Baſe G E common to both thoſe Triangles; the angles contain'd under equal right lines ſhall be equal, *viz.* ∠ G E B = ∠ G E D = ½ ∠ D E B: But ∠ G E B (= ½ ∠ D E B) is given in the fifth ſtep, and (*per prop.* 20. *Elem.* 3.) ∠ D A B is equal to ½ ∠ D E B (= ∠ G E B,) therefore ∠ D A B is given. Now in △ A D B there is given ∠ D A B, as alſo ∠ A D B and the ſide A B, therefore the ſides D B and D A, (to wit, two of the Diſtances ſought,) are given alſo.

Seventhly and laſtly, in △ D A C there is given D A, as alſo A C, and ∠ A D C, therefore D C (the third Diſtance ſought,) is given.

The End of the Fourth and laſt BOOK.

SOLI DEO GLORIA.